SOME CHARACTERIZATIONS OF TOTAL OUTER-CONNECTED DOMINATION IN GRAPHS

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Article History: Received: 14.05.2023 **Revised:** 18.06.2023 **Accepted:** 29.07.2023

Abstract

Let $G = (V(G), E(G))$ be a connected graph. A nonempty set $S \subseteq V(G)$ is a *total outer- connected dominating set* of G if $S = V(G)$ or S is a total dominating set of G and the subgraph $\langle V(G) \setminus S \rangle$ induced by $V(G) \setminus S$ is connected. The cardinality of the smallest total outer-connected dominating set of G is called *total outer-connected domination number* of G, denoted by $\tilde{\gamma}_{tc}(G)$. A total outer-connected domination number of G with cardinality equal to $\tilde{\gamma}_{tc}(G)$ is called $\tilde{\gamma}_{tc}$-*set* of G. This paper presents the characterization of total outer-connected dominating sets in a connected nontrivial graph. It also investigates the total outer-connected dominating sets in the join of two graphs and determine the corresponding value of the total outer-connected domination number.

Keywords: Total, Outer-connected, Domination, Graph, Join.

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DOI: 10.31838/ecb/2023.12.s3.787

1. Introduction

Many real life situations can be more easily described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. For example, the points could represent people, with lines joining pairs of friends, or the points might be communication centers, with lines representing communication links. A mathematical abstraction of situations of this type gives rise to the concept of a graph.

Graph theory is defined as the mathematical theory of the properties and applications of graphs. This theory was said to have its beginnings in 1736 when Euler settled a famous unsolved problem of his day called the Königsberg Bridge Problem.

One of the flourishing concepts in the field of graph theory and applied mathematics is domination. In 1958, Berge first introduced the term domination number of a graph. Moreover, the terms dominating set and domination number were used by Ore in 1962. Cockayne and Hedetniemi (1975) have made interesting and extensive analysis on the results of dominating sets in graphs.

A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G)\setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The *domination number* of G denoted by $\gamma(G)$ is the cardinality of the smallest dominating set of G. A dominating set of G with cardinality equal to $\gamma(G)$ is called a γ-*set* of G. The applications of these concepts in solving real life problems motivate the graph theorists and the researchers to extensively study the other variants of domination.

Two particular variants of domination are the total domination and outer-connected domination. The concept of total domination was introduced by Cockayne, Dawes and Hedetniemi [4]. This was further studied by Atapour and Soltankhah in [3] where characterizations of the bounds of total domination numbers were established. The concept of outer-connected domination was introduced by Cyman in [5]. Pandey studied the algorithm and hardness results for outer-connected dominating set in graphs.

Let G be a connected graph. A subset S of $V(G)$ is a *total dominating set* of G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a total dominating set in G, denoted by $\gamma_t(G)$ is the total domination number in G. A total dominating set with cardinality $\gamma_t(G)$ is called a γ_t-*set* in G. A nonempty set $S \subseteq V(G)$ is an *outer-connected dominating set* of G if $S = V(G)$ or S is a dominating set of G and the subgraph induced by $V(G) \setminus S$ is connected. The cardinality of the smallest outer-connected dominating set of G is called *outer-connected domination number* of G denoted by $\tilde{\gamma}_c(G)$. An outer-connected domination number of G with cardinality equal to $\tilde{\gamma}_c(G)$ is called $\tilde{\gamma}_c$-set of G.

The concepts mentioned gave an idea to the researcher to investigate further the concept of outer-connected domination and come up with the new variant of domination called total outer-connected domination. A nonempty set $S \subseteq V(G)$ is a *total outer-connected dominating set* of G if $S = V(G)$ or S is a total dominating set of G and the subgraph $\langle V(G) \setminus S\rangle$ induced by $V(G) \setminus S$ is connected. The cardinality of the smallest total outer- connected dominating set of G is called *total outer-connected domination number* of G, denoted by $\tilde{\gamma}_{tc}(G)$. A total outer-connected domination number of G with cardinality equal to $\tilde{\gamma}_{tc}(G)$ is called $\tilde{\gamma}_{tc}$-*set* of G.

