The Outer Connected Detour Monophonic Domination Number of a Graph

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Abstract

If $M \subseteq V$ is both an outer connected detour monophonic set and a dominating set of G, it is referred to as an *outer connected detour monophonic dominating set* of G. The *outer connected detour monophonic domination number* of G, which is represented by the symbol $\gamma_{ocdm}(G)$ is the minimum cardinality of an outer connected detour monophonic dominating set of G. The outer connected detour monophonic dominating number of G for some standard graphs is determined.

Keywords: detour monophonic path, detour monophonic number, outer connected detour monophonic number , outer connected detour monophonic domination number.

AMS Subject Classification: 05C38, 05C69.

1. Introduction

A finite, undirected connected graph with no loops or many edges is referred to as a graph $G = (V, E)$. By n and m, respectively, we indicate the order and size of G. We refer to [1] for the fundamental terms used in graph theory. If uv is an edge of G, then two vertices u and v are said to be *adjacent.* If two edges of G share a vertex, they are said to be *adjacent*.

Let $M \subset V$ be any subset of vertices of G. Then the graph with M as its vertex set and all of its edges in E having both of their end points in M is the *induced subgraph* $G[M]$. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete.

The length of the shortest path in a connected graph G is equal to the *distance* $d(u,v)$ between two vertices u and v. An $u-v$ *geodesic* is a $u-v$ path with length $d(u,v)$. An edge that connects two non-adjacent vertices of a path P is called the *chord* of P. A chordless $u-v$ path is referred to as a *monophonic path*. The *monophonic distance* $d_m(u,v)$ for two vertices u and v in a connected graph G is the length of a longest $u-v$ monophonic path in G. An $u-v$ *detour monophonic path* is one that has a length of $d_m(u,v)$. These concepts were studied in [2-7, 9,11, 13,14, 17,23].

The closed interval $J_{dm}[u,v]$ for two vertices u and v is consists of all the vertices along an $u-v$ detour monophonic path, including the vertices u and v. If $v \in E$, then $J_{dm}[u,v] = \{u,v\}$. For a set M of vertices, let $J_{dm}[M] = \cup_{u,v \in M} J[u,v]$. Then certainly $M \subseteq J_{dm}[M]$. If $J_{dm}[M] = V$, a set $M \subseteq V(G)$ is referred to as a *detour monophonic set* of G. The *detour monophonic* number $dm(G)$ of G is the minimum order of its detour monophonic sets. Any detour monophonic set of order $dm(G)$ is referred to as an dm-set of G. If $M = V$ or the subgraph $G[V - M]$ is connected, then a detour monophonic set M of a connected graph G is said to be an *outer connected detour monophonic set* of G.The *outer connected detour monophonic number* of G, indicated by the symbol $ocd_m(G)$, is the minimum cardinality of an outer connected detour monophonic set of G. The ocd_m*-set* of G is a minimum cardinality of an outer connected detour monophonic set of G. These concepts were studied in [19, 22].

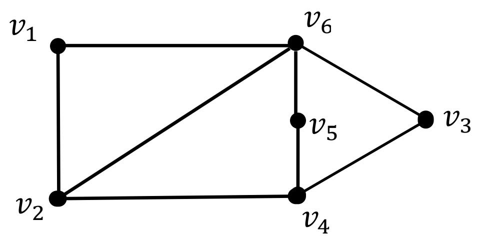
A set of vertices M in a graph G is a *dominating set* if each vertex of G is dominated by some vertex of M. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. These concepts were studied in [1, 10, 12, 14-16, 18,20].The following theorems used in sequel.

Theorem 1.1. [19] Each extreme vertex of a connected graph G belongs to every detour monophonic set of G.

2.The Outer Connected Detour Monophonic Domination Number of a Graph

Definition 2.1. If $M \subseteq V$ is both an outer connected detour monophonic set and a dominating set of G, it is referred to as an *outer connected detour monophonic dominating set* of G. The *outer connected detour monophonic domination number of* G, which is represented by the symbol $\gamma_{ocdm}(G)$ is the minimum cardinality of an outer connected detour monophonic dominating set of G. A γ_{ocdm}- set of G is an outer connected detour monophonic dominating set of cardinality $\gamma_{ocdm}(G)$.

Example 2.2. $M = \{v_1, v_3, v_5\}$ is a minimum outer connected detour monophonic dominating set of the graph G shown in figure 2. 1, resulting in $\gamma_{ocdm}(G) = 3$.



G

Figure 2.1

Remark 2.3. There can be more than one γ_{ocdm}- set of G , for the graph G given in figure 2.1 $M' = \{v_1, v_3, v_4\}$ is another γ_{ocdm}- set of G .

