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An $M/G/1$ Retrial Queue with Non-Persistent customers, Recurrent Customers, General Retrial Times with Working Vacation

S. Pazhani Bala Murugan

Associate Professor, Mathematics Section, FEAT, Annamalai University, Annamalainagar-608002, Tamilnadu, India, spbm1966@gmail.com

A. Jeba Pauline Veronica

Research Scholar, Department of Mathematics, Annamalai University, Annamalainagar-608002, Tamilnadu, India, ajpv2519@gmail.com

ABSTRACT

An $M/G/1$ retrial queue with non-persistent customers, recurrent customers, general retrial times with working vacation is get into the contemplation of this work. We consider two types of customers; recurrent and transit customers. If the arriving customer or primary customer finds that server is busy, he will either leaves the system completely without taking service or he will go to the orbit. All service times and retrial times for transit customers follow general distribution, retrial time for recurrent customers and working vacation time areassumed to have an exponential distribution, also service time of the recurrent customers follows and general distribution. The PGF and the mean of number of customers in an imperceptible waiting area are get by the effective action of supplementary variable technique. As a matter of interest, some special cases are furnished.

Keywords: Retrial queue, Recurrent customer, Transit customer, Working vacation, Non-Persistent customer, Supplementary variable technique, Probability generating function, Waiting time, Idle State

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INTRODUCTION

For the last few years, there has been extensive research into retrial queue in the field of queuing theory. *Retrial queues* are distinguished by the fact that an arriving customer discovers, a server is busy when it arrives, it is requested to leave or decides to leave the service zone and enters to an invisible place to the server, known as an *orbit*. Continuing to follow some random period, the customer makes many efforts to obtain service. This request is independent of the other customers in the orbit. In the analysis of telephone and other communication systems, these models with repeated attempts appear often. One could visit, [6,7,22] to allow a more detailed examination of the retrial queues.

Boxma OJ, Cohen JW [4] investigated an $M/G/1$ queue with specific number of recurrent customers, who rejoin the queue after completing their service (Basically in Stochastic Process, a state is considered to be *recurrent* if, whenever we leave it, we will return to it with probability one. Instead, *transient* if the probability of return is less than one.). Farahmand [9] examined the idea of recurrent customers in the retrial queue, examining two cases: the constant case, in which the call-up rate of each customer is independent of the number of customers in orbit, and the inconstant case, in which the call-uprate drops as, in orbit the number of customers rises. Moreno [14] investigated an $M/G/1$ retrial queue with a specific number K recurrent customers ($K = 1,2,3...$) which are return to orbit immediately after service; once the service complete, transit customers who exit the system will never return. Vacation queueing model was introduced as a development of the classical queueing theory in the 1970's. In this, the server may be unavailable for some timeframe for number of reasons, such as: searching for restoration, starting to work at some other queues, checking for new work, or merely by taking a break from their work, these are regular traits of many communication systems. So, this timeframe is known as *vacation* (the server is inaccessible to the primary customers).

In addition, vacation strategy deliberated by Servi and Finn [19], a new vacation tactic known as Working Vacation($\boldsymbol{WV}$) was developed. During a time period of WV, the server offers customers a lower rate of service than during the Usual Busy($\boldsymbol{UB}$) period. Moreover, suppose there is a customer in the service zone when completion of the Working Vacation(WV) period, the server can end the vacation and access to its UB period while servicing, known as *vacation interruption*. Wu and Takagi [21] investigated $M/G/1/MWV$.

For the recent development on $M/G/1$ retrial queue with vacation policy, one could see: the retrial queue with vacation was established in [10], the $M/G/1$ retrial queue with MWV was investigated in [15], $M/G/1$ retrial queue with single working vacation in [16]. Moreover, [6] provide a complete investigation of WV period.

Krishnamorthy et. al. [12] investigate an $M/G/1$ retrial queue with non-persistent customer and orbital search, in which completion of each service the server will search for the customers in orbit and arrival of primary and orbital customers finds if server is occupied then they will either leaves the system completely without taking service or they may return to the orbit. For a detailed study on non-persistent customers model one can refer Kasturi

Ramanath and Kalidass [11], Pazhani Bala Murugan and Vijayakrishnaraj [17], Peishu Chan and et. al. [18].

Here we present, characterisation of the model, steady state probabilities, performance measures, some special cases, numerical results and then conclusion.

MODEL DESCRIPTION

In this article we investigate on an $M/G/1$ retrial queue, retrial time follows general with WV period. We examine a retrial queue with single server and customers of two types: one is transit (also known as usual) customers and another one is specific number of $T(>1)$ recurrent (commonly known as permanent) customers. Recurrent customers usually back to the retrial group(orbit) once service is completed and whereas transit customers depart the system permanently. In accordance with the Poisson process the transit customers arrive with rate λ. If the arriving customer finds that server is busy, he will either leaves the system completely without taking service or he will go to the orbit. Arriving customers are decided to leave the system with probability $1-\nu$ or enter into the orbit with probability ν; arriving customers enters the orbit by FCFS discipline. Suppose that at the head of the orbit only the transit customer has access to the server then the successive inter-retrial times of the transit customer we have an arbitrary probability distribution function $R(\zeta)$, in which its corresponding probability density function(pdf) $r(\zeta)$ and Laplace-Stieldjes Transform(LST) $R^*(s)$ in WV period. Also, in UB period following $S(\zeta)$, $s(\zeta)$ and $S^*(s)$ are distribution function, pdf and LST on inter-retrial times of any transit customer respectively.

When the orbit remain only $T(>0)$ recurrent customers when the service is finished, the server takes a vacation, and the duration of the vacation period is exponentially distributed with rate η: If there are customers in the system at the end of a vacation, the server will begin a fresh hectic(busy) period. Otherwise, he waits for a customer. Such a vacation policy is called a single working vacation.

In WV period the service times of the transit customers are $i.i.d$ with a probability distribution function $H_{v_1}(\zeta)$, a pdf $h_{v_1}(\zeta)$, a LST $H^*_{v_1}(s)$. Similarly, in UB period we have $H_{u_1}(\zeta)$, $h_{u_1}(\zeta)$ and $H^*_{u_1}(s)$. The system has set number of T recurrent customers. After have been served, he straight away go back to the orbit according to FCFS discipline. We infer that at the head of the orbit the recurrent customer only approach to the server. Subsequent interretrial times of recurrent customers is exponentially distributed with parameter γ having a finite mean $1/\gamma$. This is also for WV period. In WV period the service times of the specific number T of recurrent customers are $i.i.d$ with a probability distribution function $H_{v_2}(\zeta)$, a pdf $h_{v_2}(\zeta)$, a LST $H^*_{v_2}(s)$. Similarly, in UB period we have $H_{u_2}(\zeta)$, $h_{u_2}(\zeta)$ and $H^*_{u_2}(s)$.

We assume that interarrival times, retrial times, working vacation times and service times are mutually independent. It is evident from this description that the formation of our retrial queue can be understood as, an oscillating between the periods of an idle and busy period along with WV period of the server. The customer who has to be served next is decided by a vying between two exponentially distributed rates(λ and γ) and general retrial time of transit customers(i.e., foremost customer in the orbit if any transit customer or if any recurrent customer or if any fresh arrival rivalry for service). Which are the primary distinction between waiting queue(classic) in the absence of retrials. The prime objective of our paper is to providing influence of the WV period on class of customers(recurrent and transit customers) with arrival non-persistent customers.

ANALYSIS OF STUDY STATE PROBABILITIES

Following Subsequent random variables are used in this model.

$O(\tau)$ - Size of the orbit at time "τ."

$R^0(\tau)$ - The remaining retrial time of transit customer in WV period at the head of the orbit at τ.

$H^0_{v1}(\tau), H^0_{v2}(\tau)$- The remaining service time of transit and recurrent customers in WV period at time τ.

$S^0(\tau)$ - The remaining retrial time of transit customer in UB period at time τ.