Example 2.1 Figure 1 shows that the sets $S_1 = \{b, c, d\}$ and $S_2 = \{a, b, d, e\}$ are dominating sets of P_5. The subgraph $\langle V(P_5) \setminus S_1\rangle$ is not connected while $\langle V(P_5) \setminus S_2\rangle$ is connected. Hence, S_1 is not a total outer-connected dominating set of P_5 while S_2 is a total outer-connected dominating set of P_5. Moreover, S_1 is the smallest total dominating set of P_5 and S_2 is the smallest total outer-connected dominating set of P_5. Thus, $\gamma_t(P_5) = 3 < 4 = \tilde{\gamma}_{tc}(P_5)$.

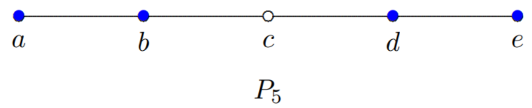

Figure 1: The path P_5 with a total outer-connected dominating set

Furthermore, the researcher investigated some properties of this new concept especially in the join of two graphs.

Statement of the Problem

The study investigated the concept of total outer-connected domination in graphs and characterized this concept in the join of two graphs.

Objectives of the Study

The study established the following specific objectives:

1. Determined the bounds or exact values of total outer-connected domination number.
2. Characterized those graphs which attained the bounds or exact values of total outer-connected domination number.
3. Characterized the total outer-connected dominating sets in the join of two graphs.
4. Determined the exact values or bounds of the total outer-connected domination numbers of the join of two graphs.

2. Results and Proofs

The study investigated the concept of total-outer connected domination in graphs and characterized this concept in the join of two graphs. This study determined the bounds or exact values of total outer-connected domination number; characterized those graphs which attained the bounds or exact values of total outer-connected domination number; characterized the total outer-connected dominating sets in the join of two graphs; and determined the exact values or bounds of the total outer-connected domination numbers of the join of two graphs. Finally, the main results obtained in this study are summarized as follows:

Remark 5.1 *Let G be a connected nontrivial graph of order* $m \geq 3$. *Then* $2 \leq \tilde{\gamma}_{tc}(G) \leq m-1$
and the bounds are sharp.

Consider the graphs K_4 and $K_{1,m-1}$ in Figure 2.

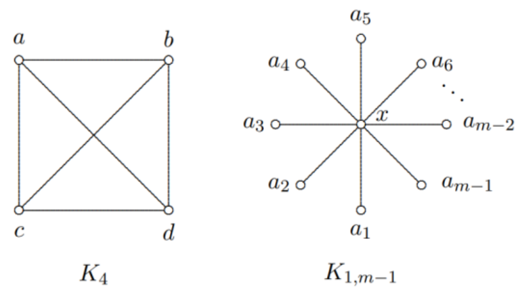

Figure 2: Graphs K_4 and $K_{1,m-1}$

Let $D_1 = \{b, c\}$ and $D_2 = \{x, a_1, a_2, ..., a_{m-2}\}$ be total dominating sets of K_4 and $K_{1,m-1}$, respectively. Observe that $(V(K_4) \setminus D_1)$ and $(V(K_{1,m-1}) \setminus D_2) = (\{a_{m-1}\})$ are connected subgraphs of K_4 and $K_{1,m-1}$, respectively. Hence, it follows that D_1 and D_2 are total outer-connected dominating sets in K_4 and $K_{1,m-1}$, respectively. Since $|D_1| = 2$, it follows that $\tilde{\gamma}_{tc}(K_4) = 2$. Observe also that D_2 is the smallest total outer-connected dominating set of $K_{1,m-1}$. Thus $\tilde{\gamma}_{tc}(K_{1,m-1}) = m-1$.

Theorem 5.2 *For any connected nontrivial graph G of order* $n \geq 3$, $\tilde{\gamma}_{tc}(G) = 2$ *if and only if there exists two adjacent vertices, a and b, that dominates G and* $(V(G) \setminus \{a, b\})$ *is connected.*

Proof: Let G be a connected nontrivial graph of order $n \geq 3$. Suppose that $\gamma_{tc}(G) = 2$. Then G has a total outer-connected dominating set of G say, S, with $|S| = 2$. Let $S = \{a, b\}$ for some $a, b \in V(G)$. Then $ab \in E(G)$ and these two vertices dominate G. Since S is an outer-connected dominating set of G, $(V(G) \setminus \{a, b\})$ is connected.

