Topological Indices and Laplacian Spectrum of Mobius Function Graph of Dihedral Groups

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Abstract

For a finite group G, the mobius function graph $M(G)$ is the simple graph whose vertices are elements of the group G with adjacency defined between vertices a and b if and only if $\mu(|a|\ |b|)=\mu(|a|)\mu(|b|)$. Topological indices and laplacian spectrum with their applications in molecular chemistry is an acute topic of research in Mathematics. This paper analyses some basic properties of mobius function graph of dihedral groups $M(D_{2n})$ and discusses topological indices, such as first and second Zagreb index, Wiener and hyper - Wiener index and finally Harary index along with the laplacian spectrum of $M(D_{2n})$.

Keywords: Mobius function graph of a group, Zagreb indices, Wiener indices , Harary index, Laplacian spectrum.

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1 Introduction

Group theory, a fundamental area of Mathematics turned out to be an essential tool in modelling many real world situations. Dihedral group, one among the numerous group structures plays an important role in group theory, geometry and many areas of Chemistry. Graph theory is another branch of Mathematics which is widely known for its applications in almost all spheres of life. Subsequent mathematical research in this area found that we can associate an algebraic structure with a graph to have a better understanding about its properties. This lead the way to the development of algebraic graph theory. We can find various ways of associating groups with graphs in literature. This is an active area in research and a lot of studies are still being made on this topic. Through this paper, we intend to study the mobius function graph of dihedral group $M(D_{2n})$ and discuss some of its basic properties.

We recall some definitions of number theory and graph theory which are essential for this paper. For the Mobius function μ(n) , $\mu(1) = 1$. And if $n > 1$, write $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k}$. Then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \cdots = a_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

A complete graph K_n is a graph with n vertices, in which each pair of distinct vertices are adjacent. For a graph Γ subset Ω of vertex set $V(\Gamma)$ is called a clique if the induced subgraph of Ω is complete. A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. Graph theory has wide applications in many fields, one among them is, in chemistry related to topological indices. Topological indices are molecular description which is computed based on molecular graph of chemical compound. Some of the the topological indices are first Zagreb index, second Zagreb index, Wiener index, hyper - Wiener index and Harary index . The spectrum of a graph also has several applications in Chemistry, Physics, Medicine, Computer Science and Information Technology. With this in mind we are interested in studying the topological indices and laplacian spectrum of mobius function graph of dihedral group D_{2n}.

The paper is structured as follows. In Section 2, we provide a brief review of relevant definitions that will be used throughout the paper. In Section 3, we analyze the characterstics of the mobius function graph of dihedral group, denoted by $M(D_{2n})$. Section 4 is dedicated to the discussion of topological indices of $M(D_{2n})$, while Section 5 presents an analysis of laplacian eigen values of $M(D_{2n})$.

2 Preliminaries

The section deals with some definitions which are helpful in our studies.

Definition 2.1. *[7] The dihedral group is generated by two elements a,b which satisfies the relations $a^n = 1$, $b^2 = 1$, $ba = a^{-1}b$. The group is denoted by D_{2n} and has 2n elements say $\{ e, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$.*

Definition 2.2. *[9] Let Γ be a simple connected graph. The first Zagreb index of Γ, denoted by $M_1(\Gamma)$, is defined as*

$$M_1(\Gamma) = \sum_{v \in V(\Gamma)} (d_\Gamma(v))^2.$$

Where $d_\Gamma(v)$ is the number of edges incident on vertex v.

Definition 2.3. *[9] Let Γ be a simple connected graph. The second Zagreb index of Γ, denoted by $M_2(\Gamma)$, is defined as*

$$M_2(\Gamma) = \sum_{uv \in E(\Gamma)} d_\Gamma(u) d_\Gamma(v).$$

Definition 2.4. *[10] Let Γ be a simple connected graph. The Wiener index of Γ, denoted by $W(\Gamma)$, is defined as*

$$W(\Gamma) = \sum_{u,v\in V(\Gamma)} d(u,v).$$

Where d(u,v) is the length of the shortest path between the vertices u and v.

Definition 2.5. *[10] Let Γ be a simple connected graph. The hyper- Wiener index of Γ, denoted by $WW(\Gamma)$, is defined as*

$$WW(\Gamma) = \frac{1}{2}\{W(\Gamma) + \sum_{u,v\in V(\Gamma)} (d(u,v))^2\}.$$

Definition 2.6. *[11] Let Γ be a simple connected graph. The Harary index of Γ, denoted by $W(\Gamma)$, is defined as*

$$W(\Gamma) = \sum_{u,v\in V(\Gamma)} \frac{1}{d(u,v)}.$$

Definition 2.7. *[2, 3, 5, 6] The laplacian matrix $L(\Gamma)$ of graph Γ given by $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ is the diagonal matrix of vertex degree of Γ and $A(\Gamma)$ is the adjacency matrix of Γ.*

3 Mobius Function Graph of the Dihedral Groups

Mobius function graph of a finite group in which we are associating group to graph by using the order of the elements of the group. Here we discuss the characteristics of mobius function graph of dihedral groups.