$H^0_{u1}(\tau), H^0_{u2}(\tau)$- The remaining service time of transit and recurrent customers in UB period at time τ.

$$\phi(\tau) = \begin{cases} 0 \quad if \quad the\ server\ is\ not\ occupied\ in\ WV\ period\ at\ time\ \tau \\ 1 \quad if \quad the\ server\ is\ occupied\ by\ transit\ customer\ in\ WV\ period\ at\ time\ \tau \\ 2 \quad if \quad the\ server\ is\ occupied\ by\ recurrent\ customer\ in\ WV\ period\ at\ time\ \tau \\ 3 \quad if \quad the\ server\ is\ not\ occupied\ in\ UB\ period\ at\ time\ \tau \\ 4 \quad if \quad the\ server\ is\ occupied\ by\ transit\ customer\ in\ UB\ period\ at\ time\ \tau \\ 5 \quad if \quad the\ server\ is\ occupied\ by\ recurrent\ customer\ in\ UB\ period\ at\ time\ \tau. \end{cases}$$

where, $\phi(\tau)$ denotes states of the server at "τ."
Now, the supplementary variables $H_{v1}^{0}(\tau), H_{v2}^{0}(\tau), H_{u1}^{0}(\tau), H_{u2}^{0}(\tau), R^{0}(\tau)$ and $S^{0}(\tau)$ are introduced to generate bivariate Markov Process $\{(S(\tau), O(\tau)); \tau \geq 0\}$.
Where, $S(\tau) = R^{0}(\tau)\ if\ \phi(\tau) = 0;\ \ H_{v1}^{0}(\tau)\ if\ \phi(\tau) = 1;\ \ H_{v2}^{0}(\tau)\ if\ \phi(\tau) = 2;\ \ S^{0}(\tau)\ if\ \phi(\tau) = 3;\ \ H_{u1}^{0}(\tau)\ if\ \phi(\tau) = 4;\ \ H_{u1}^{0}(\tau)\ if\ \phi(\tau) = 5.$
Following are the limiting probabilities

$V_{0,T} = \lim_{\tau \to \infty} \Pr\{O(\tau) = T, \phi(\tau) = 0\}$

$V_{0,m}(\zeta) = \lim_{\tau \to \infty} \Pr\{O(\tau) = m, \phi(\tau) = 0, \zeta < R^{0}(\tau) \leq \zeta + d\zeta\}; \tau \geq 0, \zeta \geq 0, m \geq T + 1$

$V_{1,m}(\zeta) = \lim_{\tau \to \infty} \Pr\{O(\tau) = m, \phi(\tau) = 1, \zeta < H_{v1}^{0}(\tau) \leq \zeta + d\zeta\}; \tau \geq 0, \zeta \geq 0, m \geq T$

$V_{2,m}(\zeta) = \lim_{\tau \to \infty} \Pr\{O(\tau) = m, \phi(\tau) = 2, \zeta < H_{v2}^{0}(\tau) \leq \zeta + d\zeta\}; \tau \geq 0, \zeta \geq 0, m \geq T + 1$

$U_{0,T} = \lim_{\tau \to \infty} \Pr\{O(\tau) = T, \phi(\tau) = 3\}$

$U_{0,m}(\zeta) = \lim_{\tau \to \infty} \Pr\{O(\tau) = m, \phi(\tau) = 3, \zeta < S^{0}(\tau) \leq \zeta + d\zeta\}; \tau \geq 0, \zeta \geq 0, m \geq T + 1$

$U_{1,m}(\zeta) = \lim_{\tau \to \infty} \Pr\{O(\tau) = m, \phi(\tau) = 4, \zeta < H_{u1}^{0}(\tau) \leq \zeta + d\zeta\}; \tau \geq 0, \zeta \geq 0, m \geq T$

$U_{2,m}(\zeta) = \lim_{\tau \to \infty} \Pr\{O(\tau) = m, \phi(\tau) = 5, \zeta < H_{v1}^{0}(\tau) \leq \zeta + d\zeta\}; \tau \geq 0, \zeta \geq 0, m \geq T + 1$

For $i = 0, 1, 2$ we define following LST and PGF of WV period and UB period,

$V_{i,m}^{*}(\theta) = \int_{0}^{\infty} e^{-\theta\zeta}\, V_{i,m}(\zeta)\ d\zeta,$ $\quad$ $U_{i,m}^{*}(\theta) = \int_{0}^{\infty} e^{-\theta\zeta}\, U_{i,m}(\zeta)\ d\zeta,$

$V_{i,m}^{*}(0) = \int_{0}^{\infty} V_{i,m}(\zeta)\ d\zeta,$ $\quad$ $U_{i,m}^{*}(\theta) = \int_{0}^{\infty} U_{i,m}(\zeta)\ d\zeta,$

$V_{i}^{*}(\ell, \theta) = \sum_{m=T+1}^{\infty} V_{i,m}^{*}(\theta)\ell^{m},$ $\quad$ $U_{i}^{*}(\ell, \theta) = \sum_{m=T+1}^{\infty} U_{i,m}^{*}(\theta)\ell^{m},$

$V_{i}^{*}(\ell, 0) = \sum_{m=T+1}^{\infty} V_{i,m}^{*}(0)\ell^{m},$ $\quad$ $U_{i}^{*}(\ell, \theta) = \sum_{m=T+1}^{\infty} U_{i,m}^{*}(0)\ell^{m},$

$V_{i}(\ell, 0) = \sum_{m=T+1}^{\infty} V_{i,m}(0)\ell^{m},$ $\quad$ $U_{i}(\ell, 0) = \sum_{m=T+1}^{\infty} U_{i,m}(0)\ell^{m}.$

For $i = 1, 2$ LST of service times in WV period and UB period are,

$H_{vi}^{*}(\theta) = \int_{0}^{\infty} e^{-\theta\zeta}\, s_{vi}(\zeta)\ d\zeta,$ $\quad$ $H_{ui}^{*}(\theta) = \int_{0}^{\infty} e^{-\theta\zeta}\, s_{ui}(\zeta)\ d\zeta$

LST of retrial time of transit customers in WV period and UB period are,

$R^{*}(\theta) = \int_{0}^{\infty} e^{-\theta\zeta}\, r(\zeta)\ d\zeta,$ $\quad$ $S^{*}(\theta) = \int_{0}^{\infty} e^{-\theta\zeta}\, s(\zeta)\ d\zeta$

Then the system of steady state equations are illustrated by the following differential difference equations:

$$\sigma V_{0,T} = V_{1,T}(0) + V_{2,T-1}(0) + U_{1,T}(0) + U_{2,T-1}(0) \tag{1}$$

$$-\frac{d}{d\zeta}V_{0,m}(\zeta) = -\sigma V_{0,m}(\zeta) + V_{1,m}(0)r(\zeta) + V_{2,m-1}(0)r(\zeta), \qquad m \geq T + 1 \tag{2}$$

$$-\frac{d}{d\zeta}V_{1,T}(\zeta) = -(\lambda\nu + \eta)V_{1,T}(\zeta) + V_{0,T+1}(0)h_{v1}(\zeta) + \lambda V_{0,T}h_{v1}(\zeta) \tag{3}$$

$$-\frac{d}{d\zeta}V_{1,m}(\zeta) = -(\lambda\nu + \eta)V_{1,m}(\zeta) + V_{0,m+1}(0)h_{v1}(\zeta) + \lambda\nu V_{1,m-1}(\zeta) + \lambda h_{v1}(\zeta)\int_{0}^{\infty} V_{0,m}(\zeta)\, d\zeta, \qquad m \geq T + 1 \tag{4}$$

$$-\frac{d}{d\zeta}V_{2,T-1}(\zeta) = -(\lambda\nu + \eta)V_{2,T-1}(\zeta) + \gamma V_{0,T}h_{v2}(\zeta) \tag{5}$$

$$-\frac{d}{d\zeta}V_{2,m}(\zeta)=-(\lambda\nu+\eta)V_{2,m}(\zeta)+\lambda\nu V_{2,m-1}(\zeta)+\gamma h_{v2}(\zeta)\int_0^\infty V_{0,m+1}(\zeta)\,d\zeta,\quad m\geq T \quad (6)$$

$$(\lambda+\gamma)U_{0,T}=\eta V_{0,T} \quad (7)$$

$$-\frac{d}{d\zeta}U_{0,m}(\zeta)=-(\lambda+\gamma)U_{0,m}(\zeta)+U_{1,m}(0)s(\zeta)+\lambda U_{2,m-1}(\zeta)s(\zeta)$$
$$+\eta\, s(\zeta)\int_0^\infty U_{0,m}(\zeta)\,d\zeta,\qquad m\geq T+1 \quad (8)$$

$$-\frac{d}{d\zeta}U_{1,T}(\zeta)=-\lambda\nu U_{1,T}(\zeta)+U_{0,T+1}(0)h_{u1}(\zeta)+\lambda U_{0,T}h_{u1}(\zeta)+\eta h_{u1}(\zeta)\int_0^\infty U_{1,T}(\zeta)\,d\zeta \quad (9)$$

$$-\frac{d}{d\zeta}U_{1,m}(\zeta)=-\lambda\nu U_{1,m}(\zeta)+U_{0,m+1}(0)h_{u1}(\zeta)+\lambda\nu U_{1,m-1}(\zeta)+\lambda h_{u1}(\zeta)\int_0^\infty U_{0,m}(\zeta)\,d\zeta$$
$$+\eta h_{u1}(\zeta)\int_0^\infty V_{1,m}(\zeta)\,d\zeta,\qquad m\geq T+1 \quad (10)$$

$$-\frac{d}{d\zeta}U_{2,T-1}(\zeta)=-\lambda\nu U_{2,T-1}(\zeta)+\gamma U_{0,T}h_{u2}(\zeta)+\eta h_{u2}(\zeta)\int_0^\infty V_{2,T-1}(\zeta)\,d\zeta \quad (11)$$

$$-\frac{d}{d\zeta}U_{2,m}(\zeta)=-\lambda\nu U_{2,m}(\zeta)+\lambda\nu U_{2,m-1}(0)+\gamma h_{u2}(\zeta)\int_0^\infty U_{0,m+1}(\zeta)\,d\zeta$$
$$+\eta h_{u2}(\zeta)\int_0^\infty V_{2,m}(\zeta)\,d\zeta,\qquad m\geq T \quad (12)$$

