THE CONNECTED OPEN DETOURMONO-PHONIC NUMBER OF A GRAPH

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Abstract*: For a graph G that has at least two vertices. If G[M] is connected, an opendetourmono-phonic set M is termed a connected open detourmono-phonic set. The lowest number of a connected open detourmono-phonic set of G is the connected open detourmono-phonic number $odm_c\ (G)$. The odm_c-set of G is any connected open detourmono-phonic set of order $odm_c\ (G)$. We determined the $odm_c\ (G)$ of certain standard graphs and achieved some findings in this work. For a connected graph G, it is demonstrated that There is an integer n such that $odm_c(G) = k$ for any pair of positive integers k,n with $3 \leq k \leq n$.*

Keywords: *mono-phonic number, open detourmono-phonic number, connected open detourmono-phonic number*

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INTRODUCTION

For a connected graph G that does not contain loops or numerous edges, m and n stand for the graph's $order$ and $size$, respectively. We direct the reader to [1] for graph theoretic terms. If the edges e connect the vertices u and v, then the vertices u and v are neighbours. $N(v)$ denotes the neighbourhood of v, the subgraph of $Ginduced$ by all vertices close to v.Thus $deg(v) = |N(v)|$.If $deg(v) = n-1$,then $v$is called a universal vertex.If the subgraph produced by $N(v)$ is complete, then v is a simplicial vertex. The no of edges in a shortest path is the distance between two vertices in a graph. If an edge links two non-neighbour vertices of a route P, the path has a chord. A mono-phonic route is one that has no chords. The mono-phonic distance $d_m(x,y)$is length of the longest $x-y$mono-phonic route. A $x-y$ $mono\ phonic\ route$ is defined as one with a length of $d_m(x,y)$ If each vertex v of G sits on a $x-y$mono-phonic path in G for any $x,y \in M$, the set M is a mono-phonic set of G. The minimum numberofmono-phonic set M of G is called the $mono\ phonic\ number\ m(G)$. The mono-phonicconceptsare studied in [2,4].

A set $M \subseteq V$ in G is an $open\ monophonic\ set$ if for each vertex v in G, either v is an internal vertex of an $x-y monophonic\ path$ for each $x,y \in S$ or v is an extreme vertex of G and $v \in M$, The open mono-phonic number om is the minimal cardinality of an open mono-phonic set of G. [6] looked into the possibility of an open mono-phonic number. If a mono-phonic set M such that $G[M]$ is connected, a set $M \subseteq v$ in a graph G called connected open mono-phonic set of G.The connectedmono-phonic number $m_c(G)$ is the least number of a connectedmono-phonic set of G. [2] explored the connectedmono-phonic number. An$u-v$detourmono-phonic path is the longest $u-v$mono-phonic path. If each vertex v is an internal vertex of $u-$vdetourmono-phonic path for some u and v in M, then the set $M \subseteq V$ of G is a detourmono-phonic set of G. The detour$mono\ phonic\ number\ dm(G)$ is the minimal cardinality of a $detour\ mono\ phonic\ set$ of G. [7] looked at the detourmono-phonic notions.

If $J_{dm}(M) = V$, a set $M \subseteq V$ is termed an $open\ detour\ mono\ phonic\ set$ of G. The odm set of G has the smallest cardinality (G). [3] looked at open detourmono-phonic notions. If v is

not an interior vertex of any $x-y$detourmono-phonic path for any $x, y \in V$, it is said to be a detourmono-phonicsimplicial vertex of G.
Definition 1.1 [5] A branch G at v is a cut vertex v in a connected$G$ and the subgraph H of $G - v$, as well as the vertex v combined with all edges joining v and $V(H)$.

