A Single Server Queue with Generalized Coxian-2 Service with Random Breakdowns and Delayed Repairs

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Abstract

The paper studies the steady state behavior of a batch arrival single service channel queue in which the first service with general service time is essential and the second service with general service time is optional. We term such a two-phase service as generalized Coxian-2 service. It is further assumed that the service channel is subject to random failures occurring only while the server is proving service. As soon as there is a breakdown, the service of a customer is suspended and unlike most of other papers which assume that the server immediately undergoes a repair process we assume that the repair process does not start immediately after a breakdown occurs but there is a delay in starting the repairs. This delay time follows an exponential distribution with mean delay time $1/\gamma,\ \gamma > 0$. As soon as the repair process of the service channel is complete, the server takes up the customer at the head of the queue and resumes providing the first essential service to the customer at the head of the queue. We obtain steady state probability generating functions for the queue size at a random epoch of time. Some particular cases of interest have been discussed.

Index Terms— Generalized Coxian-2 service, random failures, delay time, steady state, queue size.

1. Introduction

Service interruptions in queueing systems are a common phenomenon. These service interruptions may be caused by random failures, stoppages for overhauling systems or vacations taken by servers from time to time. Takine and Sengupta [12] and Nunez-Queija. [11], Vinck and Bruneel [13] studied queues with service interruptions. Fadhil, R et al [1], Hur and Ahn [3], Ke [5] and Madan and Abu Al-Rub [8] have studied queueing systems with various types of vacations. Madan [7] and Madan et al [9] studied a single server as well a two-server queues in which a service channel may fail any time, not only while it is working but it may even fail also when it is idle. However, in the present paper we assume that the server may undergo a random breakdown only when it is providing service. Federgruen and So [2], Lofti et al [6], Madan et al [8], studied different queueing systems subject to breakdowns. Jayewardene and Kella [4] studied M/G/∞ queues with altering renewal breakdowns. In this paper, we study a queueing system in which a single server provides generalized Coxian-2 service which involves first essential phase of service followed by the second optional phase of service. As a result of a random failure, there is a delay in starting the repairs. We assume that the delay time follows an exponential distribution with mean delay time $1/\gamma,\ \gamma > 0$. For papers on queuing systems with breakdown and delay in starting repairs, we refer the reader to Fadhil et al [1] and Madan [10]. The repair process follows a general repair time distribution and the customer whose service is interrupted instantly goes back to the head of the queue. As soon the repairs of the service channel are complete, the server starts providing the first essential phase to customer at the head of the queue. In case, on completion of repairs the server finds the queue empty, then it waits in the system for a new batch of customers to arrive. all over again.

The Mathematical Model

The mathematical model of our study is briefly described by the following underlying assumptions:

- Customers arrive at the system in batches of variable size in accordance with a compound Poisson process. Let $\lambda c_i dt$ ($i = 1, 2, 3, \ \cdots$) be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \le c_i \le 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. The arriving batches wait in the queue in the order of their arrival. It is further assumed that customers with each batch are pre-ordered for the purpose of service.
- There is a single server who provides generalized Coxian-2 service which means essential first phase of service followed by optional second phase of service. The first phase of service is provided to all customers one by one on a first-come, first-served basis. Let $A_1(x)$ and $a_1(x)$ respectively be the distribution function and the density function of the first phase service time and let $\mu_1(x)dx$ be the conditional probability of completion of first phase service, given that the elapsed time is x , so that

$$\mu_1(x) = \frac{a_1(x)}{1 - A_1(x)} \tag{2.1}$$

and, therefore,

$$a_1(v) = \mu_1(v) e^{-\int_0^v \mu_1(x)dx} . \tag{2.2}$$

- After completion of the first phase of service, the server provides second phase of service which is optional. A customer may take second phase of service with probability α or may leave the system with probability $1-\alpha$ Let $A_2(v)$ and $a_2(v)$ respectively be the distribution function and the density function of the second phase service time and let $\mu_2(x)dx$ be the conditional probability of completion of second phase service, given that the elapsed time is x , so that