Taking the LST from (2) to (6) and from (8) to (12) on both sides, we have following results

$$\theta V_{0,m}^*(\theta)-V_{0,m}(0)=\sigma V_{0,m}^*(\theta)-V_{1,m}(0)R^*(\theta)-V_{2,m-1}(0)R^*(\theta)\qquad m\geq T+1 \quad (13)$$

$$\theta V_{1,T}^*(\theta)-V_{1,T}(0)=(\lambda\nu+\eta)V_{1,T}^*(\theta)-V_{0,T+1}(0)\,H_{v1}^*(\theta)-\lambda V_{0,T}(0)H_{v1}^*(\theta) \quad (14)$$

$$\theta V_{1,m}^*(\theta)-V_{1,m}(0)=(\lambda\nu+\eta)V_{1,m}^*(\theta)-\lambda\nu V_{1,m-1}^*(\theta)-V_{0,m+1}(0)\,H_{v1}^*(\theta)$$
$$-\lambda V_{0,m}^*(0)H_{v1}^*(\theta),\qquad m\geq T+1 \quad (15)$$

$$\theta V_{2,T-1}^*(\theta)-V_{2,T-1}(0)=(\lambda\nu+\eta)V_{2,T-1}^*(\theta)-\gamma V_{0,T}H_{v2}^*(\theta) \quad (16)$$

$$\theta V_{2,m}^*(\theta)-V_{2,m}(0)=(\lambda\nu+\eta)V_{2,m}^*(\theta)-\lambda\nu V_{2,m-1}^*(\theta)-\gamma V_{0,m+1}^*(0)\,H_{v2}^*(\theta),\quad m\geq T \quad (17)$$

$$\theta U_{0,m}^*(\theta)-V_{0,m}(0)=(\lambda+\gamma)U_{0,m}^*(\theta)-V_{1,m}(0)S^*(\theta)-V_{2,m-1}(0)S^*(\theta)$$
$$-\eta V_{0,m}^*(0)S^*(\theta),\qquad m\geq T+1 \quad (18)$$

$$\theta U_{1,T}^*(\theta)-U_{1,T}(0)=\lambda\nu U_{1,T}^*(\theta)-U_{0,T+1}(0)\,H_{u1}^*(\theta)-\eta V_{1,T}^*(0)H_{u1}^*(\theta)-\lambda U_{0,T}H_{u1}^*(\theta) \quad (19)$$

$$\theta U_{1,m}^*(\theta)-U_{1,m}(0)=\lambda\nu U_{1,m}^*(\theta)-\lambda\nu U_{1,m-1}^*(\theta)-U_{0,m+1}(0)\,H_{u1}^*(\theta)-\eta V_{1,m}^*(\theta)H_{u1}^*(\theta)$$
$$-\lambda U_{0,m}^*(0)H_{u1}^*(\theta),\qquad m\geq T+1 \quad (20)$$

$$\theta U_{2,T-1}^*(\theta)-U_{2,T-1}(0)=\lambda\nu U_{2,T-1}^*(\theta)-\gamma U_{0,T}^*(0)\,H_{u2}^*(\theta)-\eta V_{2,T-1}^*(0)H_{u2}^*(\theta) \quad (21)$$

$$\theta U_{2,m}^*(\theta)-U_{2,m}(0)=\lambda\nu U_{2,m}^*(\theta)-\lambda\nu U_{2,m-1}^*(\theta)-\gamma U_{0,m+1}^*(0)\,H_{u2}^*(\theta)$$
$$-\eta V_{2,m}^*(0)H_{u1}^*(\theta),\qquad m\geq T \quad (22)$$

In (13) take a summation over m from $T+1$ to ∞ then multiply with ℓ^m and result,

$$(\theta-\sigma)V_0^*(\ell,\theta)=V_0(\ell,\theta)-R^*(\theta)\left[\,\ell V_1(\ell,0)+\ell V_2(\ell,0)-\left[V_{1,T}(0)+V_{2,T-1}(0)\right]\ell^T\right] \quad (23)$$

In (15) take a summation over m from $T+1$ to ∞ then multiply with ℓ^m and combine this result with (14),

$$[\theta - (\lambda_\ell + \eta)]V_1^*(\ell,\theta) = V_1(\ell,0) - H_{v1}^*(\theta)\left[\frac{V_0(\ell,0)}{\ell} + \lambda V_0^*(\ell,0) + \lambda V_{0,T}\ell^T\right] \tag{24}$$

In (17) take a summation over m from T to ∞ then multiply with ℓ^m and combine this result with (16),

$$[\theta - (\lambda_\ell + \eta)]V_2^*(\ell,\theta) = V_2(\ell,0) - \frac{\gamma H_{v2}^*(\theta)}{\ell}\left[V_0^*(\ell,0) + V_{0,T}\ell^T\right] \tag{25}$$

Placing $\theta = \sigma$ in (23), result

$$V_0(\ell,\theta) = R^*(\sigma)\left[V_1(\ell,0) + \ell V_2(\ell,0) - \ell^T\left[V_{1,T}(0) + V_{2,T-1}(0)\right]\right] \tag{26}$$

Placing $\theta = 0$ in (23) and sub., (26) in (23), results

$$V_0^*(\ell,0) = \frac{1}{\sigma}[1 - R^*(\sigma)]\left[V_1(\ell,0) + \ell V_2(\ell,0) - \ell^T\left[V_{1,T}(0) + V_{2,T-1}(0)\right]\right] \tag{27}$$

Placing $\theta = \lambda_\ell + \eta$ in (24), result

$$V_1(\ell,0) = H_{v1}^*(\lambda_\ell + \eta)\left[\frac{V_0(\ell,0)}{\ell} + \lambda V_0^*(\ell,0) + \lambda V_{0,T}\ell^T\right] \tag{28}$$

Placing $\theta = 0$ in (24), and sub., (28) in (24), result

$$V_1^*(\ell,0) = \frac{1 - H_{v1}^*(\lambda_\ell + \eta)}{\lambda_\ell + \eta}\left[\frac{V_0(\ell,0)}{\ell} + \lambda V_0^*(\ell,0) + \lambda V_{0,T}\ell^T\right] \tag{29}$$

Placing $\theta = \lambda_\ell + \eta$ in (25), results

$$V_2(\ell,0) = \frac{\gamma H_{v2}^*(\lambda_\ell + \eta)}{\ell}\left[V_0^*(\ell,0) + V_{0,T}\ell^T\right] \tag{30}$$

Placing $\theta = 0$ in (25) and sub., (30) in (25), results

$$V_2^*(\ell,0) = \frac{\gamma[1 - H_{v2}^*(\lambda_\ell + \eta)]}{\ell(\lambda_\ell + \eta)}\left[V_0^*(\ell,0) + V_{0,T}\ell^T\right] \tag{31}$$

where, $\sigma = \lambda + \gamma + \eta \ and \ \lambda_\ell = \lambda\nu - \lambda\nu\ell.$

Sub., (27) and (30) in (26), results

$$V_0(\ell,0) = \frac{\ell^{T+1}\sigma R^*(\sigma)\left[[\lambda H_{v1}^*(\lambda_\ell+\eta) + \gamma H_{v2}^*(\lambda_\ell+\eta)]V_{0,T} - \left[V_{1,T}(0) + V_{2,T-1}(0)\right]\right]}{\ell\sigma - H_{v1}^*(\lambda_\ell+\eta)[\lambda\ell + (\lambda_\ell+\gamma+\eta)R^*(\sigma)] - \gamma\ell[1 - R^*(\sigma)]H_{v2}^*(\lambda_\ell+\eta)} \tag{32}$$

Let, $f(\ell) = \ell\sigma - H_{v1}^*(\lambda_\ell + \eta)[\lambda\ell + (\lambda_\ell + \gamma + \eta)R^*(\sigma)] - \gamma\ell[1 - R^*(\sigma)]H_{v2}^*(\lambda_\ell + \eta)$

For $\ell = 0$ and $\ell = 1$ we obtain $f(0) < 0$ and $f(1) > 0 \Rightarrow \exists$ a real root $\ell_1 \in (0,1)$ such that $f(\ell_1) = 0$.

