Fuzzy Real Numbers Induced by Multiset Sequences

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Abstract: The aim of this paper is to introduce a new type of a fuzzy number that is different from the existing generalizations of fuzzy numbers. The significance of the proposed fuzzy number is to link multiset sequence with fuzzy sets in the set of all real numbers. The main advantage of the proposed fuzzy number is that it is associated with more than one increasing function and more than one decreasing function whereas all the existing generalizations of fuzzy numbers are having exactly one increasing function and one decreasing function. In certain real life problems, especially in trading, weather forecasting, medical field, share market, the values of the parameters may have several ups and downs over a specific period and they may not be modelled by the existing fuzzy numbers. The proposed fuzzy number using multiset sequence can be used to represent such situations. To validate this, several numerical examples are given.

Keywords: Fuzzy set, Fuzzy number, Multiset, Multiset sequence, Partition.

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1. Introduction

Theory of multisets was developed by Lake [16], Hickman [12] and Blizard [4]. According to them, a multiset is an unordered collection of elements that may have repeated occurrences of identical elements. Multisets have applications in certain areas of mathematics, computer science, physics, chemistry and philosophy. Singh et al.[21] proposed a systemized representation of multisets and studied the basic multiset operations with overview of the applications of multisets in mathematics and computer science. Jurgensen [14] formulated multisets, based on sets and multiplicities of their elements. Several researchers [1, 2, 3, 6,9,10,11,13, 17,19,20] studied multisets in association with other areas of mathematics namely, soft sets, filters, partially ordered sets, topology, fuzzy sets, relations, functions, modules, groups, separation axioms. As multiset theory is associated with several branches of mathematics, our aim is to link multisets with fuzzy numbers.

In the year 1965, Zadeh [22] initiated the notion of a fuzzy set, using which Chang, Zadeh [5] and Dubois, Prade [7] introduced and studied fuzzy numbers. Following this, Zhang, Guo and Chen [23] proposed generalized version of fuzzy numbers. Recently Patra [18] introduced a new technique for ranking of generalized trapezoidal fuzzy numbers. Quite recently Edvin Antony Raj et.al. [8] gave a novel approach to Arithmetic operations on Trapezoidal fuzzy numbers to optimize the transportation cost. In this paper, Zadeh's fuzzy set is linked with multisets to introduce a new type of fuzzy number, known as a multiset fuzzy real number. This approach will give another generalization of a fuzzy number which is contributed by multiset sequences.

In this paper, a fuzzy set in the set of all real numbers is known as real fuzzy set or fuzzy real set. It is obvious that all existing generalized fuzzy numbers have exactly one maximum and one minimum values which are attained in its support set. But a real fuzzy set may have more than one minimum and maximum values. The local minima and the local maxima of the membership function of a real fuzzy set are used to define a new type of a fuzzy number. Clearly local minima and local maxima of the membership function of a real fuzzy set form a sequence in which values of the terms may be repeated. Such sequences are called multiset sequences and the real numbers at which the membership function attains local maxima and local minima form a partition of the support of the underlying real fuzzy set. These two notions namely multiset sequence and its associated partition are used to introduce a new type of fuzzy number which we call multiset fuzzy real number. In this paper, the concept of a multiset fuzzy real number is introduced and its basic properties with examples are discussed.

This paper is described as follows: after the abstract and introduction, some basic concepts and results in fuzzy set theory and multiset theory are given in Sect.2. In Sect.3, multiset sequences are defined and described graphically. The notion of a multiset fuzzy real number is introduced and studied in Sect.4. In Sect.5, conclusion and further scope of the paper are briefly given.

Throughout the paper, the definitions and notations, for fuzzy related concepts and results, are taken from Klir and Yuan [15] and Zadeh [22]. $\mathbb{R}$ denotes the set of all real numbers, A is a fuzzy set in $\mathbb{R}$. Also $\mathcal{F}$ denotes the collection of all fuzzy numbers. The symbol ♦ indicates the end of the proof.

2. Preliminaries

Let X be a space of points. The members of X are denoted by x. A fuzzy set [22, page 339] A in X is characterized by a membership function $f_A(x)$ which associates with each point in X, a real number in the interval [0, 1], with the value of $f_A(x)$ at x , representing the " grade of membership " of x in A. The fuzzy set A in X can also be represented as $A=\{(x, f_A(x)): x\in X\}$ where $f_A : X \to [0, 1]$ is a function from X to [0, 1], known as the membership function of the fuzzy set A in X. For any $\alpha\in[0, 1]$, the α-cut ${}^{\alpha}A$ and the strong α-cut ${}^{\alpha+}A$ [15, page 19] are defined as the crisp sets ${}^{\alpha}A=\{x: f_A(x) \geq \alpha\}$ and ${}^{\alpha+}A =\{x: f_A(x) > \alpha\}$ respectively. The support [15, page 21] of a fuzzy set A within the universal set X is the crisp set that contains all the elements of X that have nonzero membership grades in A. Evidently the support of A is the same as the strong α-cut of A for $\alpha=0$, denoted by ${}^{0+}A$ or supp(A). The 1-cut, ${}^{1}A = \{x: f_A(x) \geq 1\} = \{x: f_A(x) =1\} = f_A^{-1}(\{1\})$ is called the core [15, page21] of A. The height, $h(A)$ [15, page 21], of a fuzzy set A is the largest membership grade obtained by any element in that set. The height of A may also be viewed as the supremum of α for which ${}^{\alpha}A\neq\emptyset$. That is $h(A) = \sup\{f_A(x) : x\in X\}$. A fuzzy set A is normal [15, page 21] when $h(A)$ =1; it is called subnormal when $h(A)<1$. A fuzzy set A is convex [22, page 347] if the α-cuts ${}^{\alpha}A=\{x: f_A(x) \geq \alpha\}$are convex for all $\alpha\in(0, 1]$.Equivalently, A is convex if and only if $f_A(\lambda x+(1-\lambda)y) \geq \text{Min}[f_A(x), f_A(y)]$ for each $\lambda\in[0, 1]$.

