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ON $nIgsemi^*$-NORMAL SPACES AND $nIgsemi^*$-REGULAR SPACES IN NANO IDEAL TOPOLOGICAL SPACES

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Abstract
The notion of nano ideal topological space was introduced by M. Parimala [8] et al. They studied its properties and characterizations. Also they introduced the concept of nano ideal generalized closed sets in nano ideal topological spaces and investigated some of its basic properties. The aim of this paper is to introduce and study $nIgsemi^*$-normal spaces and $nIgsemi^*$-regular spaces by using $nIgsemi^*$-closed sets and $nIgsemi^*$-open sets in nano ideal topological spaces. In addition, we define quasi $nIgsemi^*$- normal spaces and examine its basic properties and characterizations.

Key words: $nIgsemi^*$-normal spaces, $nIgsemi^*$-regular spaces, $nIgsemi^*$-closed sets, $nIgsemi^*$-open sets.

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1. Preliminary

Definition 1.1 [6] Let $(U, \tau_R(X))$ and $(V, \tau_{R_0}(X))$ be nano topological spaces. A mapping $f: (U, \tau_R(X)) \to (V, \tau_{R_0}(X))$ is said to be ***nano continuous*** if the inverse image of every nano open set in $(V, \tau_{R_0}(X))$ is nano open set in $(U, \tau_R(X))$.

Definition1.2 A subset H of a nano topological space $(U, \tau_R(X))$ is called

1. ***nano regular-open*** [7] if $H = N\,int(Ncl(H))$
2. ***nano π-open*** [9] is the finite union of nano regular-open sets.

Definition 1.3 [5] A nano topological space $(U, \tau_R(X))$ is said to be ***nano-normal space*** if for any pair of disjoint nano closed sets A and B, there exists disjoint nano open sets M and N such that $A \subset M$ and $B \subset N$.

Definition 1.4 [4] A nano topological space $(U, \tau_R(X))$ is said to be ***nano-regular space***, if for each nano closed set F and each point $x \notin F$, there exists disjoint nano open sets G and H such that $x \in G$ and $F \subset H$.

Definition 1.5 [6] A function $f: (U, \tau_R(X)) \to (V, \tau_R(Y))$ is defined to be ***nano open (closed) function*** if every nano open (closed) set A of U is mapped to a nano open (closed) set in V.

Definition 1.6 [1] A subset A of a nano ideal topological space (U, N, I) is said to be ***nano ideal generalized $semi^*$-closed*** (briefly, $nIgsemi^*$-closed) if $A_n^* \subseteq V$ whenever $A \subseteq V$ and V is nano $semi^*$-open. The family of all $nIgsemi^*$-closed sets of U is denoted by $nIgsemi^*C(U, N, I)$(or simply $nIgsemi^*C(U)$.

Theorem 1.7. [1] Every nano closed set is $nIgsemi^*$-closed but not conversely

Definition 1.8. [2] A map $f: (U, N, I) \to (V, M, J)$ is called **$nIgsemi^*$-continuous function**, if $f^{-1}(A)$ of each n-open set $A \subseteq (V, M, J)$ is a $nIgsemi^*$-open set in (U, N, I).

Definition 1.9. [2] Consider two nano ideal topological spaces (U, N, I) and (V, M, J) and define a function $f: (U, N, I) \to (V, M, J)$. The function f is defined to be a **$nIgsemi^*$-irresolute function**, if the inverse image $f^{-1}(A)$ is a $nIgsemi^*$-open set in (U, N, I) for every $nIgsemi^*$-open set A in (V, M, J).

Definition 1.10. [3] A map $f: (U, N, I) \to (V, M, J)$ is said to be **$nIgsemi^*$-closed map** if for every nano closed (n-closed) subset G of (U, N, I), $f(G)$ is $nIgsemi^*$-closed in (V, M, J).

2. $nIgsemi^*$-Normal Spaces

Definition 2.1 An nano ideal space (U, N, I) is said to be **$nIgsemi^*$-normal space** if for each pair of disjoint nano closed sets A and B, there exist disjoint $nIgsemi^*$-open sets G and O in U such that $A \subset G$ and $B \subset O$.

Theorem 2.2 Every nano normal space is $nIgsemi^*$-normal space.

Proof: Let (U, N, I) be nano normal space and A, B are two disjoint pair of nano closed sets. Since (U, N, I) is nano normal, there exists disjoint nano open sets M and N such that $A \subset M$ and $B \subset N$. Since every nano open set is $nIgsemi^*$-open, M and N are $nIgsemi^*$-open sets. Hence (U, N, I) is $nIgsemi^*$-normal space.

Remark 2.3 The following example shows that the converse of the above theorem need not be true.