At $\ell = \ell_1$ in (32) is converted into

$$\left[V_{1,T}(0) + V_{2,T-1}(0)\right] = [\lambda H_{v1}^*(\lambda\nu - \lambda\nu\ell_1 + \eta) + \gamma H_{v2}^*(\lambda\nu - \lambda\nu\ell_1 + \eta)]V_{0,T} = \chi_{\ell_1}V_{0,T} \tag{33}$$

Displace using (33) in (32),

$$V_0(\ell,0) = \frac{\ell^{T+1}\sigma R^*(\sigma)V_{0,T}}{Z_1(\ell)}\left[[\lambda H_{v1}^*(\lambda_\ell + \eta) + \gamma H_{v2}^*(\lambda_\ell + \eta)] - \chi_{\ell_1}\right] \tag{34}$$

By solving (26) to (31) we get followings results for WV period,

$$V_1(\ell,0) = \frac{\ell^T V_{0,T}H_{v1}^*(\lambda_\ell+\eta)}{Z_1(\ell)}\left[\sigma[\lambda\ell + \gamma H_{v2}^*(\lambda_\ell + \eta)R^*(\sigma)] - [\lambda\ell + (\lambda - \lambda\ell + \gamma +\right.$$

$$\eta)R^*(\sigma)]\chi_{\ell_1}\Big] \quad (35)$$

$$V_2(\ell,0) = \frac{\gamma\ell^{T-1}V_{0,T}H^*_{v2}(\lambda_\ell+\eta)}{Z_1(\ell)}[\sigma[\ell - H^*_{v1}(\lambda_\ell+\eta)R^*(\sigma)] - \ell[1-R^*(\sigma)]\chi_{\ell_1}] \quad (36)$$

$$V^*_0(\ell,0) = \frac{\ell^{T+1}[1-R^*(\sigma)]V_{0,T}}{Z_1(\ell)}[[\lambda H^*_{v1}(\lambda_\ell+\eta) + \gamma H^*_{v2}(\lambda_\ell+\eta)] - \chi_{\ell_1}] \quad (37)$$

$$= \frac{V_0(\ell)\ell^T}{Z_v(\ell)}V_{0,T}$$

$$V^*_1(\ell,0) = \frac{\ell^T V_{0,T}[1-H^*_{v1}(\lambda_\ell+\eta)]}{(\lambda_\ell+\eta)Z_1(\ell)}\big[\sigma[\lambda\ell + \gamma H^*_{v2}(\lambda_\ell+\eta)R^*(\sigma)] - [\lambda\ell + (\lambda - \lambda\ell + \gamma +$$

$$\eta)R^*(\sigma)]\chi_{\ell_1}\big] \quad (38)$$

$$= \frac{V_1(\ell)\ell^T}{Z_v(\ell)}V_{0,T}$$

$$V^*_2(\ell,0) = \frac{\gamma\ell^{T-1}V_{0,T}[1-H^*_{v2}(\lambda_\ell+\eta)]}{(\lambda_\ell+\eta)Z_1(\ell)}[\sigma[\ell - H^*_{v1}(\lambda_\ell+\eta)R^*(\sigma)] - \ell[1-R^*(\sigma)]\chi_{\ell_1}] \quad (39)$$

$$= \frac{V_0(\ell)\ell^{T-1}}{Z_v(\ell)}V_{0,T}$$

where,

$Z_1(\ell) = \ell\sigma - H^*_{v1}(\lambda_\ell+\eta)[\lambda\ell + (\lambda - \lambda\ell + \gamma + \eta)R^*(\sigma)] - \gamma\ell H^*_{v2}(\lambda_\ell+\eta)[1-R^*(\sigma)]$

and $Z_v(\ell) = (\lambda_\ell+\eta)Z_1(\ell)$

In (18) take a summation over m from $T+1$ to ∞ then multiply with ℓ^m and result,

$$[\theta - (\lambda+\gamma)]U^*_0(\ell,\theta) = U_0(\ell,0) - S^*(\theta)[\eta V^*_0(\ell,\theta) + U_1(\ell,0) + \ell U_2(\ell,0)$$

$$-\ell^T\big[U_{1,T}(0) + U_{2,T-1}(0)\big]\,] \quad (40)$$

In (20) take a summation over m from $T+1$ to ∞ then multiply with ℓ^m and combine this result with (19),

$$[\theta - \lambda_\ell]U^*_1(\ell,\theta) = U_1(\ell,0) - H^*_{u1}(\theta)\left[\eta V^*_1(\ell,\theta) + \frac{V_0(\ell,0)}{\ell} + \lambda U^*_0(\ell,0) + \lambda U_{0,T}\ell^T\right] \quad (41)$$

In (22) take a summation over m from T to ∞ then multiply with ℓ^m and combine this result with (21),

$$[\theta - \lambda_\ell]U^*_2(\ell,\theta) = U_2(\ell,0) - H^*_{v2}(\theta)\left[\eta V^*_2(\ell,0) + \frac{\gamma}{\ell}V^*_0(\ell,0) + \gamma U_{0,T}\ell^{T-1}\right] \quad (42)$$

Placing $\theta = \lambda + \gamma$ in (37), result

$$U_0(\ell,\theta) = S^*(\lambda+\gamma)\ \left[\eta V^*_0(\ell,0) + V_1(\ell,0) + \ell V_2(\ell,0) - \ell^T\big[U_{1,T}(0) + U_{2,T-1}(0)\big]\right] \quad (43)$$

Placing $\theta = 0$ in (37) and sub., (40) in (37), results

$$U^*_0(\ell,0) = \frac{1-S^*(\lambda+\gamma)}{\lambda+\gamma}\left[\eta V^*_0(\ell,0) + V_1(\ell,0) + \ell V_2(\ell,0) - \ell^T\big[U_{1,T}(0) + U_{2,T-1}(0)\big]\right] \quad (44)$$

Placing $\theta = \lambda_\ell$ in (38), result

$$U_1(\ell,0) = H^*_{u1}(\lambda_\ell)\left[\eta V^*_1(\ell,0) + \frac{U_0(\ell,0)}{\ell} + \lambda U^*_0(\ell,0) + \lambda U_{0,T}\ell^T\right] \quad (45)$$

Placing $\theta = 0$ in (38) and sub., (42) in (38), results

$$U_1^*(\ell,0) = \frac{1-H_{u1}^*(\lambda_\ell)}{\lambda_\ell}\left[\eta V_1^*(\ell,0) + \frac{U_0(\ell,0)}{\ell} + \lambda U_0^*(\ell,0) + \lambda U_{0,T}\ell^T\right] \tag{46}$$

Placing $\theta = \lambda_\ell$ in (39), results

$$U_2(\ell,0) = H_{u2}^*(\lambda_\ell)\left[\eta V_2^*(\ell,0) + \frac{\gamma}{\ell}U_0^*(\ell,0) + \gamma U_{0,T}\ell^{T-1}\right] \tag{47}$$

Placing $\theta = 0$ in (39) and sub., (44) in (39), results

$$U_2^*(\ell,0) = \frac{1-H_{v2}^*(\lambda_\ell)}{\lambda_\ell}\left[\eta V_2^*(\ell,0) + \frac{\gamma}{\ell}U_0^*(\ell,0) + \gamma U_{0,T}\ell^{T-1}\right] \tag{48}$$

Solving (43) to (48), we get following results for UB period

$$U_0(\ell,0) = \frac{(\lambda+\gamma)\ell S^*(\lambda+\gamma)}{Z_2(\ell)}\{\eta V_0^*(\ell,0) + H_{u1}^*(\lambda_\ell)\eta V_1^*(\ell,0) + H_{u2}^*(\lambda_\ell)\eta\ell V_2^*(\ell,0) + [\lambda H_{u1}^*(\lambda_\ell) + \gamma H_{u2}^*(\lambda_\ell)]U_{0,T}\,\ell^T - (\lambda+\gamma+\eta-\chi_{\ell_1})V_{0,T}\ell^T\} \tag{49}$$

$$U_1(\ell,0) = \frac{H_{u1}^*(\lambda_\ell)}{Z_2(\ell)}\{[\lambda\ell + (\lambda_\ell+\gamma)S^*(\lambda+\gamma)]\,\eta V_0^*(\ell,0) + [\ell(\lambda+\gamma) + \gamma\ell[1-S^*(\lambda+\gamma)]H_{u2}^*(\lambda_\ell)]\eta V_1^*(\ell,0) + \ell H_{u2}^*(\lambda_\ell)[\lambda\ell + (\lambda-\lambda\ell+\gamma)S^*(\lambda+\gamma)]\,\eta V_2^*(\ell,0) - (\lambda+\gamma+\eta-\chi_{\ell_1})V_{0,T}\ell^T[\lambda\ell + (\lambda-\lambda\ell+\gamma)S^*(\lambda+\gamma)] + \ell^T U_{0,T}(\lambda+\gamma)[\lambda\ell + \gamma H_{u2}^*(\lambda_\ell)S^*(\lambda+\gamma)]\} \tag{50}$$