Definition 3.1. *The mobius function graph of a finite group G (denoted by M(G)) is a simple graph whose vertex set is same as the elements of G and any two distinct vertices a, b are adjacent in M(G) if and only if $\mu(|a|\ |b|) = \mu(|a|)\mu(|b|)$.*

Theorem 3.2. *M(D_{2p}), p>2 prime number,is always a complete tripartite graph.*

Proof. Consider the dihedral group $D_{2p} = \{e, a, a^2, \ldots, a^{p-1}, b, ab, a^2b, \ldots, a^{p-1}b\}$,for a prime p>2. Then $| e |= 1$, $| a |=| a^2 |= \ldots =| a^{p-1} |= p$, $| b |=| ab |=| a^2b |= \ldots =| a^{p-1}b |= 2$. We partitioned the vertices of $M(D_{2n})$ into three sets, $\Omega_1 = \{e\}$, $\Omega_2 = \{u : |u| = p\} = \{a^i \mid 1 \leq i \leq p-1\}$ and $\Omega_3 = \{v : |v| = 2\} = \{a^jb \mid 0 \leq j \leq p-1\}$

In case of M(D_{2p}), the vertex associated with identity element e is adjacent to all other $(2p-1)$ vertices. That is Ω_1 is adjacent to every elements of Ω_2 and Ω_3.

For any two vertices x, y$\in \Omega_2$,

$\mu(| x |)\mu(| y |) = \mu(p)\mu(p) = (-1)(-1) = 1$

$\mu(| x || y |) = \mu(p.p) = 0$

$\mu(| x || y |) \neq \mu(| x |)\mu(| y |)$

Hence no two elements of Ω_2 are adjacent.

Similarly we can prove in case of Ω_3, no two elements of Ω_3 are adjacent.

Now consider the case when x $\in \Omega_2$ and y$\in \Omega_3$ then

$\mu(| x |)\mu(| y |) = \mu(p)\mu(2) = (-1)(-1) = 1$

$\mu(\mid x \mid\mid y \mid) = \mu(\text{p}.2) = (-1)^2 = 1$
$\mu(\mid x \mid\mid y \mid) = \mu(\mid x \mid)\mu(\mid y \mid)$
Hence every vertex of Ω_2 is adjacent to every vertex of Ω_3.
Hence we can conclude that M(D_{2p}), p >2 prime number, is a complete tripartite graph. □

Theorem 3.3. *For a prime p>2, M(D_{2p})≅ $(\overline{K_{p-1}} + \overline{K_p}) + K_1$.*

Proof. In the dihedral group D_{2p}, where p >2 a prime number. By Theorem 3.2, M(D_{2p}) is a complete tripartite graph.Then $\Omega_1 = \{e\}$, $\Omega_2 = \{u : |u| = p\}$ and $\Omega_3 = \{v : |v| = 2\}$. Also both Ω_2 and Ω_3 are independent sets having $(p-1)$ and p elements respectively. The graph of isomorphic to M(D_{2p})≅ $(\overline{K_{p-1}} + \overline{K_p}) + K_1$. □

Proposition 3.4. *M(D_{2n}) is a star graph if and only if n =2.*

Proof. Given M(D_{2n}) is a star graph.
Suppose n>2

Case 1: When n = p, a prime number.
By using Theorem 3.2 M(D_{2n}), is always a complete tripartite graph. This contradicts our assumption.

Case 2: When n = p^k , for k ≥ 2.
In a dihedral group, the rotations $\{e, a, a^2, \ldots, a^{n-1}\}$ is isomorphic to $\mathbb{Z}_n$.
Let x, y be any two elements of order p^2 so that $\mu(\mid x \mid\mid y \mid) = \mu(\mid x \mid)\mu(\mid y \mid)$.
Hence in M(D_{2n}) we get a cycle x-e-y-x. Which again contradicts our assumption.

Case 3: When n $= {p_1}^{n_1} {p_2}^{n_2} \ldots {p_r}^{n_r}$
As the rotations $\{e, a, a^2, \ldots, a^{n-1}\}$ in dihedral group is isomorphic to $\mathbb{Z}_n$.
There exist an element x of order p_i and element y of order p_j where i ≠ j.
Thus $\mu(\mid x \mid)\mu(\mid y \mid) = \mu(p_i)\mu(p_j)$ = (-1)(-1) = 1
$\mu(\mid x \mid\mid y \mid) = \mu(p_i.p_j) = (-1)^2 = 1$
$\mu(\mid x \mid\mid y \mid) = \mu(\mid x \mid)\mu(\mid y \mid)$
In this case also we get a cycle $x - e - y - x$ in M(D_{2n}). This again contradicts our assumption. From all the three case it is clear that our supposition is wrong.