Example 2.4 Consider the universal set $U = \{a, b, c, d\}$, the approximation space $U/\mathcal{R} = \{\{a\}, \{c\}, \{b, d\}\}, X = \{a, b\} \subseteq U$ with the ideal $I = \{\varphi, \{a\}\}$. The nano topology defined by U is $\tau_R(X) = \{U, \varphi, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $nIgsemi^*$-open sets are $\{\{a\}, \{b\}, \{d\}, \{a, b\}, \{b, d\}, \{a, d\}, \{b, c, d\}, \{a, b, d\}, U, \varphi\}$. Here U is $nIgsemi^*$-normal space but not nano normal space.

Theorem 2.5 If nano ideal topological space U is $nIgsemi^*$-normal then for every pair of nano open M and N whose union is U, there exist $nIgsemi^*$-closed sets A and B such that $A \subset M$, $B \subset N$ and $A \cup B = U$.

Proof: Let M and N be a pair of nano open sets in $nIgsemi^*$-normal space U such that $M \cup N = U$. Then $U - M$, $U - N$ are disjoint nano closed sets. Since U is $nIgsemi^*$-normal space, there exists two $nIgsemi^*$-open sets M_1 and N_1 such that $U - M \subset M_1$ and $U - N \subset N_1$. Let $A = U - M_1$and $B = U - N_1$ Then A and B are $nIgsemi^*$-closed sets such that $A \subset M$, $B \subset N$ and $A \cup B = U$

Theorem 2.6 If $f: (U, N, I) \to (V, M, J)$ is nano continuous bijective, $nIgsemi^*$-open function and U is nano normal space then V is $nIgsemi^*$-normal.

Proof: Let E and F be disjoint nano closed set in V. Since f is nano continuous bijective, $f^{-1}(E)$ and $f^{-1}(F)$ are disjoint nano closed in U. Now U is nano normal space, there exist disjoint nano open sets G and H such that $f^{-1}(E) \subset G$ and $f^{-1}(F) \subset H$. That is $E \subset f(G)$ and $F \subset f(H)$. Since f is $nIgsemi^*$-open function, $f(G), f(H)$ are $nIgsemi^*$-open sets in V and f is injective, $f(G) \cap f(H) = f(G \cap H) = f(\emptyset) = \emptyset$. Therefore V is $nIgsemi^*$-normal space.

Remark: 2.7 If $f:(U,N,I) \to (V,M,J)$ is nano continuous bijective, nano open function and U is nano normal space then V is $nIgsemi^*$-normal.

Theorem 2.8 If $f:(U,N,I) \to (V,M,J)$ is $nIgsemi^*$-continuous, nano closed bijection and V is a nano normal space then U is $nIgsemi^*$-normal.

Proof: Let E and F be disjoint nano closed set in U. Since f is nano closed injection, $f(E)$ and $f(F)$ are disjoint nano closed in V. Now V is nano normal space, there exist disjoint nano open sets G and H such that $f(E) \subset G$ and $f(F) \subset H$. That is $E \subset f^{-1}(G)$ and $F \subset f^{-1}(H)$. Since f is $nIgsemi^*$-continuous function, $f^{-1}(G)$ and $f^{-1}(H)$ are $nIgsemi^*$-open sets in U. Further $f^{-1}(G) \cap f^{-1}(H) = \varphi$ Therefore U is $nIgsemi^*$-normal space.

Theorem 2.9 If $f:(U,N,I) \to (V,M,J)$ is $nIgsemi^*$-irresolute, nano closed bijection and V is a $nIgsemi^*$-normal then U is $nIgsemi^*$-normal.

Proof: Let E and F be disjoint nano closed set in U. Since f is nano closed injection, $f(E)$ and $f(F)$ are disjoint nano closed in V. Now V is $nIgsemi^*$-normal space, there exist disjoint $nIgsemi^*$-open sets G and H such that $f(E) \subset G$ and $f(F) \subset H$. That is $E \subset f^{-1}(G)$ and $F \subset f^{-1}(H)$. Since f is $nIgsemi^*$-irresolute, $f^{-1}(G)$ and $f^{-1}(H)$ are $nIgsemi^*$-open sets in U. Further $f^{-1}(G) \cap f^{-1}(H) = \varphi$. Therefore U is $nIgsemi^*$-normal space.

Remark 2.10 If $f:(U,N,I) \to (V,M,J)$ is $nIgsemi^*$ irresolute, nano closed bijection and V is a nano-normal then U is also $nIgsemi^*$- normal.

3. Quasi $nIgsemi^*$- Normal Spaces

Definition 3.1 A nano ideal topological $(U,\mathcal{N},I)$ is said to be quasi $nIgsemi^*$-normal if for every pair of disjoint nano π-closed subsets A and B of U, there exist disjoint nano $nIgsemi^*$-open subsets V and W of U such that $A \subseteq V$ and $B \subseteq W$.