Conversely, suppose that there exists two adjacent vertices a and b that dominate G and $(V(G) \setminus \{a, b\})$ is connected. Let $S = \{a, b\}$. Then S is a total outer-connected dominating set of G. By Remark 5.1, $\tilde{\gamma}_{tc}(G) = |S| = 2$. ■

Corollary 5.3 *Let K_n be a complete graph of order $n \geq 3$. Then $\tilde{\gamma}_{tc}(K_n) = 2$.*

Proof: Let K_n be a complete graph of order $n \geq 3$. Let $S = \{a, b\} \subseteq V(K_n)$. Then $ab \in E(G)$ and these vertices dominate G. Moreover, since the graph is complete, $(V(G) \setminus \{a, b\})$ is connected. By Theorem 5.2, $\tilde{\gamma}_{tc}(K_n) = |S| = 2$. ■

Theorem 5.4 *Let G and H be connected nontrivial graphs. Then $S \subseteq V(G + H)$ is a total outer-connected dominating set of $G + H$ if and only if one of the following holds:*

(i) *S is a total dominating set of G;*

(ii) *S is a total dominating set of H;*

(iii) *$S = V(G + H)$ or $S = D_1 \cup D_2 \subset V(G + H)$ such that if $D_1 = V(G)$, then $(V(H) \setminus D_2)$ is connected and if $D_2 = V(H)$, then $(V(G) \setminus D_1)$ is connected.*

Proof: Let G and H be connected nontrivial graphs. Assume that S is a total outer-connected dominating set of $G + H$. Suppose that $S \subseteq V(G)$. Since S is a total dominating set of $G + H$, it follows that S is a total dominating set of G. Then, (i) holds.

Similarly, if $S \subseteq V(H)$, then S is a total dominating set of H. Hence, (ii) holds. Suppose $S = V(G + H)$. Then we are done.

Suppose that $S = D_1 \cup D_2 \subset V(G + H)$ where $\emptyset \neq D_1 \subseteq V(G)$ and $\emptyset \neq D_2 \subseteq V(H)$. Suppose further that $D_1 = V(G)$. Since $D_2 \neq \emptyset$ and S is an outer-connected dominating set of $G + H$, $(V(G + H) \setminus S) = (V(H) \setminus D_2)$ is connected. Similarly, if $D_2 = V(H)$, then $(V(G) \setminus D_1)$ is connected. Hence, (iii) holds.

For the converse, suppose first that S is a total dominating set of G. Then by definition of the join of graphs, S is a total dominating set of $G + H$. Since $(V(G + H) \setminus S) = ((V(G) \setminus S) \cup V(H))$ is connected, S is a total outer-connected dominating set of $G + H$. Similarly, if S is a total dominating set of H, then S is a total outer-connected dominating set of $G + H$.

Now, suppose that (iii) holds. If $S = V(G + H)$, then S is a total outer-connected dominating set of $G + H$. Suppose that $S = D_1 \cup D_2 \subset V(G + H)$.

Now, consider the following cases:

Case 1. $1 \leq |D_1| < |V(G)|$ and $1 \leq |D_2| < |V(H)|$
Then by definition of the join of graphs, $S = D_1 \cup D_2$ is a total dominating set of $G + H$. Moreover, $(V(G + H) \setminus S) = (V(G) \setminus D_1) \cup (V(H) \setminus D_2)$ is connected. Hence, S is a total outer-connected dominating set of $G + H$.

Case 2. $|D_1| = |V(G)|$ and $1 \leq |D_2| < |V(H)|$
Then $S = V(G) \cup D_2$ is a total dominating set of $G + H$ since $(V(G + H) \setminus S) = (V(H) \setminus D_2)$ is connected. Hence, S is a total outer-connected dominating set of $G + H$.
Case 3. $1 \leq |D_1| < |V(G)|$ and $|D_2| = |V(H)|$
Then by definition of the join of graphs, $S = D_1 \cup V(H)$ is a total dominating set of $G + H$. Since $(V(G + H) \setminus S) = (V(G) \setminus D_1)$ is connected, S is a total outer-connected dominating set of $G + H$. ■

Corollary 5.5 *For every connected nontrivial graphs G and H, $\tilde{\gamma}_{tc}(G + H) = 2$.*

Proof: Let G and H be connected nontrivial graphs.