$$\mu_2(x) = \frac{a_2(x)}{1 - A_2(x)} \tag{2.3}$$

and, therefore,

$$a_2(v) = \mu_2(v)e^{-\int_0^v \mu_{12}(x)dx} . \qquad (2.4)$$

- During the period when the server is providing the first or the second phase of service, the server is subject to random breakdowns. We assume that a breakdown occurs only while providing a service. Let βdt be the probability that there will be breakdown during the interval of time (t, dt]. As soon a breakdown occurs, the customer whose service is interrupted goes back to the head of the queue and waits for the server to return.
- On the occurrence of a breakdown, the repairs process does not start immediately. There is delay in starting the repairs. We assume that the delay time D follows an exponential distribution with mean delay time $1/\gamma,\ \gamma > 0$ and therefore $\gamma\, dt$ is the probability that the delay period is over during the period of time (t, dt]
- Let $B(v)$ and $b(v)$ respectively be the distribution function and the density function of the repair time and let $\delta(x)dx$ be the conditional probability of completion of the repair process, given that the elapsed time is x , so that

$$\delta(x) = \frac{b(x)}{1 - B(x)} \qquad (2.5)$$

and, therefore,

$$b(v) = \delta(v)e^{-\int_0^v \delta(x)\,dxx} . \qquad (2.6)$$

- As soon as the repair process is complete, the server instantly takes up the customer, if any, at the head of the queue and resumes providing service. If there is no customer waiting for service, the server joins the queue and remains idle till a new batch of customers arrives.
- Various stochastic processes involved in the system are independent of each other.

Definitions and Notations

We assume that $W_n^{(j)}(x,t)$, $j = 1, 2$ is the probability that at time t, there are $n \geq 0$ customers in the queue excluding one customer in j th phase service with elapsed service time x.

Accordingly, $W_n^{(j)}(t) = \int_{x=0}^{\infty} W_n^{(j)}(x,t)\,dx$ denotes the probability that at time t, there are $n \geq 0$ customers in the queue excluding one customer in the j th phase service irrespective of the value of x. Further, we define D_n(t) to be the probability that there are $n \geq 0$ customers in the queue and the server is in the failed state, waiting for repairs to start. Next, we define $F_n(x,t)$ to be the probability that at time t, the server is under repairs since the elapsed time x and there are $n \geq 0$ customers in the queue. Accordingly, $F_n(t) = \int_{x=0}^{\infty} F_n(x,t)\,dx$ denotes the probability that at time t, there are $n \geq 0$ customers in the queue and the service channel is under repairs irrespective of the value of x. Further, let $P_n(t) = \sum_{j=1}^{2} W_n^{(j)}(t) + F_n(t) + D_n(t)$ denote the probability that at time t there are n (≥0) customers in the queue irrespective of whether the server is providing service or waiting for repairs to start or is under repairs. Finally, let Q(t) be the probability that at time t, there is no customer in the system and the server is idle.

Steady State Equations Governing the System

Let

$\lim_{t\to\infty} W_n^{(j)}(x,t) = W_n^{(j)}(x)$, $\lim_{t\to\infty} W_n^{(j)}(t) = W_n^{(j)}$,

$\lim_{t\to\infty} F_n(x,t) = F_n(x)$, $\lim_{t\to\infty} F_n(t) = F_n$

$\lim_{t\to\infty} D(t) = D_n$,

$\lim_{t\to\infty} P_n(t) = \sum_{j=1}^{2} \lim_{t\to\infty} W_n^{(j)}(\boldsymbol{t}) + \lim_{t\to\infty} F_n(t) + \lim_{t\to\infty} D(t) = P_n$, $j = 1, 2$

$\lim_{t\to\infty} Q(t) = Q$ (4.1)

denote the corresponding steady state probabilities.

Then following usual probability reasoning based on the underlying assumptions of the model, the system has the following set of integra differential-difference forward equations:

Steady State Queue Size at a Random Epoch

We define the following probability generating functions (PGFs):

$W^{(j)}(x,z) = \sum_{n=0}^{\infty} z^n W_n^j(x)$,

$W^{(j)}(z) = \sum_{n=0}^{\infty} z^n W_n^{(j)}(x)$, j=1,2

$F(x,z) = \sum_{n=0}^{\infty} z^n F_n(x)$,

$F(z) = \sum_{n=0}^{\infty} z^n F_n$

$D(z) = \sum_{n=0}^{\infty} z^n D_n$

$P(z) = \sum_{j=1}^{2} W^{(j)}(z) + F(z) + D(z)$

$C(z) = \sum_{i=1}^{\infty} z^i c_i$, $|z| \leq 1$. (5.1)

Multiplying equation (4.1) by z^n, summing over n and adding the result to (4.2) and using (5.1) we get

$$\frac{d}{dx} W^{(1)}(x,z) + (\lambda + \beta + \mu_1(x) - \lambda C(z)) W^{(1)}(x,z) = 0 \quad (5.2)$$