Definition 2.1 A fuzzy set A in $\mathbb{R}$ is a fuzzy real number [15, page 97] if it possesses the following three properties:

(i) A must be a normal fuzzy set;

(ii) ${}^{\alpha}A$ must be a closed interval for every $\alpha\in(0, 1]$;

(iii)the support of A, ${}^{0+}A$, must be bounded.

$\mathcal{F}(\mathbb{R})$ denotes the collection of all fuzzy real numbers where fuzzy real numbers are fuzzy numbers in the sense of Klir and Yuan [15]. The next lemma gives the structure of a fuzzy real number.

Lemma 2.2 [15, Theorem 4.1, page 98] Let $A\in\mathcal{F}(\mathbb{R})$. Then, A is a fuzzy real number if and only if there exists a closed interval $[a, b]\neq\emptyset$ such that

$$f_A(x) = \begin{cases} 1 & for \quad x\in[a,b] \\ L(x) & for \quad x\in(-\infty,a) \\ R(x) & for \quad x\in(b,\infty) \end{cases} \tag{1}$$

where L is a function from $(-\infty, a)$ to [0, 1] that is monotonic increasing , continuous from the right, and such that $L(x)=0$ for every $x \in (-\infty, \omega_1)$; R is a function from (b, ∞) to [0, 1] that is monotonic decreasing, continuous from the left, and such that $R(x) = 0$ for every $x \in (\omega_2, \infty)$.

Remark 2.3 Since fuzzy real numbers are characterized by the functions L and R, they are also called L-R fuzzy real numbers.

A multiset is a collection of objects in which objects may occur more than once. The number of times an element occurs in a multiset is called its multiplicity. The cardinality of a multiset is the sum of the multiplicities of its objects. Mathematically a multiset can be defined as follows:

Let $X\neq\emptyset$ and $w>0$ be an integer. A multiset M over X is characterized by a count function $C_M:X\to\{0,1,2,3,\ldots,w\}$ which associates with each point in X, a number in $\{0,1,2,3,\ldots,w\}$ with the value of $C_M(x)$ at x , representing the " count " of x in M. The multiset M in X can also be represented as $M=\{(x, C_M(x)):x \in X\}$ where $C_M : X \to\{0,1,2,3,\ldots,w\}$ is a function from X to $\{0,1,2,3,\ldots,w\}$, known as the count function of the multiset M in X. If $X = \{a,b,c\}$ and $C_M:X \to\{0,1,2,3,\ldots,w\}$ is defined as $C_M(a) =7$, $C_M(b)=5$ and $C_M(c)=0$ then the corresponding multiset M can be represented by $\{(a, 7), (b, 5)\}$ or $7/a + 5/b$ or a^7b^5. If the count for an element of X is zero it is customary not to include it in the representation of the multiset.

3. Multiset Sequence

Let M be a multiset over a finite set X and $M^* = \{x \in X: C_M(x) > 0\}$, known as the support of M. If s is the sum of the counts of the elements in M^*, we write $S = \{1, 2, 3,\ldots, s\}$. A multiset sequence in X induced by a multiset M is a function

M_S: S $\rightarrow M^*$ such that each element of M^* occurs in the sequence exactly as many times as its multiplicity. For instance, if $M = a^3b^2$ is a multiset in $\{a,b,c\}$ then S $=\{1,2,3,4,5\}$and (b,b,a,a,a) is a multiset sequence in X determined by the multiset M where $M_S(1) = b$, $M_S(2) = b$, $M_S(3) = a$, $M_S(4) = a$ and $M_S(5) = a$. The other multiset sequences will be similarly found. The following medical signal gives an application of the multiset sequences.

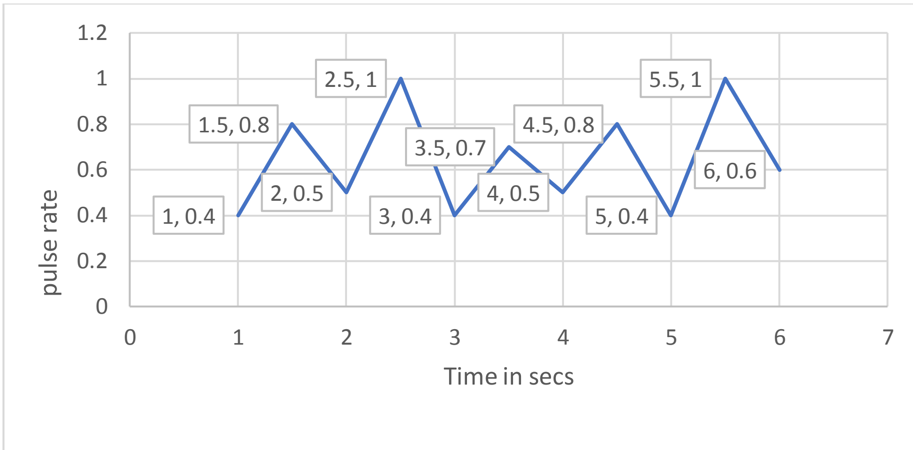

Fig.1

In the above signal, (0.4. 0.8, 0.5, 1, 0.4, 0.7, 0.5, 0.8, 0.4, 1, 0.6) is a multiset sequence of maximum and minimum heights of the signal with support set contained in [0, 6].