Converse is always true. As n= 4, we get $D_4 = \{e, a, b, ab\}$ so that $\mid e \mid = 1$ and $\mid a \mid = \mid b \mid = \mid ab \mid = 2$. Here we get M($D_4$) as star graph. □

Theorem 3.5. *For a dihedral group D_{2n} with $n = 2^k$, $k > 0$, $M(D_{2n}) \cong K_{n-1} + \overline{K_{n+1}}$.*

Proof. Let $\gamma_1 = \{u : |u| \neq 2\}$ and $\gamma_2 = \{v : |v| = 2\}$. For any a,b ∈ γ_1 and a≠ b, $\mu(\mid a \mid\mid b \mid) = 0 = \mu(\mid a \mid)\mu(\mid b \mid)$ and so by the definition of mobius function graph a is adjacent to b in $M(D_{2n})$. Hence the subgraph induced by γ_1 is complete. Also in γ_1 there are (n-1) elements.
For any c,d ∈ γ_2 and c ≠ d, by the definition of mobius function graph c is not adjacent to d in $M(D_{2n})$ and hence the subgraph induced by γ_2 in $M(D_{2n})$ is

totally disconnected. Also there are (n+1) elements of order 2.
For any $a_1 \in \gamma_1$ and $c_1 \in \gamma_2$, $\mu(\mid a_1 \mid\mid c_1 \mid) = 0 = \mu(\mid a_1 \mid)\mu(\mid c_1 \mid)$. So a_1 is adjacent to c_1 in $M(D_{2n})$. Hence every element of γ_1 is adjacent to every element in γ_2.
Thus by the above argument $M(D_{2n}) \cong K_{n-1} + \overline{K_{n+1}}$. □

Corollary 3.6. *If $n = 2^k$ with $k > 0$, then $M(D_{2n})$ is a split graph.*

Proof. As in the proof of Theorem 3.5, γ_1 is a clique and γ_2 is an independent set. Thus $M(D_{2n})$ is a split graph. □

4 Topological indices of $M(D_{2n})$

Topological indices are being studied in this section. Here we discuss about the five topological indices of mobius function graph of dihedral groups.

Theorem 4.1. *Let $M(D_{2n})$ be the mobius function graph of the dihedral group D_{2n}, then the first Zagreb index of $M(D_{2n})$ is*

$$M_1(M(D_{2n})) = \begin{cases} n(2n^2+5n-5) & for\ n\ is\ an\ odd\ prime \\ n(n-1)(5n-4) & for\ n = 2^k,\ k > 0 \end{cases}$$

Proof. Take n as a prime number with n> 2. let $\Omega_1 = \{e\}$, $\Omega_2 = \{u : |u| = n\}$ and $\Omega_3 = \{v : |v| = 2\}$. Here each element of Ω_1, Ω_2 and Ω_3 has degree (2n-1), (n+1) and n respectively. Then the first Zagreb index

$$\begin{aligned} M_1(M(D_{2n})) &= \sum_{v \in V} (d_{M(D_{2n})}(v))^2 \\ &= \sum_{e \in \Omega_1} (d_{M(D_{2n})}(e))^2 + \sum_{u \in \Omega_2} (d_{M(D_{2n})}(u))^2 + \sum_{v \in \Omega_3} (d_{M(D_{2n})}(v))^2 \\ &= (2n-1)^2 + \sum_{u \in \Omega_2} (n+1)^2 + \sum_{v \in \Omega_3} n^2 \\ &= (2n-1)^2 + (n-1)(n+1)^2 + n(n^2) \\ &= n(2n^2+5n-5). \end{aligned}$$

Also when n = 2^k where k$\geq$ 1. Let $\gamma_1 = \{u : |u| \neq 2\}$ and $\gamma_2 = \{v : |v| = 2\}$.Each element of γ_1 and γ_2 has degrees 2n-1 and n-1 respectively. Thus the first Zagreb index

$$\begin{aligned} M_1(M(D_{2n})) &= \sum_{v \in V} (d_{M(D_{2n})}(v))^2 \\ &= \sum_{u \in \gamma_1} (d_{M(D_{2n})}(u))^2 + \sum_{v \in \gamma_2} (d_{M(D_{2n})}(v))^2 \\ &= \sum_{\gamma_1} (2n-1)^2 + \sum_{\gamma_2} (n-1)^2 \\ &= (n-1)(2n-1)^2 + (n+1)(n-1)^2 \\ &= n(n-1)(5n-4) \end{aligned}$$