Consider the following cases:
Case 1. $\gamma_t(G) = 2$
Let $D_1 = \{a, b\}$ be a γ_t*-set* of G for some $a, b \in V(G)$. Then by Theorem 5.4 (i), D_1 is a total outer-connected dominating set of $G + H$. Since $2 \leq \tilde{\gamma}_{tc}(G + H) \leq |D_1| = 2$, $\tilde{\gamma}_{tc}(G + H) = 2$.
Case 2. $\gamma_t(H) = 2$
Let $D_2 = \{x, y\}$ be a $\gamma_t -$ *set* of H for some $x, y \in V(H)$. Then by Theorem 5.4 (ii), D_2 is a total outer-connected dominating set of $G + H$. Since $2 \leq \tilde{\gamma}_{tc}(G + H) \leq |D_2| = 2$, $\tilde{\gamma}_{tc}(G + H) = 2$.

Case 3. $\gamma_t(G) \neq 2$ and $\gamma_t(H) \neq 2$
Let $S = \{a, b\}$ for some $a \in V(G)$ and $b \in V(H)$. By Theorem 5.4 (*iii*), S is a total outer-connected dominating set of $G + H$. Since $2 \leq \tilde{\gamma}_{tc}(G + H) \leq |S| = 2$, $\tilde{\gamma}_{tc}(G + H) = 2$. ■

Theorem 5.6 *Let H be a connected nontrivial graph. Then $S \subseteq V(K_1 + H)$ is a total outer-connected dominating set of $K_1 + H$ if and only if one of the following holds:*

(i) *S is a total dominating set in H;*
(ii) *$S = V(K_1 + H)$ or $S = V(K_1) \cup D$ where $D \subset V(H)$ and $(V(H) \setminus D)$ is connected.*

Proof: Let H be a connected nontrivial graph. Suppose that S is a total outer-connected dominating set of $K_1 + H$. Suppose that $S \subsetneq V(H)$. Since S is a total dominating set of $K_1 + H$, it follows that S is a total dominating set in H. Hence, (*i*) holds. If $S = V(K_1 + H)$, then we are done. Suppose that $S = V(K_1) \cup D$ where $D \subset V(H)$. Since S is an outer-connected dominating set of $K_1 + H$, $(V(K_1 + H) \setminus S) = (V(H) \setminus D)$ is connected. Thus, (*ii*) holds.
For the converse, suppose first that S is a total dominating set of H. By definition of the join of graphs, S is a total dominating set of $K_1 + H$. Also, $(V(K_1 + H) \setminus S) = K_1 + (V(H) \setminus S)$ is connected. Hence, S is a total outer-connected dominating set of $K_1 + H$.

Suppose that (*ii*) holds. If $S = V(K_1 + H)$, then S is a total outer-connected dominating set of $K_1 + H$. Suppose $S = V(K_1) \cup D$ where $D \subset V(H)$ and $(V(H) \setminus D) = (V(K_1 + H) \setminus S)$ is connected. Then, S is a total outer-connected dominating set of $K_1 + H$. ■

Corollary 5.7 *Let H be a connected nontrivial graph. Then $\gamma_{tc}(K_1 + H) = 2$.*
Proof: Suppose that $S = \{x, y\}$ is a γ_t−*set* of H for some $x, y \in V(H)$. By Theorem 5.6 (*i*), S is a total outer-connected dominating set of $K_1 + H$. Hence, $\tilde{\gamma}_{tc}(K_1 + H) \leq |S| = 2$. By Remark 5.1, $2 \leq \tilde{\gamma}_{tc}(K_1 + H) \leq 2$. Hence, $\tilde{\gamma}_{tc}(K_1 + H) = 2$.