Theorem 3.2 For a nano ideal topological $(U,\mathcal{N},I)$, the following are equivalent.

1. U is quasi $nIgsemi^*$- normal space.
2. For every pair of nano π-open subsets V and W of U whose union is U, there exist nano $nIgsemi^*$-closed subsets G and H of U such that $G \subseteq V$ and $H \subseteq W$ and $G \cup H = U$.
3. For any nano π-closed set A and each nano π-open set B such that $A \subseteq B$, there exists an $nIgsemi^*$- open set V such that $A \subseteq V \subseteq nIgsemi^*cl(V) \subseteq B$.
4. For every pair of disjoint nano π-closed subsets A and B of U there exist $nIgsemi^*$- open subsets V and W of U such that $A \subseteq V, B \subseteq W$ and $nIgsemi^*cl(V) \cap nIgsemi^*cl(W) = \emptyset$.

Proof: (1)⇒ (2): Let V and W be any nano π-open subsets of a quasi $nIgsemi^*$- normal space U such that $V \cup W = U$. Then, $U\backslash V$ and $U\backslash W$ are nano π-closed subsets of U. By quasi $nIgsemi^*$ -normality of U, there exist disjoint $nIgsemi^*$-open subsets V_1 and W_1 of U such that $U\backslash V \subseteq V_1$ and $U \backslash W \subseteq W_1$. Let $G = U\backslash V_1$ and $H = U\backslash W_1$. Then G and H are $nIgsemi^*$-closed subsets of U such that $G \subseteq V$ and $H \subseteq W$ and $G \cup H = U$.

(2)⇒(3): Let A be a nano π-closed set and B be an nano π-open subset of U such that $A \subseteq B$. Then, $U \backslash A$ and B are nano π-open subsets of U and whose union is U. Then by (2), there exist $nIgsemi^*$-closed sets G and H of U such that $G \subseteq U\backslash A$ and $H \subseteq B$ and $G \cup H = U$. Then $A \subseteq U\backslash G$, $U\backslash B \subseteq U\backslash H$ and $(U\backslash G) \cap (U\backslash H) = \emptyset$. Let $V = U\backslash G$ and $W = U\backslash H$. Then V and W are disjoint $nIgsemi^*$-open sets such that $A \subseteq V$ and $U\backslash B \subseteq W$ which implies $U\backslash W \subseteq B$. Since $U\backslash W$ is $nIgsemi^*$-closed, we have $nIgsemi^*cl(V) \subseteq U\backslash W$. Thus, $A \subseteq V \subseteq nIgsemi^*cl(V) \subseteq B$.

(3)⇒ (4): Let A and B are any two disjoint nano π-closed sets of U. Then $A \subseteq U\backslash B$ where $U\backslash B$ is nano π-open. Then, by (3), there exists a $nIgsemi^*$- open subset V of U such that $A \subseteq V \subseteq nIgsemi^*cl(V) \subseteq U\backslash B$. This implies $A \subseteq V$ and $B \subseteq U\backslash nIgsemi^*cl(V)$. Let $W = U\backslash nIgsemi^*cl(V)$. Then W is $nIgsemi^*$-open subset of U. Thus, we obtain, $A \subseteq V$, $B \subseteq W$ and $nIgsemi^*cl(V) \cap nIgsemi^*cl(W) = \emptyset$.
(4)⇒(1): Obvious.

Proposition 3.3 Let $f:(U,N,I) \to (V,M,J)$ be a function then

1. The image of $nIgsemi^*$-open subset under nano open, nano continuous function is $nIgsemi^*$-open.

2. The inverse image of $nIgsemi^*$-open (resply. $nIgsemi^*$-closed) subset under nano open, nano continuous function is $nIgsemi^*$-open.
3. The image of $nIgsemi^*$-closed subset under nano open and nano closed, nano continuous bijective function is $nIgsemi^*$-open.

Theorem 3.4 The image of quasi $nIgsemi^*$-normal space under nano open and nano continuous bijective function is quasi $nIgsemi^*$-normal.

Proof: Let U be a quasi $nIgsemi^*$-normal space and let $f:(U,N,I) \to (V,M,J)$ be an nano open and nano continuous bijective function. We need to show that $f(U)$ is quasi $nIgsemi^*$-normal. Let A and B be two disjoint nano π-closed sets in $f(U)$. Since the inverse image of an nano π-closed set under nano open, nano continuous function is nano π-closed, we have $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nano π-closed subsets in U. Since U is a quasi $nIgsemi^*$-normal, there exists $nIgsemi^*$-open subsets V and W of U such that $f^{-1}(A) \subseteq V, f^{-1}(B) \subseteq W$ and $V \cap W = \phi$. Since f is nano open and nano continuous bijective function, we have $A \subseteq f(V), B \subseteq f(W)$ and $f(V) \cap f(W) = \emptyset$. Hence $f(V)$ and $f(W)$ are disjoint $nIgsemi^*$-open sets in $f(U)$ such that $A \subseteq f(V)$ and $B \subseteq f(W)$. Hence, $f(U)$ is quasi $nIgsemi^*$-open normal space.