$$U_2(\ell,0) = \frac{H_{u2}^*(\lambda_\ell)}{Z_2(\ell)}\{\gamma[1-S^*(\lambda+\gamma)]\,\eta V_0^*(\ell,0) + \gamma[1-S^*(\lambda+\gamma)]H_{u1}^*(\lambda_\ell)\eta V_1^*(\ell,0) + [\ell(\lambda+\gamma) - [\lambda\ell + (\lambda-\lambda\ell+\gamma)S^*(\lambda+\gamma)]H_{u1}^*(\lambda_\ell)]\,\eta V_2^*(\ell,0) - \gamma[1-S^*(\lambda+\gamma)](\lambda+\gamma+\eta-\chi_{\ell_1})V_{0,T}\ell^T + \gamma(\lambda+\gamma)\ell^{T-1}U_{0,T}[\ell - S^*(\lambda+\gamma)H_{u1}^*(\lambda_\ell)]\} \tag{51}$$

$$U_0^*(\ell,0) = \frac{\ell[1-S^*(\lambda+\gamma)]}{Z_2(\ell)}\{\eta V_0^*(\ell,0) + H_{u1}^*(\lambda_\ell)\eta V_1^*(\ell,0) + H_{u2}^*(\lambda_\ell)\eta\ell V_2^*(\ell,0) + [\lambda H_{u1}^*(\lambda_\ell) + \gamma H_{u2}^*(\lambda_\ell)]U_{0,T}\ell^T - (\lambda+\gamma+\eta-\chi_{\ell_1})V_{0,T}\ell^T\} \tag{52}$$

$$U_1^*(\ell,0) = \frac{1-H_{u1}^*(\lambda_\ell)}{\lambda_\ell Z_2(\ell)}\{[\lambda\ell + (\lambda_\ell+\gamma)S^*(\lambda+\gamma)]\,\eta V_0^*(\ell,0) + [\ell(\lambda+\gamma) + \gamma\ell[1-S^*(\lambda+\gamma)]H_{u2}^*(\lambda_\ell)]\eta V_1^*(\ell,0) + \ell H_{u2}^*(\lambda_\ell)[\lambda\ell + (\lambda-\lambda\ell+\gamma)S^*(\lambda+\gamma)]\,\eta V_2^*(\ell,0) - (\lambda+\gamma+\eta-\chi_{\ell_1})V_{0,T}\ell^T[\lambda\ell + (\lambda-\lambda\ell+\gamma)S^*(\lambda+\gamma)] + \ell^T U_{0,T}(\lambda+\gamma)[\lambda\ell + \gamma H_{u2}^*(\lambda_\ell)S^*(\lambda+\gamma)]\} \tag{53}$$

$$U_2^*(\ell,0)=\frac{1-H_{u2}^*(\lambda_\ell)}{\lambda_\ell Z_2(\ell)}\{\gamma[1-S^*(\lambda+\gamma)]\,\eta V_0^*(\ell,0)+\gamma[1-S^*(\lambda+\gamma)]H_{u1}^*(\lambda_\ell)\eta V_1^*(\ell,0)+[\ell(\lambda+\gamma)-[\lambda\ell+(\lambda-\lambda\ell+\gamma)S^*(\lambda+\gamma)]H_{u1}^*(\lambda_\ell)]\,\eta V_2^*(\ell,0)-\gamma[1-S^*(\lambda+\gamma)]\big(\lambda+\gamma+\eta-\chi_{\ell_1}\big)V_{0,T}\ell^T+\gamma(\lambda+\gamma)\ell^{T-1}U_{0,T}[\ell-S^*(\lambda+\gamma)H_{u1}^*(\lambda_\ell)]\}\quad(54)$$

where,

$Z_2(\ell)=(\lambda+\gamma)\ell-H_{u1}^*(\lambda_\ell)[\lambda\ell+(\lambda-\lambda\ell+\gamma)S^*(\lambda+\gamma)]-\gamma\ell H_{u2}^*(\lambda_\ell)[1-S^*(\lambda+\gamma)]$

and take $Z_u(\ell)=\lambda_\ell Z_2(\ell)$.

When server is on WV period, we define the PGF for the no., of customers in the orbit as,

$U_v(\ell)=V_0^*(\ell,0)+V_1^*(\ell,0)+V_2^*(\ell,0)+V_{0,T}\ell^T$

where, $V_0^*(\ell,0),V_1^*(\ell,0),V_2^*(\ell,0)$ are found in (37) to (39).

When server is on UB period, we define the PGF for the no., of customers in the orbit as,

$U_u(\ell)=U_0^*(\ell,0)+U_1^*(\ell,0)+U_2^*(\ell,0)+U_{0,T}\ell^T$

where, $U_0^*(\ell,0),U_1^*(\ell,0),U_2^*(\ell,0)$ are found in (52) to (54).

Define, PGF for no., of customers in the orbit as, $U(\ell)=U_v(\ell)+U_u(\ell)$.

Make use of normalizing condition $U(1)=1$, we get idle probability $V_{0,T}$ of WV period as,

$$V_{0,T}=\Bigg\{\frac{\sigma-\chi_{\ell_1}}{\eta}-\frac{1}{2\lambda Z_v(1)Z_2'(1)}\Bigg\{2\lambda\nu\Big\{\frac{\eta}{Z_v(1)}[Z_v(1)[V_0'(1)+(T+1)V_0(1)]-Z_v'(1)V_0(1)]+\frac{\eta}{Z_v(1)}[Z_v(1)[V_1'(1)+TV_1(1)]-Z_v'(1)V_1(1)]+\frac{\eta}{Z_v(1)}[Z_v(1)[V_2'(1)+(T-1)V_2(1)]-Z_v'(1)V_2(1)]-Z_v(1)T\big(\lambda+\gamma-\chi_{\ell_1}\big)\Big\}\{-[1-S^*(\lambda+\gamma)]+H_{u1}^{*'}(0)[\lambda+\gamma S^*(\lambda+\gamma)]+\gamma H_{u2}^{*'}(0)[1-S^*(\lambda+\gamma)]\}+\lambda\nu\Big\{\eta V_0(1)\{-2[1-S^*(\lambda+\gamma)]+2\lambda H_{u1}^{*'}(0)[1-S^*(\lambda+\gamma)]-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}+\eta V_1(1)\{-2[1-S^*(\lambda+\gamma)]+2H_{u1}^{*'}(0)\big[\lambda\nu[1-S^*(\lambda+\gamma)]+[\lambda+\gamma S^*(\lambda+\gamma)]\big]-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}+\eta V_2(1)\{-4[1-S^*(\lambda+\gamma)]+2H_{u1}^{*'}(0)\big[\lambda\nu[1-S^*(\lambda+\gamma)]+[\lambda+\gamma S^*(\lambda+\gamma)]\big]+2[\lambda\nu[1-S^*(\lambda+\gamma)]+[\gamma+\lambda S^*(\lambda+\gamma)]]H_{u2}^{*'}(0)-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}+2Z_v(1)\eta\Big[-2+2\lambda H_{u1}^{*'}(0)[1-\nu S^*(\lambda+\gamma)]+2\gamma S^*(\lambda+\gamma)H_{u2}^{*'}(0)-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\Big]-Z_v(1)\big(\lambda+\gamma+\eta-\chi_{\ell_1}\big)\{-2[1-S^*(\lambda+\gamma)]+2\lambda\nu H_{u1}^{*'}(0)[1-S^*(\lambda+\gamma)]-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-$$

$$S^*(\lambda+\gamma)]\}\Big\}\Big\}\Big)^{-1} \tag{55}$$

where,

$$Z_v(1) = \eta\{\sigma - H_{v1}^*(\eta)[\lambda + (\gamma+\eta)R^*(\sigma)] - \gamma H_{v2}^*(\eta)[1 - R^*(\sigma)]\} \quad \ldots\ldots\ldots(i)$$

$$Z_v'(1) = -\lambda\{\sigma - H_{v1}^*(\eta)[\lambda + (\gamma+\eta)R^*(\sigma)] - \gamma H_{v2}^*(\eta)[1 - R^*(\sigma)]\}$$
$$+\eta\{\sigma - \lambda H_{v1}^*(\eta)[1 - R^*(\sigma)] - \gamma H_{v2}^*(\eta)[1 - R^*(\sigma)] + \lambda\nu H_{v1}^{*'}(\eta)$$
$$\times[\lambda + (\gamma+\eta)R^*(\sigma)] + \lambda\nu\gamma H_{v2}^{*'}(\eta)[1 - R^*(\sigma)]\} \quad \ldots\ldots\ldots(ii)$$

$$Z_2'(1) = (\lambda+\gamma)S^*(\lambda+\gamma)\left\{1 - \left[\frac{\lambda\nu E(H_{u1})[\lambda+\gamma S^*(\lambda+\gamma)]}{(\lambda+\gamma)S^*(\lambda+\gamma)} + \frac{\lambda\nu\gamma E(H_{u2})[1-S^*(\lambda+\gamma)]}{(\lambda+\gamma)S^*(\lambda+\gamma)}\right]\right\}$$
$$= (\lambda+\gamma)S^*(\lambda+\gamma)(1-\rho)$$

where, ρ is the mean service time and the stability condition in system is obtained from (55).