Since $0 <1<1.5<2<2.5<3<3.5<4<4.5<5<5.5<6$ and $P =\{0,1,1,5,2,2.5,3,3.5,4,4.5,5,5.5,6\}$ is a partition of the interval [0, 6], it is interesting to note the following properties of the signal.

(i) The signal is parallel to the x-axis on [0, 1] which may be taken as the initialization time interval for setting the machine.

(ii) The signal is strictly increasing on [1, 1.5], [2, 2.5], [3, 3.5], [4, 4.5] and [5, 5.5].

(iii) The signal is strictly decreasing on [1.5, 2], [2.5, 3], [3.5, 4], [4.5, 5] and [5.5, 6].

The above properties of the signal motivate us to define a new type of a fuzzy number that represents the signal in Fig.1.

4. Multiset Fuzzy Real Numbers

Every fuzzy set in R determines two types of sequences namely a sequence determined by its membership grades and another determined by the support of the fuzzy set. If A is a fuzzy set in R then the local minima and local maxima of its membership function f_A, attained at the points in the support of A, form a sequence in which values of some terms may be repeated. Such sequences are called multiset sequences. The elements in the support of A, at which the membership function f_A assumes maximum/minimum value, form a partition of the support of A. These two notions of a fuzzy set in R, are used to define a new type of fuzzy number, known as a multiset fuzzy real number.

Consider a medical signal which will be computed over various time intervals. Let $[a, b]$ be a time interval in which the time parameter t varies. Let $\alpha_1, \alpha_2, \ldots,\alpha_n$ correspond to minimum heights and maximum heights of the signal. That is α_1 = lower level point, α_2 = upper level point, α_3 = lower level point, …. , α_n = lower/upper level point.
Further we assume that $0<\alpha_i \leq 1$ for i=1,2,…,n and α_i may be repeated.

Definition 4.1 Let A be a fuzzy set in $\mathbb{R}$ with membership function f_A which has support lying $[a, b]$. Then A is a multiset fuzzy real number if there exists a multiset sequence $M_s = (\alpha_1, \alpha_2, \ldots,\alpha_n)$ in (0, 1] and a partition $P = \{a_1, a_2,\ldots,a_n \}$ of $[a, b]$ with $a = a_1 \leq a_2 \leq\ldots, \leq a_n = b$ such that

(i) $f_A(t) = 0$ for all t with $t < a$ or $t >b$,
(ii) $f_A(a_i) = \alpha_i$ for i=1,2,…,n,
(iii) $f_A(a_j) = 1$ for at least one j =1,2,…,n,

(iv)The function f_A is either strictly increasing or strictly decreasing on $[a_i, a_{i+1}]$ for i=1,2,3,…

Let $\mathcal{M}$ denote the class of all multiset fuzzy real numbers.

Remark 4.2 For n=1, $M_s = (\alpha_1)$ and P= $\{a_1\}$ with $a=a_1= b$. In this case f_A is defined as $f_A(t) =0$ for $t\neq a$ and $f_A(a) =\alpha_1$. Clearly $A\in\mathcal{M}$

Remark 4.3 For n=2, $M_s = (\alpha_1, \alpha_2)$ and P $=\{a_1, a_2\}$ of $[a, b]$ with $a = a_1 \leq a_2 = b$. Further we assume that $\alpha_2 =1$. In this case f_A is defined as $f_A(t) =0$ for $t < a$ or $t >b$, $f_A(a_1) =\alpha_1$ and $f_A(a_2) =1$. f_A may be so defined such that it is strictly increasing on $[a_1 , a_2]$. Therefore $A\in \mathcal{M}$ and its graphical representation is given in Fig.2.

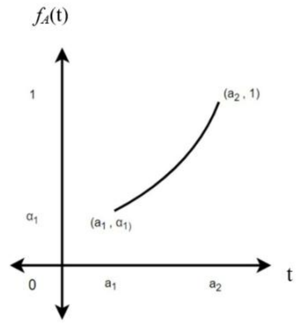

Fig.2

However, the specific example for the case n=2 is given below.

$$\text{Define } f_A(t) =\begin{cases} 0\,, & t<2 \\ \left(\frac{t}{4}\right)^2, & 2\leq t\leq 4 \\ 0, & t>4 \end{cases} .$$

It is easy to check that A satisfies the conditions of Definition 4.1 for $M_s = (0.25, 1)$ and the partition P={2, 4} of [2,4]. Then the graph of A is given in Fig.3.

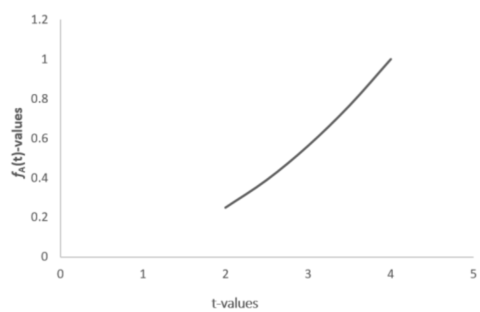

Fig.3

Remark 4.4 For n=3, $M_s = (\alpha_1, \alpha_2 , \alpha_3)$ and P=$\{a_1, a_2 , a_3\}$ of $[a, b]$ with $a = a_1\leq a_2 \leq a_3= b$. Further we assume that α_2=1. In this case f_A is defined as $f_A(t) =0$ for $t < a$ or $t >b$, $f_A(a_1) =\alpha_1$, $f_A(a_2) =1$ and $f_A(a_3) =\alpha_3$. f_A may be so defined such that it is strictly increasing on $[a_1,a_2]$ and is strictly decreasing on $[a_2,a_3]$. Therefore $A\in \mathcal{M}$ and its graphical representation is given in Fig.4.

