FUNDAMENTAL GROUP ACTION ON FIBRE SET

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Abstract:-

The structure of a topological space may be described by associating with it an algebraic system usually a group or a sequence of groups. There are several methods by which groups can be associated with topological spaces. Here we shall consider one through homotopy and the group so defined is called Fundamental group of the given topological space. The algebraic structure of the group reflects the topological and geometrical structures of the underlying space. The theory of Covering spaces is closely connected with the study of Fundamental group. Many basic topological questions about covering spaces can be reduced to purely algebraic questions about the fundamental groups of the various spaces involved. In Abstract Algebra, the action of a group on a set has many applications. We shall apply the group action concept of Abstract Algebra in Algebraic Topology. In this article we give description about the action of Fundamental group on Fibre set.

Key words:-

Fundamental Group, Fibre set, Group action, Covering spaces, Path, Loop, Path product, Loop product, Homotopy , Base point.

1.Introduction :-

Topology is an abstraction of Geometry. It deals with sets having a structure which permits the definition of continuity for functions and a concept of closeness of points and sets. Historical origins of Algebraic topology was some what different. The systematic study of Algebraic topology was initiated by French Mathematician Henri Poincare (1854-1912). Algebraic Topology or analysis situs did not develop as a branch of point set topology. It was motivated by specific geometric problems involving paths, surfaces and geometry in Euclidean spaces. Algebraic Topology describes the structure of a Topological space by associating it with an algebraic system, usually a group or a sequence of groups.

Here we give description about Fundamental group of Topological spaces. By using Fundamental group, topological problems about spaces can be reduced to purely algebraic problems about groups and homomorphisms . The theory of Covering spaces is closely connected with the study of Fundamental group. Many basic topological questions about covering spaces can be reduced to purely algebraic questions about the fundamental groups of various spaces involved. The fundamental group is an instrumental in determining and classifying topological spaces which are covering spaces of given space.The theory of covering space will allow us to determine the fundamental groups of several rather complicated spaces. In this article we give description about the action of the fundamental group $\pi(X,x)$ on the fibre set $p^{-1}(x)$. For that we define path , loop and homotopy between loops.

2. Materials and Methods

In this section we shall include the materials and methods that are required for the further work of this article. This section includes some definitions and lemma that are essential for developing main results our work. We also include some results from Abstract Algebra that are essential for this discussion.

Definition1 **[9]**

Let G be a group and let E be a nonempty set. Let there be a mapping from E X G→E which maps (x , g) → x.g for x in E and g in G such that the following two conditions hold

(1) x.1 = x

(2) $x.(g_1.g_2) = (x.g_1).g_2$ for x in E and g_1, g_2 in G .

Also 1 is the identity element of the group G.

Here E is called a **G- set** or **G- space**.

Definition 2 **[3]**

A *path* or *arc* in a topological space X is a continuous map of some closed interval I = [0,1] in to X.

Let α : I → X be a path in X. Then α(0) is called **initial point** and α(1) is called **terminal point** of the path α.

Definition 3 **[6]**

Let α and β be two paths in X such that terminal point of and initial point of are same. Then the ***path product*** *of* α and β denoted by α * β is defined as

$$\alpha * \beta(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq 1/2 \\ \beta(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

By assumption, α (1) = β (0)

Now we shall define relation between paths in a space .

Definition 4 **[3]**

Let α and β be two paths in X with common initial and terminal points. Then we say that , α is ***equivalent to*** β , that is $\alpha \sim \beta$, if there exists a continuous function

H: I x I → X such that

H(t,0) = α (t)

H(t,1) = β (t)

H(0,s) = α (0) = β (0)

H(1,s) = α (1) = β (1) , for t ,s ϵ I.

The function H is called ***Homotopy*** between α and β .

Definition 5 **[6]**

A ***loop*** in a topological space X is a closed path in X.

If α: I → X is a loop in X, then α(0)= α(1) .

The common end point is called ***base point*** of the loop.

***Definition** 6* **[6]**

Let α and β be two loops in X having base point x_0 .Then α is ***equivalent*** *to* β (α is ***homotopic to*** β) if there exists a continuous map H; I X I →X such that

H(t,0) = α (t)

H(t,1) = β (t)

H(0,s)= H(1,s) = x_0 , for t ,s є I.

***Definition** 7* **[3]**

If α and β are any two loops in X with same base point x_0 .Then the *loop product*

α * β is defined as

$$\alpha * \beta(t) = \begin{cases} \alpha(2t), & 0 \le t \le 1/2 \\ \beta(2t-1), & 1/2 \le t \le 1 \end{cases}$$

α * β is a loop in X based at x_0.

***Defintion** 8* **[7]**

The relation *Homotopy of loops* is an equivalence relation on the set of all loops in the space X with same base point. The corresponding equivalence classes are called ***homotopy classes***. Let $\pi(X,x)$ be the collection of all homotopy classes of loops in X based at x. Then $\pi(X,x)$ is a group under the operation o . This group is called ***Fundamental group*** of X at x. or ***First Poincare group*** of X at x.

Definition **9[3]**

A ***covering space*** X is a pair($\check{X}$, p) consisting of a space $\check{X}$ and a continuous map

p: $\check{X}$ → X with the property that each point x in X, there is a path connected open neighbourhood U c X such that each path component of p-1 (U) is mapped homeomorphically on to U by p.

The neighbourhood U is called ***Admissible neighbourhood*** and the mapping p is called ***covering projection***.

Definition **10[3]**

Let ($\check{X}$, p) be a covering space of X. Let f: I → X be a path in X. Thus a path

g: I →$\check{X}$ in $\check{X}$ such that pg = f is called **lifting path (Covering path)** of f.