$$V_0(1) = \eta[1 - R^*(\sigma)]\left[[\lambda H_{v1}^*(\eta) + \gamma H_{v2}^*(\eta)] - \chi_{\ell_1}\right] \quad \ldots\ldots\ldots(iii)$$

$$V_1(1) = [1 - H_{v1}^*(\eta)]\big[\sigma[\lambda + \gamma H_{v2}^*(\eta)R^*(\sigma)] - [\lambda + (\gamma+\eta)R^*(\sigma)]\chi_{\ell_1}\big]$$

$$V_2(1) = \gamma[1 - H_{v2}^*(\eta)]\big[\sigma[1 - H_{v1}^*(\eta)R^*(\sigma)] - [1 - R^*(\sigma)]\chi_{\ell_1}\big]$$

$$V_0'(1) = -\lambda\nu[1 - R^*(\sigma)]\left[[\lambda H_{v1}^*(\eta) + \gamma H_{v2}^*(\eta)] - \chi_{\ell_1} + \eta\big[\lambda H_{v1}^{*'}(\eta) + \gamma H_{v2}^{*'}(\eta)\big]\right]$$

$$V_1'(1) = \lambda\nu H_{v1}^{*'}(\eta)\big[\sigma[\lambda + \gamma H_{v2}^*(\eta)R^*(\sigma)] - [\lambda + (\gamma+\eta)R^*(\sigma)]\chi_{\ell_1}\big]$$
$$+ \lambda[1 - H_{v1}^*(\eta)]\left[\sigma\big[1 - \nu\gamma R^*(\sigma)H_{v2}^{*'}(\eta)\big] - \chi_{\ell_1}[1 - R^*(\sigma)]\right]$$

$$V_2'(1) = \lambda\nu\gamma H_{v2}^{*'}(\eta)\left[\sigma[1 - H_{v1}^*(\eta)R^*(\sigma)] - \chi_{\ell_1}[1 - R^*(\sigma)]\right]$$
$$+ \gamma[1 - H_{v2}^*(\eta)]\left[\sigma\big[1 + \lambda\nu H_{v1}^{*'}(\eta)R^*(\sigma)\big] - \chi_{\ell_1}[1 - R^*(\sigma)]\right]$$

and also, we can get idle probability $U_{0,T}$ of UB period from (7).

THE MODEL'S PERFORMANCE MEASURES

Assume that L_v and L_s are mean orbit size in WV period and UB period. W_v and W_s are mean waiting time of the customer in the orbit in WV period and UB period

$$L_v = \frac{d}{d\ell}V_v(\ell)|_{\ell=1}$$
$$= \frac{d}{d\ell}\big[V_0^*(\ell,0) + V_1^*(\ell,0) + V_2^*(\ell,0) + V_{0,T}\ell^T\big]|_{\ell=1}$$

$$L_v = \left[\frac{Z_1(1)A'(1) - A(1)Z_1'(1)}{[Z_1(1)]^2} + \frac{\eta Z_1(1)B'(1) - B(1)[\eta Z_1'(1) - \lambda\, Z_1(1)]}{[\eta Z_1(1)]^2} + \frac{\eta Z_1(1)C'(1) - C(1)[\eta Z_1'(1) - \lambda\, Z_1(1)]}{[\eta Z_1(1)]^2} + T\right] Q_{0,T}$$

where, $\eta Z_1(1) = Z_v(1)$ and $\eta Z_1'(1) - \lambda\, Z_1(1) = Z_v'(1)$ are given in (i) and (ii),

$A(1) = \frac{V_0(1)}{\eta}$, $B(1) = V_1(1)$, $C(1) = V_2(1)$,

$$A'(1) = [1 - R^*(\sigma)]\left[(T+1)\left[[\lambda H_{v1}^*(\eta) + \gamma H_{v2}^*(\eta)] - \chi_{\ell_1}\right] - \lambda\nu\left[\lambda H_{v1}^{*'}(\eta) + \gamma H_{v2}^{*'}(\eta)\right]\right]$$

$$B'(1) = \left[T[1 - H_{v1}^*(\eta)] + \lambda\nu H_{v1}^{*'}(\eta)\right]\left[\sigma[\lambda + \gamma H_{v2}^*(\eta)R^*(\sigma)] - [\lambda + (\gamma+\eta)R^*(\sigma)]\chi_{\ell_1}\right] + \lambda[1 - H_{v1}^*(\eta)]\left[\sigma\left[1 - \nu\gamma R^*(\sigma)H_{v2}^{*'}(\eta)\right] - \chi_{\ell_1}[1 - R^*(\sigma)]\right]$$

$$C'(1) = \gamma\left[(T-1)[1 - H_{v2}^*(\eta)] + \lambda\nu H_{v2}^{*'}(\eta)\right]\left[\sigma[1 - H_{v1}^*(\eta)R^*(\sigma)] - \chi_{\ell_1}[1 - R^*(\sigma)]\right] + \gamma[1 - H_{v2}^*(\eta)]\left[\sigma\left[1 + \lambda\nu H_{v1}^{*'}(\eta)R^*(\sigma)\right] - \chi_{\ell_1}[1 - R^*(\sigma)]\right]$$

and $W_v = \frac{L_v}{\lambda}$.

$$L_s = \frac{d}{d\ell} U_u(\ell)|_{\ell=1}$$
$$= \frac{d}{d\ell}\left[U_0^*(\ell,0) + U_1^*(\ell,0) + U_2^*(\ell,0) + U_{0,T}\ell^T\right]|_{\ell=1}$$
$$= \frac{-2D_2'(1)N_B'''(1) + 3D_2''(1)N_B''(1)}{12\lambda\nu\left[D_2'(1)\right]^2}$$

where,

$$D_2'(1) = (\lambda+\gamma)S^*(\lambda+\gamma) + \lambda H_{u1}^{*'}(0)[\lambda + \gamma S^*(\lambda+\gamma)] + \lambda\gamma H_{u2}^{*'}(0)[1 - S^*(\lambda+\gamma)]$$

$$D_2''(1) = 2\lambda^2 H_{u1}^{*'}(0)[1 - S^*(\lambda+\gamma)] + 2\lambda\gamma H_{u2}^{*'}(0)[1 - S^*(\lambda+\gamma)] - \lambda^2 H_{u1}''(0)[\lambda + \gamma S^*(\lambda+\gamma)] - \lambda^2\gamma H_{u2}^{*''}(0)[1 - S^*(\lambda+\gamma)]$$

$$N_B''(1) = 2\lambda\nu\{\eta V_0^{*'}(\ell,0) + \eta V_1^{*'}(\ell,0) + \eta V_2^{*'}(\ell,0) - T(\lambda+\gamma-\chi_{\ell_1})V_{0,T}\}\{-[1 - S^*(\lambda+\gamma)] + H_{u1}^{*'}(0)[\lambda + \gamma S^*(\lambda+\gamma)] + \gamma H_{u2}^{*'}(0)[1 - S^*(\lambda+\gamma)]\} + \lambda\nu\left\{\eta V_0^*(\ell,0)\{-2[1 - S^*(\lambda+\gamma)] + 2\lambda H_{u1}^{*'}(0)[1 - S^*(\lambda+\gamma)] - \lambda\nu H_{u1}^{*''}(0)[\lambda + \gamma S^*(\lambda+\gamma)] - \lambda\nu\gamma H_{u2}^{*''}(0)[1 - S^*(\lambda+\gamma)]\} + \eta V_1^*(\ell,0)\{-2[1 - S^*(\lambda+\gamma)] + 2H_{u1}^{*'}(0)\left[\lambda\nu[1 - S^*(\lambda+\gamma)] + [\lambda + \gamma S^*(\lambda+\gamma)]\right] - \lambda\nu H_{u1}^{*''}(0)[\lambda + \gamma S^*(\lambda+\gamma)] - \lambda\nu\gamma H_{u2}^{*''}(0)[1 - S^*(\lambda+\gamma)]\} + \eta V_2^*(\ell,0)\{-4[1 - S^*(\lambda+\gamma)] + 2H_{u1}^{*'}(0)\left[\lambda\nu[1 - S^*(\lambda+\gamma)] + [\lambda + \gamma S^*(\lambda+\gamma)]\right] + 2[\lambda\nu[1 - S^*(\lambda+\gamma)] + [\gamma + \lambda S^*(\lambda+\gamma)]]H_{u2}^{*'}(0) -\right.$$