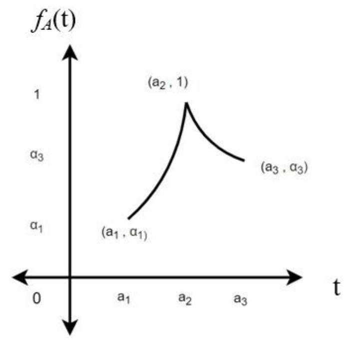

Fig.4

However, the specific example for the case n=3 is given below.

$$\text{Define} \quad f_A(t) = \begin{cases} 0 & for \quad t < 2 \\ \left(\frac{t}{4}\right)^2 & for \quad 2 \le t \le 5 \\ \left(\frac{5}{t}\right) & for \quad 5 \le t \le 8 \\ 0 & for \quad t > 8 \end{cases}$$

It is easy to check that A satisfies the conditions of Definition 4.1 for the multiset sequence Ms = (0.16, 1, 0.625) and the partition P= {2,5,8}.Then the graph of A is given in Fig. 5.

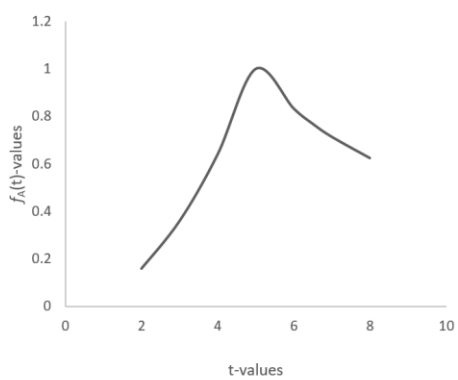

Fig.5

The next theorem gives the properties of multiset fuzzy real numbers.

Theorem 4.5 Let A be a multiset fuzzy real number.

(i) The support of A is compact.

(ii) A is normal.

(iii) For each $\alpha\in(0, 1]$, the α-cut ${}^{\alpha}A$ is a union of finitely many bounded closed intervals.

Proof By using the condition (i) of Definition 4.1, support of A is $[a, b]$ that is compact by the Heine Borel Theorem of real analysis. By using the condition (iii) of Definition 4.1, A is normal.
Let $\alpha\in(0, 1]$, Minimum values of A = $\{\alpha_1, \alpha_3, \alpha_5, \ldots\}$ and Maximum values of A = $\{\alpha_2, \alpha_4, \alpha_6, \ldots\}$.

Clearly Max $\{\alpha_2, \alpha_4, \alpha_6, \ldots\}=1$. Let Min $\{\alpha_1, \alpha_3, \alpha_5, \ldots\} = \alpha_0$ (say).
Case-1: $\alpha\le\alpha_0$.
$${}^{\alpha}A = \{x: f_A(x)\ge \alpha\} = \{x: f_A(x) \ge\alpha_0\} = [a, b]. \tag{2}$$
Therefore let $\alpha>\alpha_0$. Clearly $\alpha\le1$ which implies $\alpha_0<\alpha\le1$.
Case-2: Let $\alpha_{j-1} \le\alpha \le\alpha_j$ where $\alpha_j \in\{\alpha_2, \alpha_4, \alpha_6, \ldots\}$.
Since $f_A(a_j) = \alpha_j$, choose $y_j \in [a_{j-1}, a_j]$, $z_j \in [a_j, a_{j+1}]$ with $f_A(y_j) = \alpha = f_A(z_j)$.
Therefore $x \in [y_j, z_j] \Rightarrow f_A(x) \ge \alpha \Rightarrow x \in {}^{\alpha}A$ so that $[y_j, z_j] \subseteq {}^{\alpha}A$.

This implies that $[y_j, z_j] \subseteq {}^{\alpha}A$ for all $\alpha_j \in \{\alpha_2, \alpha_4, \alpha_6, \ldots\}$ with $\alpha_j \geq \alpha$. (3)
It is to be noted that α comes under Case-1 or Case-2.
Claim: $x \in {}^{\alpha}A \Rightarrow x \in [y_j, z_j]$ for some $\alpha_j \in \{\alpha_2, \alpha_4, \alpha_6, \ldots\}$.
Let $x \in {}^{\alpha}A$. Then $f_A(x) \geq \alpha$ that implies $f_A(x) \geq \alpha_{j-1}$ as in Case-2.
Since α_{j-1} is a minimum value of f_A we choose $\alpha_j \in \{\alpha_2, \alpha_4, \alpha_6, \ldots\}$ with $\alpha_{j-1} \leq \alpha \leq f_A(x) \leq \alpha_j$.
As in Case-2, choose $y_j \in [a_{j-1}, a_j]$, $z_j \in [a_j, a_{j+1}]$ with $f_A(y_j) = \alpha = f_A(z_j)$.
Now $f_A(x) \geq \alpha \Rightarrow f_A(x) \geq f_A(y_j)$ and $f_A(x) \geq f_A(z_j)$. Since $y_j \leq z_j$, it follows that $y_j \leq x \leq z_j$.
Therefore $x \in [y_j, z_j]$ (4)
Using (2), (3) and (4), we see that ${}^{\alpha}A$ is a union of finitely many bounded closed intervals which proves (iii)♦