□

Theorem 4.2. *Let $M(D_{2n})$ be the mobius function graph of the dihedral group D_{2n}, then the second Zagreb index of $M(D_{2n})$ is*

$$M_2(M(D_{2n})) = \begin{cases} n^4+4n^3-3n^2-2n+1 & for\ n\ is\ an\ odd\ prime \\ \dfrac{n(n-1)(2n-1)(4n-5)}{2} & for\ n=2^k,\ k>0 \end{cases}$$

Proof. Take n, a prime number with p> 2. let $\Omega_1 = \{e\}$, $\Omega_2 = \{u : |u| = n\}$ and $\Omega_3 = \{v : |v| = 2\}$. Here each element of Ω_1, Ω_2 and Ω_3 has degree (2n-1), (n+1) and n respectively. Then the second Zagreb index

$$M_2(M(D_{2n})) = \sum_{uv\in E}(d_{M(D_{2n})}(u))(d_{M(D_{2n})}(v)).$$
$$= \sum_{\substack{eu\in E\\ u\in\Omega_2}}(d_{M(D_{2n})}(e))(d_{M(D_{2n})}(u)) + \sum_{\substack{ev\in E\\ v\in\Omega_3}}(d_{M(D_{2n})}(e))(d_{M(D_{2n})}(v))$$
$$+ \sum_{\substack{uv\in E\\ u\in\Omega_2\\ v\in\Omega_3}}(d_{M(D_{2n})}(u))(d_{M(D_{2n})}(v))$$
$$= \sum_{\Omega_2}(2n-1)(n+1) + \sum_{\Omega_3}(2n-1)n + \sum_{\Omega_2\Omega_3}(n+1)n$$
$$=(n-1)[(2n-1)(n+1)] + n[(2n-1)(n)] + n[(n-1)[(n+1)(n)]]$$
$$=n^4+4n^3-3n^2-2n+1$$

Also when n = 2^k where k$\geq$ 1. Let $\gamma_1 = \{u : |u| \neq 2\}$ and $\gamma_2 = \{v : |v| = 2\}$. Each element of γ_1 and γ_2 has degrees 2n-1 and n-1 respectively. Thus the second Zagreb index

$$M_2(M(D_{2n})) = \sum_{uv\in E}(d_{M(D_{2n})}(u))(d_{M(D_{2n})}(v)).$$
$$= \sum_{\substack{uv\in E\\ u,v\in\gamma_1}}(d_{M(D_{2n})}(u))(d_{M(D_{2n})}(v)) + \sum_{\substack{uv\in E\\ u\in\gamma_1\\ v\in\gamma_2}}(d_{M(D_{2n})}(u))(d_{M(D_{2n})}(v))$$
$$= \sum_{u,v\in\gamma_1}(2n-1)(2n-1) + \sum_{\gamma_1,\gamma_2}(2n-1)(n-1)$$
$$=\binom{n-1}{2}(2n-1)^2 + (n-1)(n+1)(2n-1)(n-1)$$
$$=\frac{n(n-1)(2n-1)(4n-5)}{2}$$

□

Theorem 4.3. *Let $M(D_{2n})$ be the mobius function graph of the dihedral group D_{2n}, then the Wiener index of $M(D_{2n})$ is*

$$W(M(D_{2n})) = \begin{cases} 3n^2-3n+1 & for\ n\ is\ an\ odd\ prime \\ \dfrac{n(5n-1)}{2} & for\ n=2^k,\ k>0 \end{cases}$$

Proof. Take n, a prime number with p> 2. let $\Omega_1 = \{e\}$, $\Omega_2 = \{u : |u| = n\}$ and $\Omega_3 = \{v : |v| = 2\}$. Then the Wiener index

$$W(M(D_{2n})) = \sum_{u,v\in V} d(u,v).$$
$$= \sum_{\substack{e\in\Omega_1\\ u\in\Omega_2}}(d(e,u)) + \sum_{\substack{e\in\Omega_1\\ v\in\Omega_3}}(d(e,v)) + \sum_{\substack{u\in\Omega_2\\ v\in\Omega_3}}(d(u,v)) + \sum_{u,v\in\Omega_2}(d(u,v))$$
$$+ \sum_{u,v\in\Omega_3}(d(u,v))$$
$$= \sum_{\substack{e\in\Omega_1\\ u\in\Omega_2}}1 + \sum_{\substack{e\in\Omega_1\\ v\in\Omega_3}}1 + \sum_{\substack{u\in\Omega_2\\ v\in\Omega_3}}1 + \sum_{u,v\in\Omega_2}2 + \sum_{u,v\in\Omega_3}2$$
$$= (n-1) + n + n(n-1) + \binom{n-1}{2}2 + \binom{n}{2}2$$