Theorem 2.4. Every outer connected detour monophonic dominating set of a graph G contains every extreme vertex that belongs to that graph.

Proof. Because every outer connected detour monophonic dominating set is a monophonic set, the conclusion comes from Theorem 1.1 .

■

Corollary 2.5. For the complete graph $K_n(n \geq 2), \gamma_{ocdm}(K_n) = n$.

Proof. The vertex set of $K_n(n \geq 2)$ is the unique outer connected detour monophonic dominating set of K_n because every vertex of the complete graph $K_n(n \geq 2)$ is an extreme vertex. Hence $\gamma_{ocdm}(K_n) = n$. ■

Theorem 2.6. For the path $G = P_n(n \geq 4), \gamma_{ocdm}(G) = n - 2$.

Proof. Let $P_n : v_1, v_2, v_3, \ldots, v_n$. Let $M = \{v_1, v_4, v_5, \ldots, v_n\}$. Then M is an outer connected detour monophonic dominating set of G such that $\gamma_{ocdm}(G) \leq n - 2$. We demonstrate that $\gamma_{ocdm}(G) = n - 2$. Contrarily, consider that $\gamma_{ocdm}(G) \leq n - 3$. Then $|M'| \leq n - 3$, there exists a γ_{ocdm}-set M' . When $G[V - M']$ is connected, M' is not a dominating set of G. There is a contradiction if $G[V - M']$ is not connected since then M'cannot be an outer connected detour monophonic set of G. Hence M' is not a detour monophonic dominating set of G, Therefore,$\gamma_{ocdm}(P_n) = n - 2$ ■

Theorem 2.7. For the cycle $C_n(n \geq 5)$,$\gamma_{ocdm}(G) = n - 2$.

Proof: Let $v_1, v_2, v_3 \ldots, v_n, v_1$ be the cycle C_n. Let $M = \{v_1, v_4, v_5, \ldots, v_n, v_1\}$. Then M is an outer connected detour monophonic dominating set of G such that $\gamma_{ocdm}(G) \leq n - 2$. We prove that $\gamma_{ocdm}(G) = n - 2$. Contrarily, consider that $\gamma_{ocdm}(G) \leq n - 3$. Then there exists a γ_{ocdm}-set M' such that $|M'| \leq n - 3$. If $G[V - M']$ is connected, then M' is not a dominating set of G if $G[V-M']$ is disconnected, then $M'$is not an outer connected detour monophonic set of G, which is a contradiction.Therefore M' is not an outer connected detour monophonic dominating set of G. Hence $\gamma_{ocdm}(G) = n - 2$. ■

Theorem2.8. For the complete bipartite graph $G = K_{r,s}(1 \le r \le s)$,

$$\gamma_{ocdm}(G) = \begin{cases} s\,, & if\ r = 1, s \ge 2 \\ 3\,, & if\ r = s = 2 \\ s+1, & if\ r = 2, s \ge 2 \\ 4\,, & otherwise. \end{cases}$$

.**Proof.** Let $X = \{x_1, x_2, \dots\dots, x_r\}$, $Y = \{y_1, y_2, \dots\dots, y_s\}$ be a bipartition of G. Let $G = K_{r,s}$ be a star with r end vertices when $r = 1$ and $s \ge 2$. After that $\gamma_{ocdm}(G) = r$. $G = K_{2,2}$ is the cycle C_4 for $r = s = 2$. As a result, $M = V - \{v\}$ is a γ_{ocdm}- set of G , and $\gamma_{ocdm}(G) = 3$. $M = \{x_1\} \cup Y$ is an outer connected detour monophonic dominating set of G, for $r = 2, s \ge 3$, and as a result, $\gamma_{ocdm}(G) \le s + 1$. We show that $\gamma_{ocdm}(G) = s + 1$. Assume that $\gamma_{ocdm}(G) \le s$. Then, a γ_{ocdm}- set M'exists such that $|M'| < s$. If $M' \subseteq Y$, then either $G[V - M']$ is disconnected or $M'$is not a dominating set of G, which is a contradiction . If $M' \subseteq X \cup Y$, then $G[V - M']$ is disconnected. As a result, $\gamma_{ocdm}(G) = s + 1$. Now let $3 \le m \le n$. Let G be bipartitioned into $X = \{x_1, x_2, \dots\dots, x_r\}$, $Y = \{y_1, y_2, \dots\dots, y_s\}$ Then $M = \{x_i, x_j, y_r, y_s\}$, where $i \ne j, r \ne s$ is the G minimal detour monophonic dominating set . M is a minimum outer connected detour monophonic dominating set of G. since the subgraph $G[V - M]$ is connected. So $\gamma_{ocdm}(G) \le 4$. We demonstrate that $\gamma_{ocdm}(G) = 4$. Assume that $\gamma_{ocdm}(G) \le 3$. A γ_{ocdm}-set M_1 of G exists, and it has the property $|M_1| \le 3$. If $M_1 \subseteq X$, the connection between $M_1$and V is broken. $M_1 \subseteq Y$ indicates that $G[V - M_1]$ is unconnected. Consequently, $M_1 \subseteq X \cup Y$. Then $x \in G$ has a vertex such that $J[M_1] \ne V$ exists. Because of this contradiction, M_1 cannot be an outer connected detour monophonic dominating set of G. As a result $\gamma_{ocdm}(G) = 4$.