4. $nIgsemi^*$-Regular Spaces

Definition 4.1. A nano ideal topological space (U,N,I) is said to be ***nIgsemi**- regular space**, if for each nano closed set F and each point $x \notin F$, there exists disjoint $nIgsemi^*$- open sets G and H such that $x \in G$ and $F \subset H$.

Theorem 4.2. Every nano regular space is $nIgsemi^*$- regular space.

Proof: Let F be a nano closed set and $x \notin F$ be a point of a nano regular space (U,N,I). Since U is nano regular space there exist two disjoint nano open sets G and H such that $x \in G$ and $F \subset H$. Since every nano open set is $nIgsemi^*$- open set, G and H are $nIgsemi^*$-open sets such that $x \in G$ and $F \subset H$. Hence (U,N,I) is $nIgsemi^*$-regular space.

Remark 4.3. Every $nIgsemi^*$-regular space need not be nano regular as given in the following example.

Example 4.4 Let $U = \{a,b,c,d\}$, the approximation space $U/\mathcal{R} = \{\{a\},\{b\},\{c,d\}\}, X = \{b,d\} \subseteq U$ with the ideal $I = \{\varphi,\{c\}\}$. The nano topology defined by U is $\tau_R(X) = \{U,\varphi, \{b\},\{c,d\},\{b,c,d\}\}$ and $nIgsemi^*$-closed sets are $\{U,\varphi,\{a\},\{c\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$. Here U is $nIgsemi^*$-regular space but not nano regular space.

Theorem 4.5. If $f:(U,N,I) \to (V,M,J)$ is nano continuous bijective, $nIgsemi^*$ -open function and U is a nano regular space then V is $nIgsemi^*$-regular.

Proof: Let F be a nano closed set in V and $y \notin F$. Let $y = f(x)$ for some $x \in U$. Since f is nano continuous, $f^{-1}(F)$ is nano closed in U such that $x \notin f^{-1}(F)$. Now U is nano regular space, there exist disjoint nano open sets G and H such that $x \in G$ and $f^{-1}(F) \subset H$. That is $y = f(x) \in f(G)$ and $F \subset f(H)$. Since f is $nIgsemi^*$-open function, $f(G)$ and $f(H)$ are $nIgsemi^*$-open sets in V. $f(G) \cap f(H) = f(G \cap H) = \emptyset$. Therefore V is $nIgsemi^*$- regular space.

Theorem 4.6 If $f:(U,N,I) \to (V,M,J)$ is $nIgsemi^*$- continuous, nano closed bijection and V is a nano regular space then U is $nIgsemi^*$-regular.

Proof: Let F be a nano closed set in U and $x \notin F$. Since f is nano closed injection, $f(F)$ is nano closed set in V such that $f(x) \notin f(F)$. Now V is nano regular, there exist disjoint nano open sets G and H such that $f(x) \in G$ and $f(F) \subset H$. Thus $x \in f^{-1}(G)$ and $F \subset f^{-1}(H)$. Since f is $nIgsemi^*$ continuous function, $f^{-1}(G)$ and $f^{-1}(H)$ are $nIgsemi^*$-open sets in U. Also $f^{-1}(G) \cap f^{-1}(H) = \varphi$. Hence U is $nIgsemi^*$-regular space.

Remark 4.7 If $f:(U,N,I) \to (V,M,J)$ is nano continuous, nano closed bijection and V is a nano regular space then U is $nIgsemi^*$-regular.

Theorem 4.8 If $f:(U,N,I) \to (V,M,J)$ is $nIgsemi^*$-irresolute, nano closed bijection and V is a $nIgsemi^*$-regular then U is also $nIgsemi^*$-regular.

Proof: Let F be a nano closed set in U and $x \notin F$. Since f is nano closed, $f(F)$ is nano closed set in V. But V is $nIgsemi^*$- regular, hence there exist disjoint $nIgsemi^*$ -open sets G and H in V such that $f(x) \in G$ and $f(F) \subset H$. This implies $x \in f^{-1}(G)$ and $F \subset f^{-1}(H)$, where $f^{-1}(G)$ and $f^{-1}(H)$ $nIgsemi^*$-open sets in U since f is

$nIgsemi^*$ irresolute. Also $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint sets. Hence U is $nIgsemi^*$-regular space.

Remark 4.9 If $f:(U,N,I) \to (V,M,J)$ is $nIgsemi^*$-irresolute, nano closed bijection and V is a nano-regular then U is also $nIgsemi^*$-regular.

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