THE CONNECTED OPEN DETOURMONO-PHONIC NO.OF A GRAPH
Definition 2.1.*An open detour mono-phonic set* M is called a *connected open detour mono-phonic set* if $G[M]$ is *connected*.Theconnected open detourmono-phonic number $odm_c(G)$is *the minimum cardinality* of a connected open detour*mono* − *phonic set of G*. Any open detour*mono-phonic* set of order $odm_c(G)$is called odm_c-set of *G*.
Example 2.2. In Figure 2.1, $M = \{v_1, v_2, v_9, v_{10}\}$ is a odm-set of G *which is not connected*$M_1 = \{v_1, v_4, v_5, v_6, \ v_9, v_{10}\}$ is a odm_c -set of $G odm_c(G) = 6$.

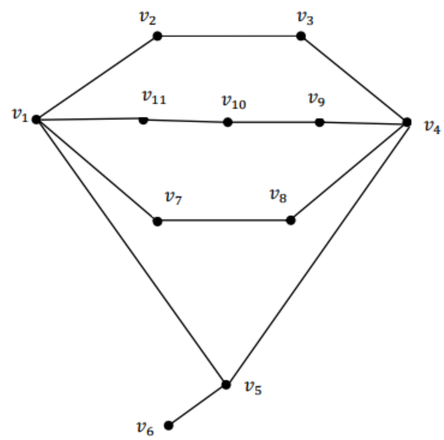

Figure 2.1

Observation 2.3. Let G be a connected graph with $n \geq 2$.
(i) Eachodm_c- set inG containsdetourmono-phonicsimplicial vertex of *G*
(ii) $2 \leq dm(G) \leq odm(G) \leq odm_c(G) \leq n$.
(iii) Every subgraphof $G - v$contains anelement of*M*, *ifv* is a cut-vertex and *M*is a odm_c-set of *G*.
(iv) If*G*is of order $n \geq 4$ and*G*does notcontains detourmono-phonicsimplicial vertices, then $odm(G) \geq 3$.
(v) For any graph G with exactly one universal vertex, say x. If $d_m(G - x) \geq 3$,then $x$is a detourmono-phonicsimplicial vertex of *G*.
(vi) Let G be a complete graph, $odm_c(G) = n$.
Theorem 2.4. For a connected graph G. Every minimum odm_c-set G contains each cut vertex of G.
Proof.Let *M*be a minimum odm_c- set of *G*and $v$be a cut-vertex of *G*. Let $G_1, G_2, \ldots, G_i, i \geq 2$ be the components of $G - v$. By Observation 2.3(iii) M contains at least one vertex from each G_i. We prove that $v \in M$. On the contrary $v \notin M$. Then *G[M]*is disconnected, which is aimpossible. Therefore $v \in M$.
Corollary 2.5. For any connected graph G of order n. If *every vertex* of G is either a detourmono-phonic*simplicial vertex* or *a cut* − *vertex of G*. Then $odm_c(G)$= n.
Proof.This comes from Observation 2.3(i) and Theorem 2.4.
Remark 2.6.The reverse part of the corollary 2.5 need not be true. A graph illustrating the failure of converse of Corollary 2.5. In Figure 2.2, $M = \{v_1, v_2, v_3, v_4, v_5\}$ is the only minimum odm_c-set of *G*,$odm_c(G) = 5 = n$.

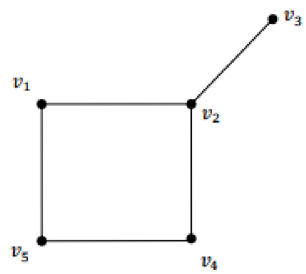

Figure 2.2

The Connected Open detourmono-Phonic Numberof Some Standard Graphs

Theorem 3.1. If $G = C_n$ is a cycle on n-vertices, then

$$odm_c(G) = \begin{cases} 3 \ if n = 3 \\ 4 \ if n = 4,5 \\ 5 \ if n = 6 \\ 6 \ if \ n \geq 7 \end{cases}$$