Similar operation on equations (4.3) and (4.4), (4.5) and (4.6), and (4.7) and (4.8) yield

$$\frac{d}{dx} W^{(2)}(x,z) + (\lambda + \beta + \mu_2(x) - \lambda C(z)) W^{(2)}(x,z) = 0 \quad (5.3)$$

$$\big(\lambda + \gamma - \lambda C(z)\big) D(z) = \beta z (W^{(1)}(z) + W^{(2)}(z)) \quad (5.4)$$

$$\frac{d}{dx} F(x,z) + (\lambda + \delta(x) - \lambda C(z)) F(x,z) = 0 \quad (5.5)$$

Next, we perform the similar operations on the boundary conditions (4.9), (4.10) and (4.11) make use of equation (4.8). Thus, we get

$$zW^{(1)}(0,z) = (1-\alpha)\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx$$
$$+\int_0^\infty W^{(2)}(x)\mu_2(x)dx +$$
$$\int_0^\infty F(x,z)\delta(x)dx - (\lambda - \lambda C(z))Q \quad (5.6)$$
$$W^{(2)}(0,z) = \alpha\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx \quad (5.7)$$
$$F(0,z) = \gamma D(z) \quad (5.8)$$

Now we integrate equations (5.2), (5.3) and (5.5) between the limits *0* and x and obtain

$$W^{(1)}(x,z) = W^{(1)}(0,z)\, exp\left[-kx - \int_0^x \mu_1(t)dt\right] \quad (5.9)$$
$$W^{(2)}(x,z) = W^{(2)}(0,z)\, exp\left[-kx - \int_0^x \mu_2(t)dt\right] \quad (5.10)$$
$$F(x,z) = F(0,z)\, exp\left[-lx - \int_0^x \delta(t)dt\right] \quad (5.11)$$

Where $k = (\lambda + \beta - \lambda C(z)),\ l = (\lambda - \lambda C(z))$ and $W^{(1)}(0,z)$, $W^{(2)}(0,z)$ and $F(0,z)$ are given above in equations (5.6) and (5.7) respectively.

Next, we again integrate equations (5.9) to (5.11) with respect to *x* by parts and obtain

$$W^{(1)}(z) = W^{(1)}(0,z)\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right) \quad (5.12)$$
$$W^{(2)}(z) = W^{(2)}(0,z)\left(\frac{1-\bar{A}^{(2)}[k]}{k}\right) \quad (5.13)$$
$$F(z) = F(0,z)\left(\frac{1-\bar{B}[l]}{l}\right) \quad (5.14)$$

Where $\bar{A}^{(j)}[k] = \int_0^\infty e^{-kx} dA^{(j)}(x)$, $j = 1,2$ is the Laplace-Steiltjes transform of the jth phase service time and $\bar{B}[l] = \int_0^\infty e^{-lx} dB(x)$ is the Laplace-Steiltjes transform of the repair time.

Now we shall determine the integrals $\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx$, $\int_0^\infty W^{(2)}(x,z)\mu_2(x)dx$ and $\int_0^\infty F(x,z)\delta(x)dx$ appearing in the right sides of equations (5.6) and (5.7). For this purpose, we multiply equations (5.9), (5.10) and (5.11) by $\mu_1(x)$, $\mu_2(x)$ and $\delta(x)$ respectively and integrate each with respect to *x*. Thus, we obtain

$$\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx = W^{(1)}(0,z)\bar{A}^{(1)}[k] \quad (5.15)$$
$$\int_0^\infty W^{(2)}(x,z)\mu_2(x)dx = W^{(2)}(0,z)\bar{A}^{(2)}[k] \quad (5.16)$$
$$\int_0^\infty F(x,z)\delta(x)dx = F(0,z)\bar{B}[l] \quad (5.17)$$

Utilizing equations (5.15) to (5.17) into equations (5.6) and (5.7), we get on simplifying

$$\left[z - (1-\alpha)\bar{A}^{(1)}[k]\right] W^{(1)}(0,z)$$

$$= W^{(2)}(0,z)\bar{A}^{(2)}[k] + F(0,z)\bar{B}[l] - lQ \tag{5.18}$$

$$W^{(2)}(0,z) = \alpha W^{(1)}(0,z)\bar{A}^{(1)}[k], \tag{5.19}$$

Further, using (5.4). equation (5.8) gives

$$F(0,z) = \left[\frac{\gamma\beta z}{(\lambda+\gamma-\lambda C(z))}\right]\left[W^{(1}(z) + W^{(2)}(z)\right], \tag{5.20}$$