$= 3n^2 - 3n + 1$

Similarly way we consider the case when n= 2^k with k $\geq$ 2. Take $\gamma_1 = \{u : |u| \neq 2\}$ and $\gamma_2 = \{v : |v| = 2\}$. Then corresponding Wiener index

$$\begin{aligned} W(M(D_{2n})) &= \sum_{u,v\in V} d(u,v). \\ &= \sum_{\substack{u\in\gamma_1 \\ v\in\gamma_2}} (d(u,v)) + \sum_{u,v\in\gamma_1} (d(u,v)) + \sum_{u,v\in\gamma_2} (d(u,v)) \\ &= \sum_{\substack{u\in\gamma_1 \\ v\in\gamma_2}} 1 + \sum_{u,v\in\gamma_1} 1 + \sum_{u,v\in\gamma_2} 2 \\ &= (n-1)(n+1) \ + \ \binom{n-1}{2} 1 \ + \ \binom{n+1}{2} 2 \\ &= \frac{n(5n-1)}{2} \end{aligned}$$

□

Theorem 4.4. *Let $M(D_{2n})$ be the mobius function graph of the dihedral group D_{2n}, then the hyper Wiener index of $M(D_{2n})$ is*

$$WW(M(D_{2n})) = \begin{cases} 4n^2 - 5n + 2 & for\ n\ is\ an\ odd\ prime \\ 3n^2 & for\ for\ n = 2^k,\ k > 0 \end{cases}$$

Proof. Take n, a prime number with p> 2. Let $\Omega_1 = \{e\}$, $\Omega_2 = \{u : |u| = n\}$ and $\Omega_3 = \{v : |v| = 2\}$.

$$\begin{aligned} \sum_{u,v\in V} d(u,v)^2 &= \sum_{\substack{e\in\Omega_1 \\ u\in\Omega_2}} (d(e,u))^2 + \sum_{\substack{e\in\Omega_1 \\ v\in\Omega_3}} (d(e,v))^2 + \sum_{\substack{u\in\Omega_2 \\ v\in\Omega_3}} (d(u,v))^2 \\ &+ \sum_{u,v\in\Omega_2} (d(u,v))^2 + \sum_{u,v\in\Omega_3} (d(u,v))^2 \\ &= \sum_{\substack{e\in\Omega_1 \\ u\in\Omega_2}} 1 + \sum_{\substack{e\in\Omega_1 \\ v\in\Omega_3}} 1 + \sum_{\substack{u\in\Omega_2 \\ v\in\Omega_3}} 1 + \sum_{u,v\in\Omega_2} 2^2 + \sum_{u,v\in\Omega_3} 2^2 \\ &= (n-1) + n + n(n-1) + \binom{n-1}{2} 4 \ + \ \binom{n}{2} 4 \\ &= 5n^2 - 7n + 3 \end{aligned}$$

For finding the hyper Wiener index , referring Theorem 4.3 we have

$$\begin{aligned} WW(M(D_{2n})) &= \frac{1}{2}\{W(M(D_{2p})) + \sum_{u,v\in V} (d(u,v))^2\}. \\ &= \frac{1}{2}\{(3n^2 - 3n + 1) \ + \ (5n^2 - 7n + 3)\} \\ &= 4n^2 - 5n + 2 \end{aligned}$$

Similarly way we consider the case when n= 2^k , where k $\geq$ 2.Take $\gamma_1 = \{u : |u| \neq 2\}$ and $\gamma_2 = \{v : |v| = 2\}$.Then corresponding

$$\begin{aligned} \sum_{u,v\in V} d(u,v)^2 &= \sum_{\substack{u\in\gamma_1 \\ v\in\gamma_2}} (d(u,v))^2 + \sum_{u,v\in\gamma_1} (d(u,v))^2 + \sum_{u,v\in\gamma_2} (d(u,v))^2 \\ &= \sum_{\substack{u\in\gamma_1 \\ v\in\gamma_2}} 1 + \sum_{u,v\in\gamma_1} 1 + \sum_{u,v\in\gamma_2} 2^2 \\ &= (n-1)(n+1) \ + \ \binom{n-1}{2} 1 \ + \ \binom{n+1}{2} 2 \\ &= \frac{n(7n+1)}{2}. \end{aligned}$$