Proof. Let $G = C_n, n \geq 3$. For $G = C_3 = K_3$, by Observation 2.3(iv) $odm_c(G) = 3$. For $G = C_4$or C_5. Since Ghas no detourmono-phonicsimplicial vertices, by Observation 2.3(iv), $odm(G) \geq 3$. Hence by Observation 2.3(ii) $odm_c(G) \geq 3$.Clearly,subsets of vertices $V(G)$ *with* three element is not an odm_c-set of G. Let M= $\{v_1, v_2, v_3, v_4\}$is an odm_c-set of Gso that $odm_c(G) = 4$.For $G = C_n, n \geq 6$. It is clear that there is three element or 4 element which is subset of $V(G)$ is not an odm_c- set of G. For n=6 $M_1 = \{v_1, v_2, v_3, v_4, v_5\}$ is an odm_c-set of G. Therefore$odm_c(G) = 5$.For $n \geq 7$,since $v_3 \notin J_{dm}(M_1)$,M_1 is not an odm_c-set of G. Therefore $M_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is anodm_c −set of G, $odm_c(G) = 6$.

Theorem 3.2. If $G = K_{r,s}$ is a complete bipartite graph $(2 \leq r \leq s)$, then $odm_c(G) = 4$.

Proof.Let $K_{r,s}(2 \leq r \leq s)$. Let $X = \{x_1, x_2, .., x_r\}$ and $Y = \{y_1, y_2, .., y_s\} \in G$. Since Ghas no detourmono-phonicsimplicial vertices, by Observation 2.3(ii) and (iv) $odm_c(G) \geq 3$. Clearly, three element subset of $V(G)$ is not an odm_c −set of G, $odm_c(G) \geq 4$. Let $M = \{x, y, u, v\}$, where $x, y \in X$and $u, v \in Y$. Then $M$is a odm_c − set of Gso that $odm_c(G) = 4$.

Theorem 3.3. If $G = W_n = K_1 + C_{n-1}$ is the wheel graph $(n \geq 4)$, then $odm_c(G) = \begin{cases} 4 \ \ if n = 4,5 \\ 5 \ \ if n = 6 \\ 6 \ \ if n \geq 7 \end{cases}$

Proof.Let $V(K_1) = x$and $V(C_{n-1}) = \{v_1, v_2, .., v_{n-1}\}$. If$n = 4$, then $G = K_4$, by Observation 2.3 (vi), $odm_c(G) = 4$.

For $n = 5$, it can be easily seen that $M = \{v_1, v_2, v_3, v_4\}$ is anodm_c-set of G, $odm_c(G) = 4$. Let $n \geq 6$. Since $d_m(G - x) \geq 3$, by Observation 2.3(v), x is a *detour mono-phonic simplicial vertex* of $G$and so x contains every odm_c-set of G. Let $M_1 = \{x, v_1, v_2, v_3, v_4\}$ is a odm_c-set of G, $odm_c(G) \geq 5$. Let$n \geq 7$, every odm_c- of Gcontains minimum five vertices from $V(C_{n-1})$ and so $odm_c(G) \geq 6$. Let $M_2 = \{x, v_1, v_3, v_{n-2}, v_n\}$ is an odm_c-set of G, $odm_c(G) = 5$.

Theorem 3.4.If $G = L_n = K_2 \times P_n$ is the ladder graph $(n \geq 3)$, then $odm_c(G) = n + 2$.

Proof. Let$V(G) = \{v_{11}, v_{21}, .., v_{n1}\} \cup \{v_{12}, v_{22}, .., v_{n2}\}$. Then $M = \{v_{11}, v_{21}, ..., v_{n1}\} \cup \{v_{12}, v_{n2}\}$ is a odm_c-set of G,$odm_c(G) \leq n + 2$. To showthat $odm_c(G) = n + 2$. Suppose, assume that $odm_c(G) \leq n + 1$. Then there is an odm_c-set M′ such that $|M'| \leq n + 1$. Let $z \in M'$and $z \notin M$. If $M' \subset M$, then $G[M']$ so that $z \notin M$. If $M' \subset M$, then $G[M']$is not connected. Therefore $M' \nsubseteq M$. Since$|M'| \leq n + 1$, $M'$is not an odm_c- set of G, which is impossible. Hence$odm_c(G) = n + 2$.