■

Theorem 2.9. For the wheel $G = W_n = K_1 + C_{n-1}, (n \geq 5)$,

$$\gamma_{ocdm}(G) = \begin{cases} 2, & if\ 4 \leq n \leq 6 \\ 3, & if\ 7 \leq n \leq 8 \\ 4, & if\ n \geq 9. \end{cases}$$

Proof. Let C_{n-1} be the cycle with $v_1, v_2, \dots\dots, v_{n-1}, v_1$. Let $4 \leq n \leq 6$,consider two non-adjacent vertices of G , x and y. So that $\gamma_{ocdm}(G) = 2$, $M = \{x, y\}$ is an outer connected detour monophonic dominating set of G. Let $7 \leq n \leq 8$. It is simple to demonstrate that $\gamma_{ocdm}(G) = 3$. Thus, let $n \geq 9$, it is simple to demonstrate that $|M| \leq 3$ and M do not form an outer connected detour monophonic dominating set. Let $M_1 = \{v_1, v_2, v_{n-1}, x\}$. So that $\gamma_{ocdm}(G) = 4$ M_1 is an outer connected detour monophonic dominating set of G.

■

Theorem 2.10. For a connected graph G of order n, then $2 \leq ocd_m(G) \leq \gamma_{ocdm}(G) \leq n$

Proof. Any outer connected detour monophonic set requires a minimum of two vertices , and so does $ocd_m(G) \geq 2$. Every outer connected detour monophonic dominating set is likewise an outer connected detour monophonic set, therefore $ocd_m(G) \leq \gamma_{ocdm}(G)$. Additionally, it is evident that $\gamma_{ocdm}(G) \leq n$ because $V(G)$ is an outer connected detour monophonic dominating set, Consequently $2 \leq ocd_m(G) \leq \gamma_{ocdm}(G) \leq n$. ■

Remark 2.11. Theorem 2.10 has precise bounds. The set of two end vertices of $G = P_4$ is the unique outer connected detour monophonic set of G ,resulting in $ocd_m(P_n) = 2$. For the cycle $G = C_4, ocd_m(G) = \gamma_{ocdm}(G) = 3$. For the complete graph $G = K_n$, $\gamma_{ocdm}(G) = n$. For the graph G shown in figure 2.1. the inequalities in Theorem 2.10 can also be strict.

$G = W_9, ocd_m(G) = 3, \gamma_{ocdm}(G) = 4$ and $p = 9$ for the wheel. Consequently, $2 < ocd_m(G) < \gamma_{ocdm}(G) < n$.

Theorem 2.12. $\gamma_{ocdm}(G) = 3$ for the graph $G = K_1 + P_{n-1}$ $(n \geq 5)$

Proof. Suppose that P_{n-1} is $v_1, v_2, \ldots\ldots, v_{n-1}$ and $V(K_1) = x$. Then, the only two extreme vertices of G are v_1 and v_{n-1} . A ocd_m-set of G is $M = \{v_1, v_{n-1}\}$. However, M is not a set that dominates G, therefore $\gamma_{ocdm}(G) \geq 3$. Let $M_1 = M \cup \{x\}$. Following that, M_1 is an ocd_m-set of G such that $\gamma_{ocdm}(G) = 3$. ■

Theorem 2.13. Suppose $G = \overline{K}_2 + P_{n-2}(n \geq 5)$. Then, $\gamma_{ocdm}(G) = 2$.

Proof. Let's say $V(\overline{K}_2) = \{x, y\}$. Then $V(\overline{K}_2)$ is a dominating set of G as well as an outer connected detour monophonic set . Therefore $\gamma_{ocdm}(G) = 2$. ■

Theorem 2.14. $G = K_{n_1} + K_{n_2}$where $n_1, n_2 \geq 2, \gamma_{ocdm}(G) = n_1 + n_2$.

Proof. We know that $G = K_{n_1} + K_{n_2}$ is the complete graph $G = K_{n_1 + n_2}$.Then $M = V(G)$ is the unique γ_{ocdm}-set of G and $\gamma_{ocdm}(G) = n_1 + n_2$ as a result ■

Theorem 2.15. Let $G = P_{n_1} + P_{n_2}$, where $n_1 \geq 3$ and $n_2 \geq 3$. Therefore, $\gamma_{ocdm}(G) = 4$.