Theorem 4.6 Let A be a multiset fuzzy real number induced by a multiset sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and a partition $P=\{a_1, a_2, \ldots, a_n\}$ of $[a, b]$ with $a = a_1 \leq a_2 \leq \ldots \leq a_n = b$ where n=2k+1, k=1,2,3….Then there are 2k real valued functions $L_i:(-\infty, a_{i+1}) \rightarrow [0,1]$, $R_i:(a_{i+1}, \infty) \rightarrow [0,1]$ such that $L_i(t)=0$ for all $t<a_i$, $L_i(a_{i+1}) = \alpha_i$, L_i is increasing and continuous from the right and $R_i(t)=0$ for all $t > a_{i+2}$, and $R_i(a_{i+1}) = \alpha_i$, R_i is decreasing and continuous from the left where i=1,3,…,2k-1.

Proof: Let A be a multiset fuzzy real number induced by a multiset sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and a partition $P=\{a_1, a_2, \ldots, a_n\}$ of $[a, b]$ with $a = a_1 \leq a_2 \leq \ldots \leq a_n = b$ where n=2k+1, k=1,2,3….

Define $f_i(t) = \begin{cases} 0 & for \quad t < a_i \\ f_A(t) & for \quad a_i \leq t \leq a_{i+2} \\ 0 & for \quad t > a_{i+2} \end{cases}$. Then each f_i behaves like a membership function of a fuzzy number where '1' is replaced by 'α_i'. By applying Lemma 2.2, there are real functions, L_i and R_i satisfying the following conditions.

$L_i(t) = \begin{cases} 0 & for \quad t < a_i \\ f_A(t) & for \quad a_i \leq t \leq a_{i+1} \end{cases}$ and $R_i(t) = \begin{cases} f_A(t) & for \quad a_{i+1} \leq t \leq a_{i+2} \\ 0 & for \quad t > a_{i+2} \end{cases}$. Also, each L_i is increasing and continuous from the right and each R_i is decreasing and continuous from the left.♦

Theorem 4.7 Let A be a multiset fuzzy real number induced by a multiset sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and a partition $P=\{a_1, a_2, \ldots, a_n\}$ of $[a, b]$ with $a = a_1 \leq a_2 \leq \ldots \leq a_n = b$ where n=2k, k=1,2,3,…Then there are k real valued functions $L_i:(-\infty, a_{i+1}) \rightarrow [0,1]$ where i=1,3,…,2k-1 and k-1 real valued functions $R_j:(a_{j+1}, \infty) \rightarrow [0,1]$ where j=1,3,…,2k-3 such that $L_i(t)=0$ for all $t<a_i$, $L_i(a_{i+1}) = \alpha_i$, L_i is increasing and continuous from the right and $R_j(t)=0$ for all $t > a_{j+2}$ and $R_j(a_{j+1}) = \alpha_j$, R_j is decreasing and continuous from the left .

Proof: Analogous to Theorem 4.6. ♦

It is interesting to infer from Theorem 4.6 and Theorem 4.7 that every multiset fuzzy real number is associated with some special type of the real valued functions L_i , R_i .Therefore, multiset fuzzy real numbers can also be represented by such functions. This motivates to define a linear type multiset fuzzy real number. A multiset fuzzy real number is of linear type if it's associated real functions L_i , R_i are linear. Otherwise, it is of non-linear type. If A is a multiset fuzzy real number determined by the multiset sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and the partition $P=\{a_1, a_2, \ldots, a_n\}$ of $[a, b]$ and it is of linear type then it is represented by the notation $A=\langle(\alpha_1, \alpha_2, \ldots, \alpha_n); \{a_1, a_2, \ldots, a_n\}\rangle$. The following are examples of linear type multiset fuzzy real numbers.

Example 4.8 Let n=2, multiset sequence $M_s = (\alpha_1, \alpha_2)$ where $\alpha_2=1$ and partition $P=\{a_1, a_2\}$ with $a_1 \leq a_2$. Then

$f_A(t) = \begin{cases} 0 & for \quad t < a_1 \\ \alpha_1 + \frac{(1-\alpha_1)}{(a_2-a_1)}(t - a_1) & for \quad a_1 \leq t \leq a_2 \\ 0 & for \quad t > a_2 \end{cases}$ will represent the linear type multiset fuzzy real number which is shown below.

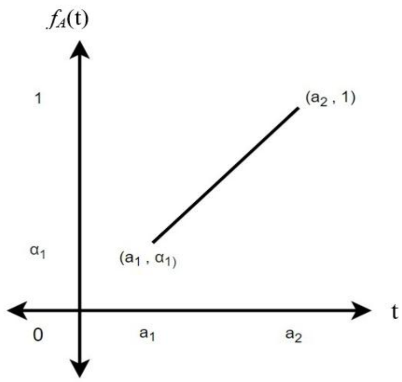

Fig.6

For M_s =(0.25,1) with P={2, 4}, define $f_A(t) = \begin{cases} 0 & for \quad t < 2 \\ 0.375t - 0.5 & for \quad 2 \le t \le 4 \\ 0 & for \quad t > 4 \end{cases}$. The graph of this linear type multiset fuzzy real number is Fig.7.

