A Study on special (α, β)-metric

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Abstract – Studying the locally dually flat (ldf) special (α, β)-metric F satisfying $F(\alpha,\beta) = c\alpha^2 + \beta + c\beta^2$, where c ϵ (0,1) is undertaken by considering a special (α, β)-metric where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian Metric (RM) and $\beta = b_i(x)y^i$ is a differential one form.

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1. Introduction

It was M. Matsumoto who introduced the concept of (α, β)- metrics in 1972. Further study was done by M. Hashiguchi, Y. Ichijyo, S. Kikuchi, C. Shibata and others ([2]-[5] and [7,8]).

Using Finsler metrics, we study locally dual flatness in information geometry. S.I. Amari and H. Nagaoka [1] describe an approach to locally dually flat metrics. Later, when studying information geometry on Riemannian spaces, Z. Shen [6] developed an idea of ldf to Finsler metrics. A study of ldf metrics and ldf metrics with curvature properties was conducted by X Cheng, Z Shen, Y Zhou and Y Tian [9, 10, 11].

2. Preliminaries

2.1. Definition

We say a Finsler metric F as ldf if there exists a system of coordinates (x^i) at every point, where the spray coefficients look like

$$G^i = -\frac{1}{2} g^{ij} H_{y^j} \qquad (2.1)$$

In an adapted coordinate system, H = H(x, y) is a local scalar function on M's tangent bundle TM.

2.2. Definition

A Finsler metric on a manifold M is a smooth function L: TM\{0} → [0,∞) with the subsequent conditions:

(a) L is smooth on TM\{0} (Regularity

(b) For λ > 0, L(x, λy) = λL(x, y) (Positive homogeneity)

(c) For all (x, y) ∈ TM\{0}, fundamental tensor g^{ij} (x, y) is positive definite, where $g^{ij} = \frac{1}{2}[L^2]_{y^i y^j}(x,y)$. (Strong convexity)

2.3. Definition

An (α, β)-metric L of a Finsler space F^n = (M, D, L) is a positively homogeneous function of degree one in two arguments $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$, where α is a Riemannian metric and β is a differential one form.

2.4. Lemma ([6])

The Finsler metric L = L(x, y) on an open subset ℧ ⊂ R^n is ldf if and only if it meets the following conditions:

$$[L^2]_{x^k y^l} y^k - 2[L^2]_{x^m} y^m = 0 \tag{2.2}$$

In (2.1), H = H(x, y) is given by H = $\frac{1}{6}[L^2]_{x^m} y^m$.

3. Locally dually flatness of special (α, β)-metric

This section examines the dual flatness of the given special (α, β)-metric locally and proves the following theorem.

3.1. Theorem

The special (α, β)-metric F on a manifold M satisfying $F(\alpha,\beta) = c\alpha^2 + \beta + c\beta^2$, where c ϵ (0,1) , $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential one form is locally dually flat if and only if the following conditions are satisfied by α and β in an adapted coordinate system.

$$G_\alpha^m = 2\beta\theta^m - \beta^2 b^m \tau - b^m r_{oo} + \frac{1}{2c}(3s_o^m - \theta^m - r_o{}^m) \tag{3.1}$$

$$r_{oo} = \beta^2\tau - b^m\theta^m\left(2\beta + \frac{1}{2c}\right) + \frac{1}{2c}(3s_o^m - r_o^m) \tag{3.2}$$

where the scalar function is τ = τ (x) and the one form on M $\theta^m = a^{mi}\theta_i$ is $\theta = \theta_k y^k$.

Proof: Let the special (α, β)-metric F satisfy $F(\alpha,\beta) = c\alpha^2 + \beta + c\beta^2$ where c ϵ (0,1), is locally dually flat on an open subset ℧ ⊂ R^n . We have the identities:

$$\alpha_{x^k} = \frac{y_m}{\alpha}\frac{\partial G_\alpha^m}{\partial y^k} \text{ and } \beta_{x^k} = b_{m/k} y_m + b_m \frac{\partial G_\alpha^m}{\partial y^k} \tag{3.3}$$

where $y^k = a_{jk}y^j$. By direct computation, we obtain

$$[F]_{x^k} = \frac{\partial G_\alpha^m}{\partial y^k}(2cy_m + b_m + 2c\beta b_m) + b_{m/k} y_m(1 + 2c\beta) \tag{3.4}$$

$$[F]_{x^k y^k} y^l = G_\alpha^m(2ca_{mk} + 2cb_k b_m) + \frac{\partial G_\alpha^m}{\partial y^k}(2cy_m + b_m + 2c\beta b_m) + b_{k/o}(1 + 2c\beta) + 2cb_k r_{oo} \tag{3.5}$$

Substituting (3.4) and (3.5) into (2.2), we obtain

$$A_1 G_\alpha^m - B_1 \frac{\partial G_\alpha^m}{\partial y^k} + C_1 r_{oo} + D_1\left(b_{k/o} - 2b_{0/K}\right) = 0 \tag{3.6}$$

where,

$$A_1 = (2ca_{mk} + 2cb_k b_m)$$
$$B_1 = (2cy_m + b_m + 2c\beta\, b_m)$$
$$C_1 = 2cb_k$$
$$D_1 = (1 + 2c\beta)$$