$\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}+(\lambda+\gamma)U_{0,T}\{-2+2\lambda H_{u1}^{*'}(0)[1-\nu S^*(\lambda+\gamma)]+2\gamma S^*(\lambda+\gamma)H_{u2}^{*'}(0)-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}-(\lambda+\gamma+\eta-\chi_{\ell_1})\{-2[1-S^*(\lambda+\gamma)]+2\lambda\nu H_{u1}^{*'}(0)[1-S^*(\lambda+\gamma)]-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}\Big\}$

$N_B'''(1)=3\lambda\nu\{\eta V_0^{*''}(\ell,0)+\eta V_1^{*''}(\ell,0)+\eta V_2^{*''}(\ell,0)-T(T-1)(\lambda+\gamma-\chi_{\ell_1})V_{0,T}\}\{-[1-S^*(\lambda+\gamma)]+H_{u1}^{*'}(0)[\lambda+\gamma S^*(\lambda+\gamma)]+\gamma H_{u2}^{*'}(0)[1-S^*(\lambda+\gamma)]\}+3\lambda\nu\Big\{\eta V_0^{*'}(\ell,0)\{-2[1-S^*(\lambda+\gamma)]+2\lambda H_{u1}^{*'}(0)[1-S^*(\lambda+\gamma)]-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}+\eta V_1^{*'}(\ell,0)\{-2[1-S^*(\lambda+\gamma)]+2H_{u1}^{*'}(0)\big[\lambda\nu[1-S^*(\lambda+\gamma)]+[\lambda+\gamma S^*(\lambda+\gamma)]\big]-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}+\eta V_2^{*'}(\ell,0)\{-4[1-S^*(\lambda+\gamma)]+2H_{u1}^{*'}(0)\big[\lambda\nu[1-S^*(\lambda+\gamma)]+[\lambda+\gamma S^*(\lambda+\gamma)]\big]+2[\lambda\nu[1-S^*(\lambda+\gamma)]+[\gamma+\lambda S^*(\lambda+\gamma)]]H_{u2}^{*'}(0)-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}+(\lambda+\gamma)U_{0,T}\{-2+2\lambda H_{u1}^{*'}(0)[1-\nu S^*(\lambda+\gamma)]+2\gamma S^*(\lambda+\gamma)H_{u2}^{*'}(0)-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}-(\lambda+\gamma+\eta-\chi_{\ell_1})V_{0,T}\{-2[1-S^*(\lambda+\gamma)]+2\lambda\nu H_{u1}^{*'}(0)[1-S^*(\lambda+\gamma)]-\lambda\nu H_{u1}^{*''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-\lambda\nu\gamma H_{u2}^{*''}(0)[1-S^*(\lambda+\gamma)]\}\Big\}+\lambda\nu\Big\{\eta V_0^*(\ell,0)\{-3\lambda^2\nu H_{u1}^{*''}(0)[1-S^*(\lambda+\gamma)]+\lambda^2\nu^2 H_{u1}^{*'''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]+\lambda^2\nu^2\gamma H_{u2}^{*'''}(0)[1-S^*(\lambda+\gamma)]\}+\eta V_1^*(\ell,0)\{6\lambda\nu H_{u1}^{*'}(0)[1-S^*(\lambda+\gamma)]+6\lambda\nu\gamma H_{u1}^{*'}(0)H_{u2}^{*'}(0)[1-S^*(\lambda+\gamma)]-3\lambda\nu H_{u1}^{*''}(0)\big[\lambda[1-S^*(\lambda+\gamma)]+[\lambda+\gamma S^*(\lambda+\gamma)]\big]+\lambda^2\nu^2 H_{u1}^{*'''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]+\lambda^2\nu^2\gamma H_{u2}^{*'''}(0)[1-S^*(\lambda+\gamma)]\}+\eta V_2^*(\ell,0)\{-6[1-S^*(\lambda+\gamma)]+6\lambda H_{u1}^{*'}(0)[1-S^*(\lambda+\gamma)]+12\lambda\nu H_{u2}^{*'}(0)[1-S^*(\lambda+\gamma)]-6\lambda\nu H_{u1}^{*'}(0)H_{u2}^{*'}(0)[\lambda+\gamma S^*(\lambda+\gamma)]-3\lambda\nu H_{u1}^{*''}(0)\big[\lambda[1-S^*(\lambda+\gamma)]+[\lambda+\gamma S^*(\lambda+\gamma)]\big]-3\lambda\nu\big[\lambda\nu[1-S^*(\lambda+\gamma)]+[\gamma+\lambda S^*(\lambda+\gamma)]\big]H_{u2}^{*''}(0)+\lambda^2\nu^2 H_{u1}^{*'''}(0)[\lambda+\gamma S^*(\lambda+\gamma)]+\lambda^2\nu^2\gamma H_{u2}^{*'''}(0)[1-S^*(\lambda+\gamma)]\}-3\lambda(\lambda+\gamma)U_{0,T}\Big[-6\gamma S^*(\lambda+\gamma)H_{u2}^{*'}(0)-6\lambda\nu\gamma S^*(\lambda+\gamma)H_{u1}^{*'}(0)H_{u2}^{*'}(0)-3\lambda^2\nu H_{u1}^{*''}(0)[1-\nu S^*(\lambda+\gamma)]-\lambda\nu\gamma S^*(\lambda+\gamma)H_{u2}^{*''}(0)+$

$$\lambda^2\nu^2 H_{u1}^{*'''}(0)[\lambda+\gamma S^*(\lambda+\gamma)] + \lambda^2\nu^2\gamma H_{u2}^{*'''}(0)[1-(\lambda+\gamma)]\Big] - \big(\lambda+\gamma+\eta-\chi_{\ell_1}\big)V_{0,T}\Big\{-3\lambda^2\nu H_{u1}^{*''}(0)[1-S^*(\lambda+\gamma)] + \lambda^2\nu^2 H_{u1}^{*'''}(0)[\lambda+\gamma S^*(\lambda+\gamma)] + \lambda^2\nu^2\gamma H_{u2}^{*'''}(0)[1-S^*(\lambda+\gamma)]\Big\}\Big\}$$

here, $V_i^{*'}(\ell,0) = \frac{V_{0,T}}{Z_v(1)^2}\{Z_v(1)[Q_i'(1)+kQ_i(1)] - Z_v'(1)Q_i(1)\}$ such that,

$$k = \begin{cases} T+1, & i=0 \\ T, & i=1 \\ T-1, & i=2 \end{cases}$$

And $V_i^{*''}(\ell,0) = \frac{V_{0,T}}{Z_v(1)^2}\Big\{-2Z_v'(1)[Q_i'(1)+kQ_i(1)] + Z_v(1)[Q_i''(1)+k(k-1)Q_i(1) + 2kQ_i'(1)] - Z_v''(1)Q_i(1) + \frac{2Z_v'(1)^2Q_i(1)}{Z_v(1)}\Big\}$ such that, $k = \begin{cases} T+1, & i=0 \\ T, & i=1 \\ T-1, & i=2 \end{cases}$

and $W_s = \frac{L_s}{\lambda}$.

NUMERICAL RESULTS

The curved graph constructed in Figure 1 and the values tabulated in the Table 1 are obtained by setting the fixed values $\mu_{v1} = 7.2$, $\mu_{v2} = 7.5$, $\mu_{u1} = 4$, $\mu_{u2} = 2.5$, $\mu_v = 8$, $\mu_u = 4$, $\eta = 1.1$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of ν from 0.1 to 0.4 in steps of 0.15, we observed that as λ rises L_{v1} also rises which shows the stability of the model.