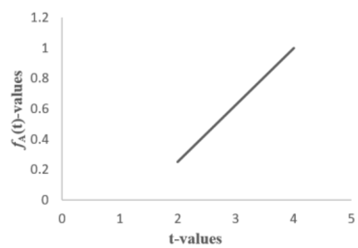

Fig.7

Example 4.9 Let n=3, M_s =(α_1,α_2, α_3) where α_2=1 and partition P={a_1,a_2,a_3} with $a_1 \le a_2 \le a_3$. Then

$$f_A(t) = \begin{cases} 0 & for \quad t < a_1 \\ \alpha_1 + \frac{(1-\alpha_1)}{(a_2-a_1)}(t - a_1) & for \quad a_1 \le t \le a_2 \\ 1 + \frac{(\alpha_3-1)}{(a_3-a_2)}(t - a_2) & for \quad a_2 \le t \le a_3 \\ 0 & for \quad t > a_3 \end{cases}$$

will represent the linear type multiset fuzzy real number which is shown below.

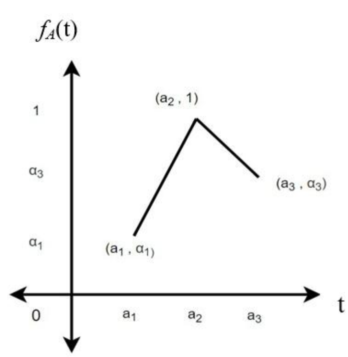

Fig.8

For M_s =(0.16, 1, 0.625) with P={2, 5, 8}, define $f_A(t) = \begin{cases} 0 & for \quad t < 2 \\ 0.28t - 0.4 & for \quad 2 \le t \le 5 \\ -0.125t + 1.625 & for \quad 5 \le t \le 8 \\ 0 & for \quad t > 8 \end{cases}$ whose graph is given in Fig.9.

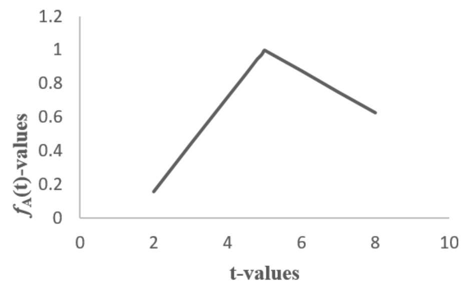

Fig.9

Example 4.10: Let n=4, M_s =(α_1,α_2, α_3, α_4) where α_2=1 and partition P={a_1,a_2,a_3,a_4} with $a_1 \le a_2 \le a_3 \le a_4$.

Then $f_A(t) = \begin{cases} 0 & for \quad t < a_1 \\ \alpha_1 + \frac{(1-\alpha_1)}{(a_2-a_1)}(t - a_1) & for \quad a_1 \le t \le a_2 \\ 1 + \frac{(\alpha_3-1)}{(a_3-a_2)}(t - a_2) & for \quad a_2 \le t \le a_3 \\ \alpha_3 + \frac{(\alpha_4-\alpha_3)}{(a_4-a_3)}(t - a_3) & for \quad a_3 \le t \le a_4 \\ 0 & for \quad t > a_4 \end{cases}$

will represent the linear type multiset fuzzy real number which is shown below.

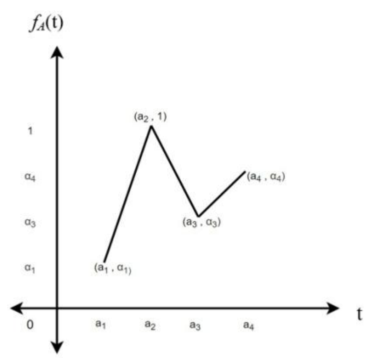

Fig.10

For M_s =(0.16, 1, 0.625, 0.859) with P={2, 5, 8,11}, define $f_A(t) = \begin{cases} 0 & for \quad t < 2 \\ 0.28t - 0.4 & for \quad 2 \le t \le 5 \\ -0.125t + 1.625 & for \quad 5 \le t \le 8 \\ 0.078t + 0.001 & for \quad 8 \le t \le 11 \\ 0 & for \quad t > 11 \end{cases}$

the graph of which is given in Fig.11.

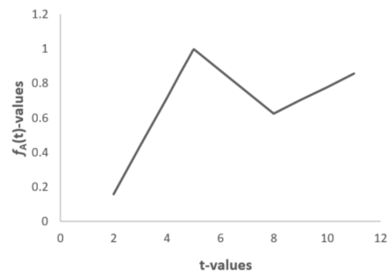

Fig.11

Example 4.11 Let n=5, M_s =(α_1,α_2, α_3, α_4, α_5) where α_2=1 and partition P={a_1,a_2,a_3,a_4,a_5} with $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$.

$$\text{Then } f_A(t) = \begin{cases} 0 & for \quad t < a_1 \\ \alpha_1 + \frac{(1-\alpha_1)}{(a_2-a_1)}(t-a_1) & for \quad a_1 \leq t \leq a_2 \\ 1 + \frac{(\alpha_3-1)}{(a_3-a_2)}(t-a_2) & for \quad a_2 \leq t \leq a_3 \\ \alpha_3 + \frac{(\alpha_4-\alpha_3)}{(a_4-a_3)}(t-a_3) & for \quad a_3 \leq t \leq a_4 \\ \alpha_4 + \frac{(\alpha_5-\alpha_4)}{(a_5-a_4)}(t-a_4) & for \quad a_4 \leq t \leq a_5 \\ 0 & for \quad t > a_5 \end{cases}$$

will represent the linear type multiset fuzzy real number which is shown below.