By Theorem 4.3 we get hyper Wiener index

$$WW(M(D_{2n})) = \frac{1}{2}\{W(M(D_{2p})) + \sum_{u,v\in V}(d(u,v))^2\}.$$
$$= \frac{1}{2}\{\frac{n(5n-1)}{2} \ + \ n(7n+1)2\}$$
$$= 3n^2$$

□

Theorem 4.5. *Let $M(D_{2n})$ be the mobius function graph of the dihedral group D_{2n}, then the Harary index of $M(D_{2n})$ is*

$$H(M(D_{2n})) = \begin{cases} \dfrac{3n^2-1}{2} & for\ n\ is\ an\ odd\ prime \\ \dfrac{n(7n-5)}{4} & for\ n = 2^k,\ k > 0 \end{cases}$$

Proof. Take n, a prime number with n > 2. Let $\Omega_1 = \{e\}$, $\Omega_2 = \{u : |u| = n\}$ and $\Omega_3 = \{v : |v| = 2\}$.

The Harary index

$$H(M(D_{2n})) = \sum_{u,v\in V} \frac{1}{d(u,v)}.$$
$$= \sum_{\substack{e\in\Omega_1\\u\in\Omega_2}} \frac{1}{d(e,u)} + \sum_{\substack{e\in\Omega_1\\v\in\Omega_3}} \frac{1}{d(e,v)} + \sum_{\substack{u\in\Omega_2\\v\in\Omega_3}} \frac{1}{d(u,v)} + \sum_{u,v\in\Omega_2} \frac{1}{d(u,v)}$$
$$+ \sum_{u,v\in\Omega_3} \frac{1}{d(u,v)}$$
$$= \sum_{\substack{e\in\Omega_1\\u\in\Omega_2}} 1 + \sum_{\substack{e\in\Omega_1\\v\in\Omega_3}} 1 + \sum_{\substack{u\in\Omega_2\\v\in\Omega_3}} 1 + \sum_{u,v\in\Omega_2} \frac{1}{2} + \sum_{u,v\in\Omega_3} \frac{1}{2}$$
$$= (n-1) + n + n(n-1) + \binom{n-1}{2}\frac{1}{2} \ + \ \binom{n}{2}\frac{1}{2}$$
$$= \frac{3n^2-1}{2}$$

Similarly way we consider the case when n= 2^k, where k $\geq$ 2. Take $\gamma_1 = \{u : |u| \neq 2\}$ and $\gamma_2 = \{v : |v| = 2\}$. Then corresponding Harary index

$$H(M(D_{2n})) = \sum_{u,v\in V} d(u,v).$$
$$= \sum_{\substack{u\in\gamma_1\\v\in\gamma_2}} \frac{1}{d(u,v)} + \sum_{u,v\in\gamma_1} \frac{1}{d(u,v)} + \sum_{u,v\in\gamma_2} \frac{1}{d(u,v)}$$
$$= \sum_{\substack{u\in\gamma_1\\v\in\gamma_2}} 1 + \sum_{u,v\in\gamma_1} 1 + \sum_{u,v\in\gamma_2} \frac{1}{2}$$
$$= (n-1)(n+1) \ + \ \binom{n-1}{2} 1 \ + \ \binom{n+1}{2}\frac{1}{2}$$
$$= \frac{n(7n-5)}{4}$$

□

5 Laplacian Spectrum of $M(D_{2n})$

Section deals with the eigenvalues of laplacian matrix of mobius function graph of dihedral groups. Let G be a graph with n vertices, for the vertices $v_1, v_2, \cdots, v_i$ in G, $L_{v_1,v_2,\cdots,v_i}(G)$ is defined as the principal submatrix of $L(G)$ formed by deleting the rows and columns corresponding to the vertices $v_1, v_2, \cdots, v_i$. In particular if i=n , then $\Theta(L_{v_1,v_2,\cdots,v_n}(G), x)$= 1 .

Theorem 5.1. *For n= 2 ^{k}, the laplacian eigenvalue of M(D_{2n}) are 0, (n-1) and 2n of multiplicity 1, n and n-1 respectively.*

Proof. When n= 2^k, the vertices of $M(D_{2n})$ can be partitioned as $\gamma_1 = \{u : |u| \neq 2\} = \{a_1, a_2, \cdots, a_{n-1}\}$, each of degree (2n-1) and γ_2 =$\{v : |v| = 2\}$ = $\{b_1, b_2, \cdots, b_{n+1}\}$, each of degree (n-1). Then the laplacian matrix is

$$\text{L(M}(D_{2n})) = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ b_1 \\ b_2 \\ \vdots \\ b_{n+1} \end{array} \begin{array}{c} \begin{array}{cccccccc} a_1 & a_2 & \cdots & a_{n-1} & b_1 & b_2 & \cdots & b_{n+1} \end{array} \\ \begin{pmatrix} 2n-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & 2n-1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 2n-1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & n-1 & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & n-1 \end{pmatrix}_{2n\times 2n} \end{array}$$