Table 1: :L_{v1} with turnover of λ

$\boldsymbol{\lambda}$	$v = \mathbf{0.1}$	$v = \mathbf{0.25}$	$v = \mathbf{0.4}$
1.0	3.1619	2.8951	2.5978
1.2	3.2614	2.9603	2.6019
1.4	3.3794	3.0446	2.6126
1.6	3.5201	3.1539	2.6314
1.8	3.6893	3.2971	2.6607
2.0	3.8951	3.4883	2.7047

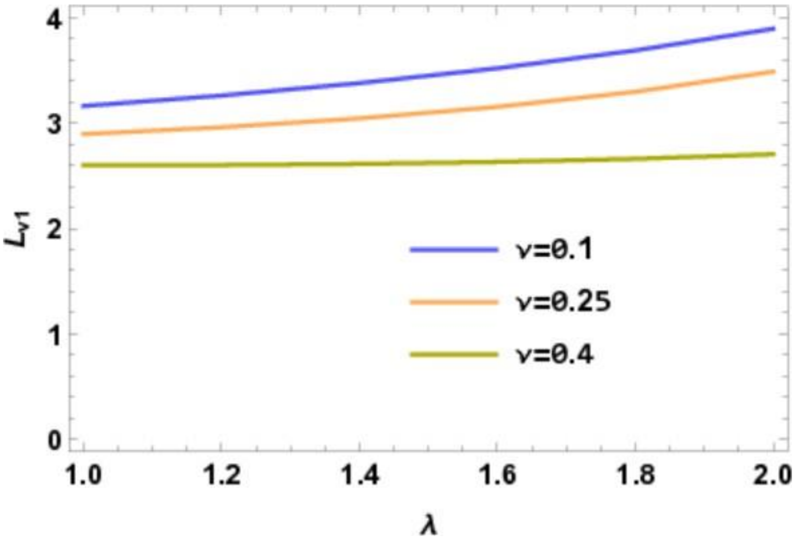

Figure 1:L_{v1} with turnover of λ

The curved graph constructed in Figure 2 and the values tabulated in the Table 2 are obtained by setting the fixed values $\mu_{v1} = 1.2, \mu_{v2} = 1.5, \mu_{u1} = 2, \mu_{u2} = 1.5, \mu_v = 4, \mu_u = 3.7, \eta = 0.1$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of ν from 0.6 to 0.8 in steps of 0.1, we observed that as λ rises L_{v2} also rises which shows the stability of the model.

Table 2:L_{v2} with turnover of λ

λ	$v = \mathbf{0.6}$	$v = \mathbf{0.7}$	$v = \mathbf{0.8}$
1.0	0.3370	0.9633	2.1253
1.2	2.0036	1.7588	3.0719
1.4	2.9546	2.3135	3.7497
1.6	3.6009	2.7319	4.2695
1.8	4.0826	3.0631	4.6858
2.0	4.4622	3.3341	5.0293

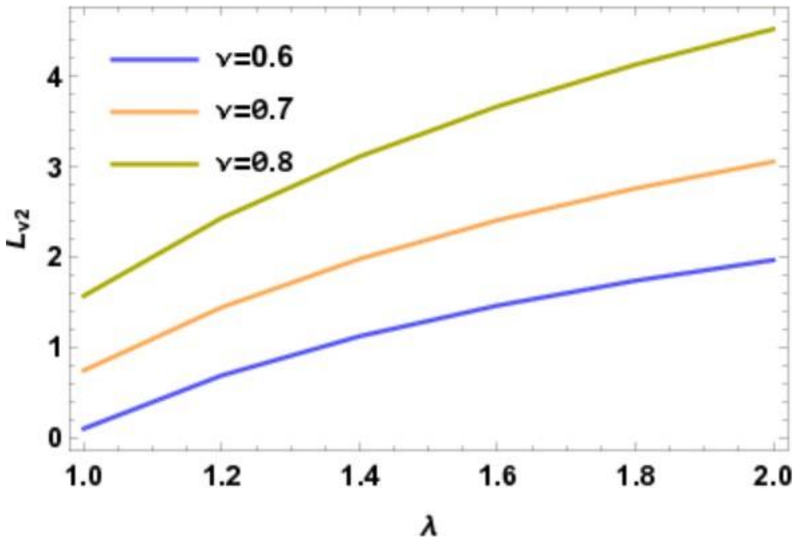

Figure 2:L_{v2} with turnover of λ

The curved graph constructed in Figure 3 and the values tabulated in the Table 3 are obtained by setting the fixed values $\mu_{v1} = 3.7, \mu_{v2} = 3.5, \mu_{u1} = 5.1, \mu_{u2} = 5, \mu_v = 2.8, \mu_u = 5.3, \eta = 0.5$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of ν from 0.3 to 0.7 in steps of 0.2, we observed that as λ rises L_{u1} also rises which shows the stability of the model.

Table 3: L_{u1} with turnover of λ

λ	$v = 0.3$	$v = 0.5$	$v = 0.7$
1.0	6.0198	5.2180	4.8889
1.2	6.5140	5.8689	6.2647
1.4	7.0253	6.7607	8.5576
1.6	7.5642	8.0163	12.4344
1.8	8.1442	9.8332	19.3322
2.0	8.7835	12.5487	33.0173

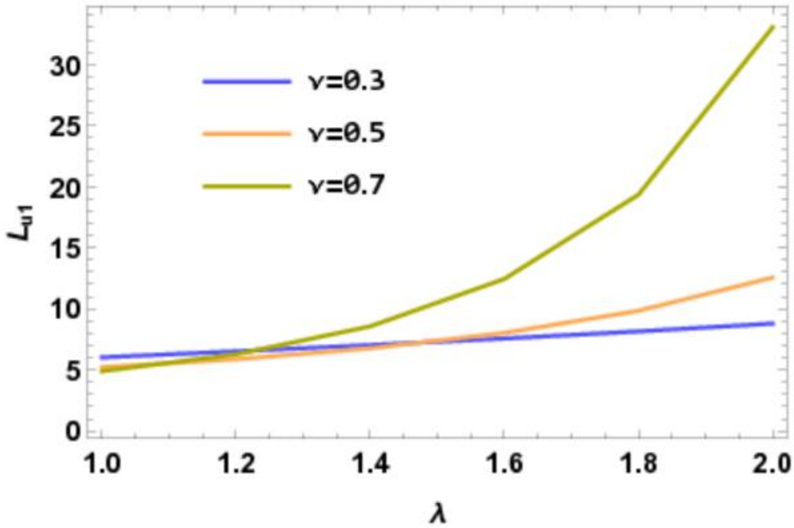

Figure 3: L_{u1} with turnover of λ

The curved graph constructed in Figure 4 and the values tabulated in the Table 4 are obtained by setting the fixed values $\mu_{v1} = 3.9, \mu_{v2} = 2, \mu_{u1} = 1.8, \mu_{u2} = 2.5, \mu_v = 4.3, \mu_u = 4.8, \eta$=0.7 and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of ν from 0.3 to 0.4 in steps of 0.05, we observed that as λ rises L_{u2} also rises which shows the stability of the model.

Table 4:L_{u2} with turnover of λ

λ	$v = 0.3$	$v = 0.35$	$v = 0.4$
1.0	8.1457	8.3140	8.5259
1.2	8.2646	8.6703	9.1984
1.4	8.7772	9.5911	10.7406
1.6	9.7644	11.3550	13.9457
1.8	11.4715	14.7647	21.7884
2.0	14.5073	22.5222	56.9134

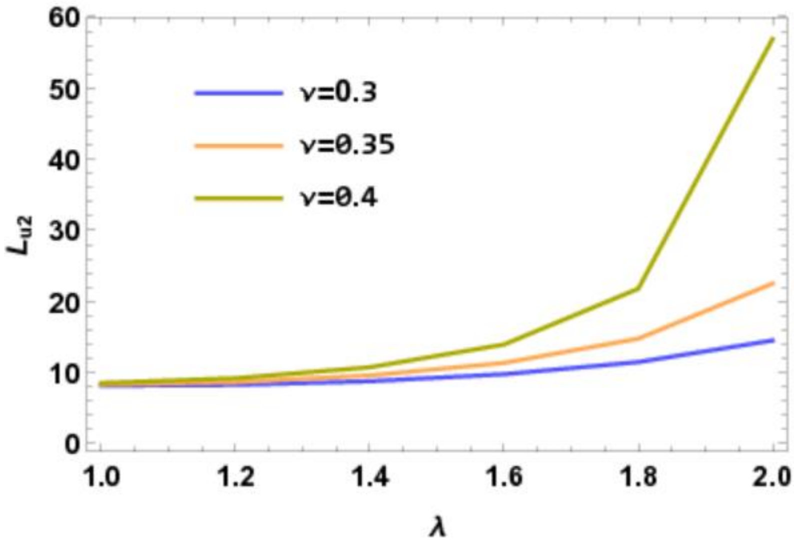

Figure 4:L_{u2} with turnover of λ

SPECIAL CASES:

(a) If there is no vacation and arriving customers are deciding to enter the orbit without leaving the system i.e., $\eta \to \infty$ and $\nu = 1$ then the present model will be remodelled as [14].

(b) If there is no recurrent customers and arriving customers are deciding to enter the orbit without leaving the system i.e., T=0 and $\nu = 1$ then the present model will be remodelled as [16].

(c) If service time follows exponential distribution also no recurrent customers, no retrial and arriving customers are deciding to enter the orbit without leaving the system i.e., T=0, $\gamma \to 0$ and $\nu = 1$ then the present model will be remodelled as [19]

CONCLUSION:

In this paper, an $M/G/1$ retrial queue with non-persistent customers, recurrent customers, general retrial times, and working vacation is evaluated. We obtain the PGF for the number of customers and the mean number of customers in the orbit. We worked out the waiting time distributions. We also derive the performance measures. We perform numerical results and some special cases.

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