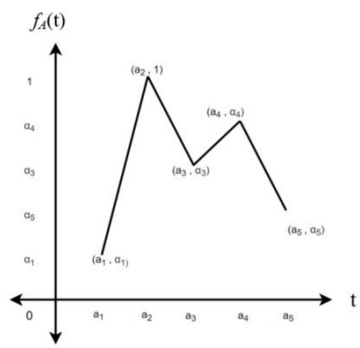

Fig.12

For M_s =(0.16, 1, 0.625, 0.859, 0.762) with P={2, 5, 8, 11, 13},

$$\text{define } f_A(t) = \begin{cases} 0 & for \quad t < 2 \\ 0.28t - 0.4 & for \quad 2 \leq t \leq 5 \\ -0.125t + 1.625 & for \quad 5 \leq t \leq 8 \\ 0.078t + 0.001 & for \quad 8 \leq t \leq 11 \\ -0.0665t + 1.5905 & for \quad 11 \leq t \leq 13 \\ 0 & for \quad t > 13 \end{cases} \text{ whose graph is given in Fig.13.}$$

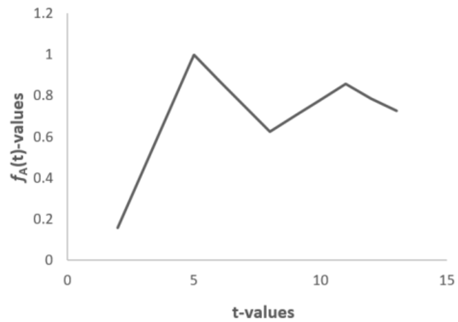

Fig.13

The following theorem characterizes the linear type multiset fuzzy real numbers.

Theorem 4.12 Let $a = a_1 \leq a_2 \leq \ldots \leq a_n = b$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ are in (0,1].

(i) $\langle (1); \{a\} \rangle$ represents a real number.

(ii) $\langle (\alpha_1, 1, \alpha_1); \{a_1, a_2, a_3\} \rangle$ is a $\wedge$ - shape multiset fuzzy real number.

(iii) $\langle (\alpha_1, 1, \alpha_1, 1); \{a_1, a_2, a_3, a_4\} \rangle$ is a N - shape multiset fuzzy real number.

(iv) $\langle (\alpha_1, 1, \alpha_1, 1, \alpha_1); \{a_1, a_2, a_3, a_4, a_5\} \rangle$ is a M- shape multiset fuzzy real number.

Proof The assertion (i) follows from Remark 4.2.

Let A be a multiset fuzzy real number induced by a multiset sequence $(\alpha_1, 1, \alpha_1)$ and a partition P=$\{a_1, a_2, a_3, a_4\}$ of $[a, b]$.Since $\langle (\alpha_1, 1, \alpha_1); \{a_1, a_2, a_3\} \rangle$ is a linear type multiset fuzzy real number, the corresponding μ can be defined as

$$f_A(t) = \begin{cases} 0 & for \quad t < a_1 \\ \alpha_1 & for \quad t = a_1 \\ \alpha_1 + \left(\frac{1-\alpha_1}{a_2 - a_1}\right)(t - a_1) & for \quad a_1 \leq t \leq a_2 \\ 1 & for \quad t = a_2 \\ 1 + \left(\frac{\alpha_3 - 1}{a_3 - a_2}\right)(t - a_2) & for \quad a_2 \leq t \leq a_3 \\ & for \quad t > a_3 \\ 0 & \end{cases}$$

Using Theorem 4.7, $L_1(t) = \begin{cases} 0 & for \quad t < a_1 \\ \mu(t) & for \quad a_1 \leq t \leq a_2 \end{cases}$ and $R_1(t) = \begin{cases} \mu(t) & for \quad a_2 \leq t \leq a_3 \\ 0 & for \quad t > a_3 \end{cases}$

Clearly the graph of μ looks like the graph of Example 4.9, which completes the proof of (ii).

Let μ be a multiset fuzzy real number induced by a multiset sequence $(\alpha_1, 1, \alpha_1, 1)$ and a partition P=$\{a_1, a_2, a_3, a_4\}$ of $[a, b]$.

Since $\langle (\alpha_1, 1, \alpha_1, 1); \{a_1, a_2, a_3, a_4\} \rangle$ is a linear type multiset fuzzy real number, the corresponding μ can be defined as

$$\mu(t) = \begin{cases} 0 & for \quad t < a_1 \\ \alpha_1 & for \quad t = a_1 \\ \alpha_1 + \left(\frac{1-\alpha_1}{a_2 - a_1}\right)(t - a_1) & for \quad a_1 \leq t \leq a_2 \\ 1 & for \quad t = a_2 \\ 1 + \left(\frac{\alpha_3 - 1}{a_3 - a_2}\right)(t - a_2) & for \quad a_2 \leq t \leq a_3 \\ \alpha_1 & for \quad t = a_3 \\ \alpha_3 + \left(\frac{\alpha_4 - \alpha_3}{a_4 - a_3}\right)(t - \alpha_3) & for \quad a_3 \leq t \leq a_4 \\ 1 & for \quad t = a_4 \\ 0 & for \quad t > a_4 \end{cases}$$

Using Theorem 4.6, $L_1(t) = \begin{cases} 0 & for \quad t < a_1 \\ \mu(t) & for \quad a_1 \leq t \leq a_2 \end{cases}$ and $R_1(t) = \begin{cases} \mu(t) & for \quad a_2 \leq t \leq a_3 \\ \alpha_1 & for \quad t > a_3 \end{cases}$

$$L_3(t) = \begin{cases} \mu(t) & for \quad a_3 \leq t \leq a_4 \\ 0 & for \quad t > a_4 \end{cases}$$

Clearly the graph of μ looks like the graph of Example 4.10 which completes the proof of (iii).