Theorem 3.5. If $G = H_n = K_1 \circ W_n$is the helm graph $(n \geq 3)$, then$odm_c(G) = \begin{cases} 2n if n = 3,4,5,6 \\ 2n + 1 \ if n \geq 7 \end{cases}$

Proof. Let H_n contains a central vertex,say x. For $n = 3,4,5,6$. It can be easily proved that $odm_c(G) = 2n$. So let $n \geq 7$.Let the set S be cut vertices and end vertices of $G$and Mbe an odm_c- set of G. Then $S \subseteq M$. Since $x \notin J_{dm}(M)$, $M$is not an odm_c- set of $G$and so $odm_c(G) \leq 2n+1$. Hence$M = V(G)$is the unique odm_c-set of Gsuch that $odm_c(G) = 2n+1$.

Theorem 3.6.If $G = F_n = K_1 + P_{n-1}$ is the fan graph $(n \geq 5)$, then $odm_c(G) = 3$.

Proof. Let $V(K_1) = x$and $V(P_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$. Then $M = \{x, v_1, v_{n-1}\}$ is a set of all detourmono-phonicsimplicial vertices of G and so $odm_c(G) \geq 3$. Since $G[M]$is connected. $M$is an odm_c- set of G and$odm_c(G) = 3$.

Theorem 3.7. For a connected graph G. Let k, n be positive integers with $3 \leq k \leq n$, then there is an integer n such that $odm_c(G) = k$.

Proof. Case(i) Suppose $k = 3$. Let $V(P) = \{x, y, z\}$. Let P be a path on 3 vertices. Let G beconstructed from $P$by introducing new vertices $u_1, u_2, \dots, u_{n-3}$ and join x, y and z with each $u, (1 \leq i \leq n-3)$ and introduce the edge $u_{i-1}u_i$,
$(1 \leq i \leq n-3)$and isshown in Figure 3.1.
Let $Y = \{x, z\}$ be the detour*mono-phonic simplicial* vertices of G, then Y $\subseteq odm_c$-set of Gsuch that G[Y] is disconnected and so $odm(G) \geq 3$. Hence$M = \{x, y, z\}$is *minimum* odm_c-set of G,$odm_c(G) = 3$.

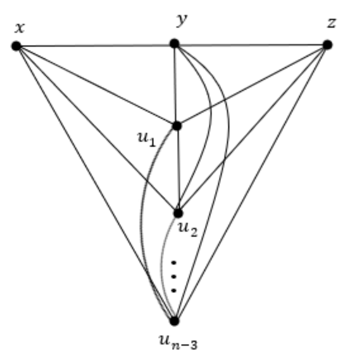

Figure 3.1

Case(ii)Suppose$k = 4$. Let $P: u, v, w$be a path on 3 vertices. Let the graph G canbe constructed from path P by introducing new vertices $v_1, v_2, \dots, v_{n-3}$ and join $u$and $w$with each $v_i, (1 \leq i \leq n-3)$and introduce the edge $v_{i-1}v_i$ andisgiven in Figure 3.2.