Corresponding characteristic polynomial is given by

$\Theta(L(M(D_{2n})), x)$

$$= \begin{vmatrix} x-(2n-1) & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & x-(2n-1) & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & x-(2n-1) & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & x-(n-1) & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & \cdots & x-(n-1) \end{vmatrix}$$

Multiplying first row by (x-1) and then applying the row operations, $R_1 = R_1 - R_2 - \ldots - R_{2n}$ then the above determinant will be

$\Theta(L(M(D_{2n})), x) = \dfrac{x(x-2n)}{x-1}\Theta(L_e(M(D_{2n})), x)$.

Again multiplying the first row of $\Theta(L_e(M(G)), x)$ by (x-2) and then applying the row operations $R_1 = R_1 - R_2 - \ldots - R_{2n-1}$ we get

$\Theta(L_e(M(D_{2n})), x) = \dfrac{(x-1)(x-2n)}{x-2}\Theta(L_{e,a}(M(D_{2n})), x)$.

Proceeding in this way $\Theta(L_{e,a,a^2,\ldots,a^{n-1}}(M(D_{2n})), x) = \dfrac{x(x-2n)^{n-1}}{x-(n-1)}\Theta(L_{e,a,a^2,\ldots,a^{n-1},b}(M(D_{2n})), x)$.

and $\Theta(L_{e,a,a^2,\ldots,a^{n-1},b}(M(D_{2n})),x)=\begin{vmatrix} x-(n-1) & 0 & \cdots & 0 \\ 0 & x-(n-1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x-(n-1) \end{vmatrix}$

Hence we get $\Theta(L(M(D_{2n})),x) = x(x-2n)^{n-1}(x-(n-1))^n$

Thus $M(D_{2n})$ has laplacian eigenvalues 0, (n-1) and 2n of multiplicity 1, n and n-1 respectively. □

Theorem 5.2. *The laplacian eigenvalues of M(D_{2p}) where p is a prime number are 0,p ,p+1 and 2p of multiplicity 1, (p-1),(p-2) and 2 respectively.*

Proof. By using Theorem 3.2, $\Omega_1 = \{e\}$, $\Omega_2 = \{u : |u| = p\} = \{a, a^2, \cdots, a^{p-1}\}$ and $\Omega_3 = \{v : |v| = 2\} = \{b, ab, \cdots, a^{p-1}b\}$. The laplacian matrix $M(D_{2p})$ is given by

$$\text{L(M}(D_{2p})) = \begin{array}{c} \\ e \\ a \\ \vdots \\ a^{p-1} \\ b \\ ab \\ \vdots \\ a^{p-1}b \end{array} \begin{array}{c} \begin{array}{cccccccc} e & a & \cdots & a^{p-1} & b & ab & \cdots & a^{p-1}b \end{array} \\ \begin{pmatrix} 2p-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & p+1 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & p+1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & p & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & p \end{pmatrix}_{2p\times 2p} \end{array}$$

Corresponding characteristic polynomial is

$\Theta(L(M(D_{2p})),x)$

$$= \begin{vmatrix} x-(2p-1) & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & x-(p+1) & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & x-(p+1) & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & x-p & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & x-p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & x-p \end{vmatrix}$$

Multiplying first row by (x-1) and then applying the row operations, $R_1 = R_1 - R_2 - \ldots - R_{2p}$ then the above determinant will be

$\Theta(L(M(D_{2p})),x) = \dfrac{x(x-2p)}{x-1}\Theta(L_e(M(D_{2p})),x)$.

$$\Theta(L_e(M(D_{2p})),x) = \begin{vmatrix} x-(p+1) & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & x-(p+1) & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x-(p+1) & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & x-p & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & x-p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & x-p \end{vmatrix}$$

$$=|A||D - B^T A^{-1} B| \qquad \cdots\cdots \ (1)$$

Where $A = \begin{bmatrix} x-(p+1) & 0 & \cdots & 0 \\ 0 & x-(p+1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x-(p+1) \end{bmatrix}_{(p-1)\times(p-1)}$

$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{(p-1)\times p}$

and $D = \begin{bmatrix} x-p & 0 & \cdots & 0 \\ 0 & x-p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x-p \end{bmatrix}_{p\times p}$