Multiplying (3.6) by α^3 and rewriting as a polynomial in α, we have

$$\alpha^3 A_2 = 0 \tag{3.7}$$

where,

$$A_2 = G_\alpha^m(2ca_{mk} + 2cb_k b_m) - \frac{\partial G_\alpha^m}{\partial y^k}(2cy_m + b_m + 2c\beta\, b_m) + 2cb_k r_{oo} +$$
$$(1 + 2c\beta)(3s_{ko} - r_{ko})$$

From (3.7), we have the coefficients of α zero. This results in obtaining the coefficients of α^3 as zero too. That is

$$A_2 = G_\alpha^m(2ca_{mk} + 2cb_k b_m) - \frac{\partial G_\alpha^m}{\partial y^k}(2cy_m + b_m + 2c\beta\, b_m) + 2cb_k r_{oo} +$$

$$(1 + 2c\beta)(3s_{ko} - r_{ko}) = 0 \ \ (i.e, A_2 = 0) \tag{3.8}$$

Note that

$$y_m \frac{\partial G_\alpha^m}{\partial y^k} = \frac{\partial (y_m G_\alpha^m)}{\partial y^k} - a_{mk} G_\alpha^m \tag{3.9}$$

$$b_m \frac{\partial G_\alpha^m}{\partial y^k} = \frac{\partial (b_m G_\alpha^m)}{\partial y^k} \tag{3.10}$$

Contracting (3.8) b_k with and by using (3.9) and (3.10), we get

$$6cb_m G_\alpha^m + 2cr_{oo} + (1 + 2c\beta)(3s_o - r_o) = 2c\, y_m \frac{\partial G_\alpha^m}{\partial y^k} b_k +$$
$$(1 + 2c\beta)\, b_m \frac{\partial G_\alpha^m}{\partial y^k} b_k \tag{3.11}$$

Multiplying (3.11) by β^2 and reducing it into,

$$(0\,\alpha^2 - 2c\beta^2)\, A_3 = (0\,\alpha^2 - \beta^2) B_3 \tag{3.12}$$

where,

$$A_3 = -(\, y_m \frac{\partial G_\alpha^m}{\partial y^k} b_k + \beta b_m \frac{\partial G_\alpha^m}{\partial y^k} b_k - 3b_m G_\alpha^m - r_{oo} - \beta(3s_o - r_o))$$
$$B_3 = -(\, b_m \frac{\partial G_\alpha^m}{\partial y^k} b_k + (3s_o - r_o))$$

We know that ($\alpha^2 - 2c\beta^2$) and ($\alpha^2 - \beta^2$) are irreducible polynomial of (y^i) and there is a function τ = τ (x) on M such that

$$\tau(2c\beta^2) = b_m \frac{\partial G_\alpha^m}{\partial y^k} b_k + (3s_o - r_o) \tag{3.13}$$

$$\tau(\beta^2) = y_m \frac{\partial G_\alpha^m}{\partial y^k} b_k + \beta b_m \frac{\partial G_\alpha^m}{\partial y^k} b_k - 3b_m G_\alpha^m - r_{oo} - \beta(3s_o - r_o) \quad (3.14)$$

Then equation (3.14) can be written as

$$b_m G_\alpha^m = \frac{1}{3}\left(y_m \frac{\partial G_\alpha^m}{\partial y^k} b_k + \beta b_m \frac{\partial G_\alpha^m}{\partial y^k} b_k - \beta^2 \tau - r_{oo} - \beta(3s_o - r_o)\right) \quad (3.15)$$

$$b_m G_\alpha^m = \theta \quad (3.16)$$

where $\theta = \theta_k y^k$ is a 1-form on M. Then

$$b_m \frac{\partial G_\alpha^m}{\partial y^k} = \theta_k \quad (3.17)$$

$$y_m \frac{\partial G_\alpha^m}{\partial y^k} = 2\theta_k \beta + \beta^2 b_k \tau + b_k r_{oo} + \beta(3s_o - r_o) b_k \quad (3.18)$$

By using (3.16), (3.17) and (3.18), the equations (3.8) become

$$2ca_{mk} G_\alpha^m - 4c\theta_k \beta - 2c\beta^2 b_k \tau - 2cb_k r_{oo} - \theta_k + (3s_o - r_o) = 0 \quad (3.19)$$

Contracting (3.19) with a^{lk} yields

$$2cG_\alpha^l - 4c\beta\theta^l - 2c\beta^2 b^l \tau - 2cb^l r_{oo} + 3s_o^m - \theta^m - r_o^m \quad (3.20)$$

Then, from (3.20), we obtain the necessary condition (3.1) of the theorem.

Substituting (3.1) into (3.16), we obtain final required condition (3.2) of the theorem.

By assuming that the three conditions (3.1) and (3.2) hold true, we can prove the sufficient condition by direct computation. This completes the proof of the Theorem 3.1.

4. Conclusion

There is an important role for locally dually flat Finsler metrics in studying a flat Finsler information structure, for they have a special geometric approach. The purpose of the paper is to identify the necessary and sufficient conditions that allow a special metric to be locally dual.

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