Lemma **11:-**

Let ($\check{X}$, p) be a covering space of X , $\widetilde{x_0}$ is an element of $\check{X}$ and p$\widetilde{x_0}$ = x_0.

Then for any path f: I $\rightarrow$ X with initial point x_0, there exists a unique path

g : I $\rightarrow$ $\breve{X}$ with initial point $\breve{x}_0$ such that pg = f.

3. RESULTS

Action of Fundamental group on Fibre set

This section includes the main results of this article. Here we shall give description about the action of Fundamental group on fibre set.

Definition 12:-

For any space X, if ($\breve{X}$, p) is a covering space of X, then the set p^{-1} (x) consists of all $\breve{x}$ in $\breve{X}$ such that p($\breve{x}$)= x is called **fibre set.**

The *action of Fundamental group on Fibre set* is defined as follows.

Let α be a loop in X based at x.

Then [α] $\in \pi(X, x)$

For $\breve{x} \in$ p^{-1}(x), there is a unique path $\breve{\alpha}$ in $\breve{X}$ with initial point $\breve{x}$ such that

p$\breve{\alpha}$ = α

Then p $\breve{\alpha}$ (1) = α(1) = x, so that p $\breve{\alpha}$ (1) $\in$ p-1(x).

This gives an association of an element in p^{-1}(x) with the pair ($\breve{x}$, [α]) .

Thus we get a correspondence from p^{-1}(x) X $\pi(X, x)$ to p-1(x).

The map so defined has the following properties.

For [α] , [β] in $\pi(X, x)$ and $\breve{x} \in$ p-1(x) ,

($\breve{x}$. [α]) . [β] = $\breve{\beta}$(1) where $\breve{\beta}$ is the path in $\breve{X}$ with initial point

$\breve{x}$. [α] = $\breve{\alpha}$ (1) and such that p $\breve{\beta}$ = β

Now $\breve{\alpha}$ and $\breve{\beta}$ are paths in $\breve{X}$ such that $\breve{\alpha}$ (1) = $\breve{\beta}$(0)

Hence the product $\breve{\alpha} * \breve{\beta}$ is defined in $\breve{X}$ such that

$$\breve{\alpha} * \breve{\beta}(t) = \begin{cases} \breve{\alpha}(2t), & 0 \leq t \leq 1/2 \\ \breve{\beta}(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

Now $\breve{\alpha} * \breve{\beta}$ (0) = $\breve{\alpha}$ (0) = $\breve{x}$

Also $p(\breve{\alpha} * \breve{\beta}) = \begin{cases} p\breve{\alpha}(2t), & 0 \leq t \leq 1/2 \\ p\breve{\beta}(2t-1), & 1/2 \leq t \leq 1 \end{cases}$

$= \begin{cases} \alpha(2t), & 0 \leq t \leq 1/2 \\ \beta(2t-1), & 1/2 \leq t \leq 1 \end{cases}$

$= \alpha * \beta$

Thus $\breve{\alpha} * \breve{\beta}$ is a path in $\breve{X}$ with initial point $\breve{x}$ such that $p(\breve{\alpha} * \breve{\beta}) = \alpha * \beta$

So by definition, $\breve{x}.([\alpha].[\beta]) = \breve{x}.([\alpha * \beta])$

$= (\breve{\alpha} * \breve{\beta})(1)$

$= \breve{\beta}(1)$

Hence we get $(\breve{x}.[\alpha]).[\beta] = \breve{x}.([\alpha].[\beta])$

The identity element of the fundamental group $\pi(X, x)$

Is [Cx] where Cx is the constant loop defined by Cx (t) = x , for all , $0 \leq t \leq 1$.

The lifting $\widetilde{C_x}$ has initial point $\breve{x}$ and satisfies $p\widetilde{C_x} = Cx$.

But the path $C_{\breve{x}}$ defined by $C_{\breve{x}}(t) = \breve{x}$ for all t has the initial point $\breve{x}$ and satisfies $pC_{\breve{x}}(t) = p\breve{x} = x = Cx(t)$, for all t.

So $pC_{\breve{x}} = Cx$

Hence $\widetilde{C_x} = C_{\breve{x}}$

So $\breve{x}.1 = \breve{x}.[Cx]$

$= \widetilde{C_x}(1)$

$= C_{\breve{x}}(1)$

$= \breve{x}$

So the correspondence from $p^{-1}(x) \times \pi(X,x) \to p^{-1}(x)$ is actually an action of $\pi(X,x)$ on $p^{-1}(x)$.

4. ***Discussion***

In Abstract Algebra , if a group G acts on a set S, then S is called as a *G- set*. In this article ,we have discussed about action of the group on a set in Algebraic Topology. In our discussion, the concerned group is the Fundamental group $\pi(X,x)$ and the set we concerned is the fibre set $p^{-1}(x)$. That is we have showed that $p^{-1}(x)$ is a $\pi(X,x)$ - set . For the comprehensive study of this article, we needed some results from Covering Spaces which also included in the discussion. The detailed study of Fundamental group and covering spaces is obtained from [4], [5], [8] &[14].

5. Conclusion

In this article we have discussed about the action from the Fundamental group on the fibre set by using results of fundamental group and covering spaces. This result has many scope for future study. As in Abstract Algebra, from this group action , we can think of orbit, stabilizer etc. The study of orbit and stabilizer will pave the way for obtaining outstanding results in Algebraic Topology. Here we correlates Abstract Algebra to Algebraic Topology. Researchers can correlate this results to other branches of Mathematics. This discussion is a simple step for that study.

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7. References

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