The proof of assertion (iv) is analog. ♦
It is interesting to know that the real numbers and fuzzy real numbers are identified with multiset fuzzy real numbers as seen in the next theorem.

Theorem 4.13:

(i) Every real number is a multiset fuzzy real number.
(ii) Every fuzzy real number is a multiset fuzzy real number.

5. Conclusion and Further Scope

In this paper, we have proposed a new generalization of a fuzzy number, known as a multiset fuzzy real number which will be used to model a data set of parameters whose values have fluctuations during the specified time interval. Enough examples are given to understand the concept. The standard and topological properties of multiset fuzzy real numbers are also studied. In future, arithmetic operations of existing fuzzy numbers may be extended to multiset fuzzy real numbers. The basic properties and applications of fuzzy real numbers can be compared with those of existing fuzzy numbers.

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References

[1] Adam F and Hassan N (2019) Q-Cubic Soft Multiset Theory, *Computer Science*. Jan 1:14(1):103-19.
[2] Anusuya Ilamathi VS Vimala J and Davvaz B (2019) Multiset filters of residuated lattices and its application in medical diagnosis, *Journal of Intelligent & Fuzzy Systems*. Jan 1:36(3):2297-305. **DOI:** 10.3233/JIFS-169940
[3] Balogun F Singh D and Tella Y (2020) Realizers of Partially Ordered Multisets, *Theory and Applications of Mathematics & Computer Science*. Aug 28:10(2):1-6.
[4] Blizard WD (1989) Multiset theory, *Notre Dame Journal of formal logic*. Jan:30(1):36-66.
[5] Chang SS and Zadeh LA 1972 On fuzzy mapping and control, *IEEE Transactions on Systems, Man, and Cybernetics*. Jan (1):30-4. **DOI:** 10.1109/TSMC.1972.5408553
[6] Das R (2022) Generalized multiset and multiset ideal continuous function, *Applied Sciences*. Jan 1:12.
[7] Dubois D and Prade H (1978) Operations on fuzzy numbers, *International Journal of systems science*. Jun 1:9(6):613-26.
https://doi.org/10.1080/00207727808941724
[8] Raj ME Sivaraman G and Vishnukumar P (2022) A Novel Kind of Arithmetic Operations on Trapezoidal Fuzzy Numbers and Its Applications to Optimize the Transportation Cost, *International Journal of Fuzzy Systems*. Nov 17:1-8.
https:/doi.org/10.1007//s40815-022-01425-6
[9] Ejegwa PA Jana C and Pal M (2022) Medical diagnostic process based on modified composite relation on pythagorean fuzzy multi-sets, Granular Computing. Jan:7:15-23.
[10] Felisiak PA Qin K and Li G (2020) Generalized multiset theory, *Fuzzy Sets and Systems*. Feb 1:380:104-30.
[11] Girish KP and John SJ (2009) Relations and functions in multiset context, *Information sciences*. Mar 1:179(6):758-68.
[12] Hickman JL(1980) A note on the concept of multiset, *Bulletin of the Australian Mathematical society*. Oct :22(2):211-7.

[13] John SJ Suma P and Athira TM (2022) Multiset modules, *Journal of Computational and Cognitive Engineering*. Feb 1:1(1):37-41. https://doi.org/10.47852/bonviewJCCE6152118205514

[14] Jürgensen H (2020) Multisets, heaps, bags, families: What is a multiset?, *Mathematical Structures in Computer Science*. Feb:30(2):139-58. https://doi.org/10.1017/S0960129519000215

[15] Klir G and Yuan B (1995) Fuzzy sets and fuzzy logic, *New Jersey*: *Prentice hall*; Jan.

[16] Lake J (1976) Sets, fuzzy sets, multisets and functions, *Journal of the London Mathematical Society*. Feb 1:2(3):323-6.

[17] Orellana R and Zabrocki M (2023) Howe duality of the symmetric group and a multiset partition algebra, *Communications in Algebra*. Jan 2:51(1):393-413. https://doi.org/10.1080/00927872.2022.2099884

[18] Patra K (2022) Fuzzy risk analysis using a new technique of ranking of generalized trapezoidal fuzzy numbers, *Granular Computing*. Jan :7(1):127-40.

[19] PEZÜK ŞK and SENEL G (2021) An Application of Soft Multisets to a Decision-Making Problem Concerning Side Effects of COVID-19 Vaccines, *Journal of New Theory*. (35):103-13. https://doi.org/10.53570/jnt.949227

[20] Ray GC and Dey S (2021) Mixed multiset topological space and separation axioms, *Indian Journal of Pure and Applied Mathematics*.1-8.

[21] Singh D Ibrahim AM Yohanna T and Singh JN (2007) An overview of the applications of multisets, *Novi Sad Journal of Mathematics*.37 (2):73-92.

[22] Zadeh LA (1965) Fuzzy sets, *Information and control*. Jun 1:8(3):338-53.

[23] Zhang DL Guo C and Chen D (2019) On generalized fuzzy numbers, *Iranian Journal of Fuzzy Systems*. Feb 1:16(1):61-73.