Now we get $|A| = (x-(p+1))^{(p-1)}$

$$|D - B^T A^{-1} B| = \begin{vmatrix} (x-p) - \frac{(p-1)}{x-(p+1)} & -\frac{(p-1)}{x-(p+1)} & \cdots & -\frac{(p-1)}{x-(p+1)} \\ -\frac{(p-1)}{x-(p+1)} & (x-p) - \frac{(p-1)}{x-(p+1)} & \cdots & -\frac{(p-1)}{x-(p+1)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{(p-1)}{x-(p+1)} & -\frac{(p-1)}{x-(p+1)} & \cdots & (x-p) - \frac{(p-1)}{x-(p+1)} \end{vmatrix}$$

$$=(x-p) - \frac{p(p-1)}{x-(p+1)}(x-p)^{(p-1)}$$

$$=\frac{(x-1)(x-2p)}{x-(p+1)}(x-p)^{(p-1)}$$

Substituting in (1) we get

$$\Theta(L_e(M(D_{2p})), x) = (x-(p+1))^{(p-1)} \frac{(x-1)(x-2p)}{x-(p+1)}(x-p)^{(p-1)}$$

$$=(x-(p+1))^{(p-2)}(x-1)(x-2p)(x-p)^{(p-1)}.$$

Hence $\Theta(L(M(D_{2p})), x) = x(x-2p)^2(x-(p+1))^{(p-2)}(x-p)^{(p-1)}$

□

References

[1] Bapat,R.B.,*Graph And Matrices*,Vol 27, Springer, New York 2010.

[2] Grone,R. and Merris,R.,*The Laplacian Spectrum of a graph II*, SIAM Journal of Discrete Mathematics. 7 (1994) 221-229.

[3] Grone,R.,Merris,R.and Sunder,V.S.,*The Laplacian Spectrum of a graph*, SIAM Journal of Matrix Analysis. 11 (1990) 218-238

[4] Gutman,I.,*The energy of regular graph*, Ber.Math-Statist.Sekt.Forchungsz.Graz,103,pp.1-22,1978.

[5] Merris,R., *A survey of graph laplacians*, Linear Multilinear Algebra, 39 (1995) 19 - 31

[6] Merris,R., *Laplacian Matrices of a graphs: A survey*, Linear Algebra Appl.197-198(1994) 143 -176

[7] Gallian,J .A.,*Contemporary abstract algebra*, Fourth edition, Narosa publishing house.

[8] Mohar,B.,*The Laplacian Spectrum of graphs*. In: Alavi Y, Chartrand G, Oellermann OR, Schwenk AJ, editors. Graph Theory, combinatorics and applications. Vol.2.New York: Wiley; 1991.p.871-898.

[9] Das, K.C., Xu, K. and Nam, J., *Zagreb indices of graphs* , Frontiers of Mathematics in China, 10(3)(2015), 567582.

[10] Dobrynin, A.A., Entringer, R. and Gudman, I., *Wiener index of trees: theory and applications*, Acta Applicandae Mathematics, 66(3)(2001), 211-249

[11] Xu, K., and Das, K.C.,*On harary index of graphs*, Discrete applied mathematics, 159(15)(2011), 1631 -1640.

[12] Chattopadhyay,Sriparna and Panigrahi,Pratima *On laplacian spectrum of power graphs of finite cyclic and dihedral groups*, linear and multilinear algebra, 2015, vol 63.

[13] Aravinth,R. H., Vignesh, R., *Mobius Function Graph* $M_n(G)$, Inter. J. Innovative Tech. Explor. Engi., 8 (10) (2019), 1481–1484.

[14] Balakrishnan,R., Ranganathan, K. *A Textbook of Graph Theory*, Springer, New York, 2012.

[15] Cameron,P. J., *The power graph of a finite group II*, J. Group Theory, 13 (6) (2010), 779–783.

[16] Cayley,P., Desiderata and suggestions: No.2. The Theory of Groups: Graphical Representation, *Amer. J. Math.*, 1 (2) (1878), 174–176.

[17] Folders,S, Hammer, P. L., Split Graphs, *Proc. 8th Southeastern Conf. on Combinatorics, Graph Theory and Computing* (F. Koffman et al. eds.), Louisiana State Univ., Baton Rouge, Louisianna, (1977), 311-315.

[18] Jose, Rani, Selvakumar, K and Susha, D,*On the Mobius function graph of finite groups*, communicated (2023).

[19] Fraleigh,J. B., *A first course in abstract algebra*, Third Edition, Narosa Publishing House, India, 2013.

[20] Herstein, I. N. *Topics in Algebra*, Second Edition, John Wiley and Sons, UAS, 1975.

[21] Sattanathan,M, Kala,R., *An introduction to order prime graphs*, Int. J. Contemp. Math. Science, 4 (10) (2009), 467–474.

[22] Apostol,T.M., *Introduction to Analytic Number Theory*, Springer, New York, 2010.

[23] Williams, J. S., *Prime graph components of finite groups*, J. Algebra, 69 (2) (1981), 487–513.