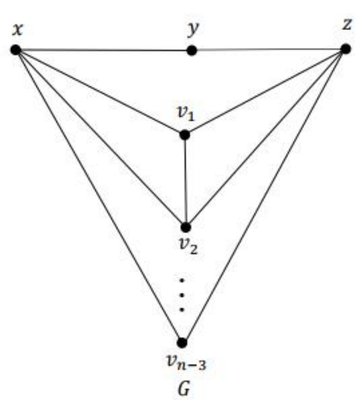

Figure 3.2

*G*has no detourmono-phonicsimplicial vertices and so by Observaton 2.3(ii) $odm_c(G) \geq 3$. Clearly G has no three element $subset$ of V(G)is a odm_c-set of G. It is easily verified that $M = \{u, v, w, v_i\}, (1 \leq i \leq n-3)$is a minimum$odm_c$- set of *G*, $odm_c(G) = 4$.
Case (iii) Suppose $k \geq 5$. Let $V(K_1) = \{y, z\}$and $V(K_{n-k+2}) = \{x_1, x_2, \ldots x_{n-k+2}\}$. Let the graph *G*constructed from $K_2$and $K_{n-k+2}$and introducing the vertices $y_1, y_2, \ldots, y_{k-4}$ and join z with each $y_i, (i = 1, 2, 3, \ldots k-4)$ and $x_j (j = 1, 2, 3, \ldots n-k+2)$and theedges yx_j are introduced which is shown in Figure 3.3.

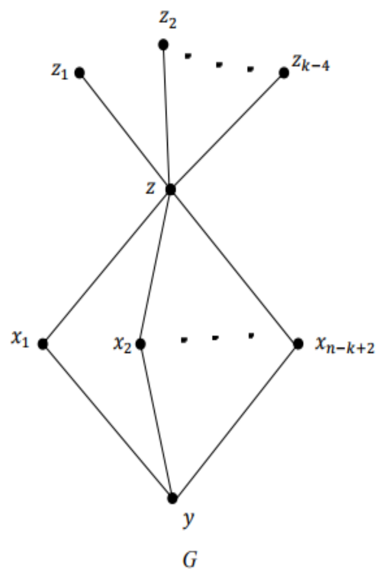

Figure 3.3

The set of all detour mono-phonic simpliicial vertex of G is $Y = \{y_1, y_2, \ldots, y_{k-4}\}$ Because Y ⊆ of every connected open detour mono-phonic set of G. Y is not an odm_c-set of G since $J_{dm}(Y) \neq V$. $Y \cup X$where$|X| \leq 3$ is not a connected open $\text{detourmono} - \text{phonic}$ set of G, therefore $odm_c(G) \geq k$. Allow $M = Y \cup \{y, z, x_1, x_2\}$. M is therefore an odm_c-set of *G*with$odm_c(G) = k$.

Theorem 3.8. Let v be a $cut - vertex$ and n the order of connected graph G.
(i) If each branch of $G - v$is complete, then $odm_c(G) = n$.
(ii) If r branches of *G*is a copy of $K_p - \{e\}$, *p*= 4 and the remaining branches are complete, then $odmc(G) = n - r$.

Proof. If $v$is a cut-vertex of*G*.Then by Theorem 2.4, $v$belongs to each odm_c-set of *G*.
(i) If $G - v$is complete, $V(G) - \{v\}$is the set of all detourmono-phonicsimplicial vertices of G. Therefore *M* = *V(G)*is the unique odm_c -set of *G*, $odm_c(G) = n$.
(ii) Let $K_{P_i} - \{e\}(1 \leq i \leq r)$be a copy of $K_p - \{e\}$and $e = x_i y_i$. Let $x_i, z_i y_i$ $(1 \leq i \leq r)$be a detourmono-phonic path in $K_p - \{e\}$. Then the set $M = V(G) - \{z_1, z_2, \ldots, z_r\}$ is thedetour mono-phonic simplicial vertices of G. SinceM is a subset of allodm_c-set of *G*so that $odm_c(G) \geq n - r$.Since M is anodm_c-set of *G*, $odmc(G) = n - r$.

Theorem 3.9. Let *G* be a connected graph.For every positive integers *n* ≥ *4*, there exists an integer n with $odm(G) = odm_c(G) = 4$.

Proof.Let $K_2 = \{x, y\}$and $K_{n-2} = \{v_1, v_2, \ldots, v_{n-2}\}$.Let $G = K_2 + K_{n-2}$. Since G has no detourmono-phonicsimplicialvertices $odm_c(G) \geq 3$. Clearly no three element subset of vertices of Gis a odm-set of *G*. It is verified that $M = \{x, y, v_1, v_2\}$is a $minimumodm$-set of *G* and so $odm(G) = 4$. Since *G[M]*is connected, $odm_c(G) = 4$.

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