Suppose that $S = V(K_1) \cup \{a\}$ where $(V(H) \setminus \{a\})$ is connected. By Theorem 5.6 (*ii*), S is a total outer-connected dominating set of $K_1 + H$. By Remark 5.1, $2 \leq \tilde{\gamma}_{tc}(K_1 + H) \leq |S| = 2$. Hence, $\tilde{\gamma}_{tc}(K_1 + H) = 2$. ■

Theorem 5.8 *Let $m \geq 2$ and $n \geq 2$ be integers. Then $S \subseteq V(K_{m,n})$ is a total outer-connected dominating set of $K_{m,n}$ if and only if one of the following holds:*

(i) *$S = V(K_{m,n})$; or*
(ii) *$S = D_1 \cup D_2$ ($\subset V(K_{m,n})$) such that if $D_1 = V(\overline{K}_m)$, then $\langle V(\overline{K}_n)\setminus D_2\rangle$ is a singleton subgraph of $\overline{K}_n$ and if $D_2 = V(\overline{K}_n)$, then $\langle V(\overline{K}_m)\setminus D_1\rangle$ is a singleton subgraph of $\overline{K}_m$.*

Proof: Let $m \geq 2$ and $n \geq 2$ be integers. Suppose that S is a total outer-connected dominating set of $K_{m,n}$. If $S = V(K_{m,n})$, then (*i*) holds. Suppose $S \subset V(K_{m,n})$. Then $S = D_1 \cup D_2$ where $D_1 \subseteq V(\overline{K}_m)$ and $D_2 \subseteq V(\overline{K}_n)$. Since S is a total outer-connected dominating set, $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. Suppose $D_1 = V(\overline{K}_m)$. Since $\overline{K}_m$ is an empty graph and S is outer-connected, it follows that $\langle V(\overline{K}_n)\setminus D_2\rangle$ is connected. This implies that $\langle V(\overline{K}_n)\setminus D_2\rangle$ is a singleton subgraph of $\overline{K}_n$. Similarly, if $D_2 = V(\overline{K}_n)$, $\langle V(\overline{K}_m)\setminus D_1\rangle$ is a singleton subgraph of $\overline{K}_m$.

For the converse, suppose that (*i*) holds. Then $S = V(K_{m,n})$ is a total outer-connected dominating set of $K_{m,n}$. Suppose that (*ii*) holds. Then $S = D_1 \cup D_2 \subset V(K_{m,n})$ is a total dominating set of $K_{m,n}$.

Consider the following cases:

Case 1. $1 \leq |D_1| < |V(\overline{K}_m)|$ and $1 \leq |D_2| < |V(\overline{K}_n)|$
Then $(V(K_{m,n}) \setminus S)$ is connected. Hence, S is a total outer-connected dominating set of $K_{m,n}$.

Case 2. $|D_1| = V(\overline{K}_m)$
By (*ii*), $\langle V(\overline{K}_n)\setminus D_2\rangle$ is a singleton subgraph of $\overline{K}_n$ and hence, a connected subgraph of $\overline{K}_n$. Thus, $S = V(\overline{K}_m) \cup D_2$ is a total outer-connected dominating set of $K_{m,n}$.

Case 3. $|D_2| = V(\overline{K}_n)$
By (*ii*), $\langle V(\overline{K}_m)\setminus D_1\rangle$ is a singleton subgraph of $\overline{K}_m$ and hence, a connected subgraph of $\overline{K}_m$. Thus, $S = D_1 \cup V(\overline{K}_n)$ is a total outer-connected dominating set of $K_{m,n}$. ■

Corollary 5.9 *Let* $m \geq 2$ *and* $n \geq 2$ *be integers. Then* $\tilde{\gamma}_{tc}(K_{m,n}) = 2$.

Proof: Let $m \geq 2$ and $n \geq 2$ be integers. Suppose $S = \{a, z\}$ where $a \in V(\overline{K}_m)$ and $z \in V(\overline{K}_m)$. By Theorem 5.8 (*ii*), S is a total outer-connected dominating set of $K_{m,n}$. By Remark 5.1, $2 \leq \tilde{\gamma}_{tc}(K_{m,n}) \leq |S| = 2$. Therefore, $\tilde{\gamma}_{tc}(K_{m,n}) = 2$. ■

Recommendations

The study recommends some of the following interesting problems:

1. Characterize the existence of connected graphs which have a total outer-connected dominating set.
2. Characterize the total outer-connected dominating sets in the corona and composition of graphs and 3.
3. determine the bounds or exact values of the corresponding parameters.

Show that every pair of two positive integers is realizable as total outer-connected domination number.

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