Proof. Let $P_{n_1}$and P_{n_2} be $v_1, v_2, \ldots\ldots, v_{n_1}$ and $u_1, u_2, \ldots\ldots, u_{n_2}$, respectively. Then $M = \{v_1, v_{n_1}, u_1, u_{n_2}\}$ is an outer connected detour monophonic dominating set of G. And so $\gamma_{ocdm}(G) \leq 4$. We prove that $\gamma_{ocdm}(G) = 4$ .On the contrary suppose that $\gamma_{ocdm}(G) \leq 3$.Then there exists a γ_{ocdm}-set M' such that $|M'| < 3$. Then $J_{dm}[G] \neq V(G)$,which is a contradiction. Therefore $\gamma_{ocdm}(G) = 4$.

■

Theorem 2.16. Let $G = C_{p_1} + C_{p_2}$ where $p_1 \geq 4$ and $p_2 \geq 4$. Then $\gamma_{ocdm}(G) = 3$.

Proof. Let x be a vertex of $C_{p_1}$and y be a vertex of C_{p_2} Then there exists a vertex x_1 in G such that $x_1 \notin J_{dm}[x,y]$ and so $\gamma_{ocdm}(G) \geq 3$, Let $M = \{x, y_1, x_1\}$ Then M is a γ_{ocdm}-set of G.So that $\gamma_{ocdm}(G) = 3$. ■

Theorem 2.17. $G = P_{n_1} + C_{n_2}$, $n_1 \geq 3, n_2 \geq 4$. When this happens

$$\gamma_{ocdm}(G) = \begin{cases} 2, & \begin{matrix} n_1 = 3, n_2 = 4 \\ or \\ n_1 = 3, n_2 = 5 \\ or \\ n_1 = 4, n_2 = 5 \end{matrix} \\ 3, & n_1 \geq 4, n_2 \geq 6 \end{cases}$$

Proof. Suppose $P_{n_1}$is $v_1, v_2, \ldots\ldots, v_{n_1}$ and C_{n_2}, is $u_1, u_2, \ldots\ldots, u_{n_2}$.

Case (i): $n = 3\ and\ m = 4$. Let $M = \{u_1, u_3\}$. So that $\gamma_{ocdm}(G) = 2$, M is an outer connected detour monophonic dominating set of G .

Case (ii): $n = 3\ and\ m = 5$. . Let $M = \{u_1, u_4\}$. So that $\gamma_{ocdm}(G) = 2$, M is an outer connected detour monophonic dominating set of G .

Case (iii): $n = 4$ and $m = 5$ Let $M = \{u_1, u_3\}$. So that $\gamma_{ocdm}(G) = 2$, M is an outer connected detour monophonic dominating set of G .

Case (iv): Let $M = \{v_1, v_{p_1}, u_1\}$. So that $\gamma_{ocdm}(G) \leq 3$, M is an outer connected detour monophonic dominating set of G . We establish that $\gamma_{ocdm}(G) = 3$. Assume that $\gamma_{ocdm}(G) = 2$. When this occurs, a γ_{ocdm}-set M_1 exists such that $|M_1| = 2$. If $M_1 \subseteq V(P_1)$, then since $p_1 \geq 4$, none of the items in P_1 are completely dominated by M_1 elements. If $M_1 \subseteq V(C_{p_2})$, Since $p_2 \geq 6$, the elements of C_2 are not dominated by M_1 if $M_1 \subseteq V(C_{p_2})$, A contradiction results if $M_1 \subseteq [V(P_{n_1}) \cup V(C_{n_2})]$, then M_1 is a chord. Therefore, $\gamma_{ocdm}(G) = 3$. ■

Theorem 2.18. Let G be a connected graph with $n \geq 3$ nodes. Then $\gamma_{ocdm}(G) = 2$ if and only if there exists a $\gamma_{ocdm} -$ set $M = \{u, v\}$ such that $d_m(u, v) \leq 3$.

Proof. Consider that $\gamma_{ocdm}(G) = 2$. Suppose that $M = \{u, v\}$ is a γ_{ocdm} −set of G. Assume that $d_m(u, v) \geq 4$. Consequently, the monophonic diametral path has a minimum of three internal vertices. Hence, $\gamma_{ocdm}(G) \geq 3$, which is inconsistent . Thus, $d_m(u, v) \leq 3$. The converse is obvious. ■

Conclusion

This article established a novel detour monophonic distance parameter called the outer connected detour monophonic domination number of a graph. We will develop this concept to incorporate more distance considerations in a subsequent investigation.

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