Now, we use (5.19) into (5.18) and simplify. Thus, we obtain

$$\left[z-(1-\alpha)\bar{A}^{(1)}[k] - \alpha\bar{A}^{(1)}[k]\bar{A}^{(2)}[k]\right] W^{(1)}(0,z) = F(0,z)\bar{B}[l] - lQ \tag{5.21}$$

Which gives

$$W^{(1)}(0,z) = \frac{F(0,z)\bar{B}[l]-lQ}{\left[z-(1-\alpha)\bar{A}^{(1)}[k]-\alpha\bar{A}^{(1)}[k]\bar{A}^{(2)}[k]\right]} \tag{5.22}$$

Next, utilizing (4.12) and (4.20) into (4.22), we obtain

$$\frac{W^{(1)}(z)}{\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)} = \frac{\left[\frac{\gamma\beta z}{(\lambda+\gamma-\lambda C(z))}\right]\left[W^{(1}(z)+W^{(2)}(z)\right]\bar{B}[l]-lQ}{\left[z-(1-\alpha)\bar{A}^{(1)}[k]-\alpha\bar{A}^{(1)}[k]\bar{A}^{(2)}[k]\right]} \tag{5.23}$$

Which gives

$$W^{(1)}(z) = \frac{\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)\left[\frac{\gamma\beta z}{(\lambda+\gamma-\lambda C(z))}\right]\left[W^{(1}(z)+W^{(2)}(z)\right]\bar{B}[l]-lQ}{\left[z-(1-\alpha)\bar{A}^{(1)}[k]-\alpha\bar{A}^{(1)}[k]\bar{A}^{(2)}[k]\right]} \tag{5.24}$$

Now, from (5.19), using (5.12) and (5.13), we get

$$\frac{W^{(2)}(z)}{\left(\frac{1-\bar{A}^{(2)}[k]}{k}\right)} = \alpha\bar{A}^{(1)}[k]\left[\frac{W^{(1)}(z)}{\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)}\right] \tag{5.25}$$

Which gives

$$W^{(2)}(z) = \alpha\bar{A}^{(1)}[k]\left[\frac{W^{(1)}(z)}{\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)}\right]\left(\frac{1-\bar{A}^{(2)}[k]}{k}\right) \tag{5.26}$$

Next, using (5.26) into (5.24), we obtain

$$\left[1 - \frac{\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)\left[\frac{\gamma\beta z}{(\lambda+\gamma-\lambda C(z))}\right]\bar{B}[l]\left[1+\frac{\alpha\bar{A}^{(1)}[k]\left(\frac{1-\bar{A}^{(2)}[k]}{k}\right)}{\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)}\right]}{\left[z-(1-\alpha)\bar{A}^{(1)}[k]-\alpha\bar{A}^{(1)}[k]\bar{A}^{(2)}[k]\right]}\right] W^{(1)}(z)$$

$$= \frac{-lQ}{\left[z-(1-\alpha)\bar{A}^{(1)}[k]-\alpha\bar{A}^{(1)}[k]\bar{A}^{(2)}[k]\right]}$$

Which gives

$$W^{(1)}(z) = \frac{-lQ}{\left\{\begin{matrix}\left[z-(1-\alpha)\bar{A}^{(1)}[k]-\alpha\bar{A}^{(1)}[k]\bar{A}^{(2)}[k]\right] \\ -\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)\left[\frac{\gamma\beta z}{(\lambda+\gamma-\lambda C(z))}\right]\bar{B}[l]\left[1+\frac{\alpha\bar{A}^{(1)}[k]\left(\frac{1-\bar{A}^{(2)}[k]}{k}\right)}{\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)}\right]\end{matrix}\right\}} \tag{5.27}$$

Now, we use (5.27) into (5.26), simplify and get

$$W^{(2)}(z)=\frac{-lQ\left[\frac{\alpha\bar{A}^{(1)}[k]}{1-\bar{A}^{(1)}[k]}\right]\langle 1-\bar{A}^{(2)}[k]\rangle}{\left\{\begin{array}{c}\left[z-(1-\alpha)\bar{A}^{(1)}[k]-\alpha\bar{A}^{(1)}[k]\bar{A}^{(2)}[k]\right]\\ -\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)\left[\frac{\gamma\beta z}{(\lambda+\gamma-\lambda C(z))}\right]\bar{B}[l]\left[1+\frac{\alpha\bar{A}^{(1)}[k]\left(\frac{1-\bar{A}^{(2)}[k]}{k}\right)}{\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)}\right]\end{array}\right\}} \tag{5.28}$$

Then using (5.14), (5.26) and (4.27) into (5.20) we obtain

$$F(z)=\left[\frac{\gamma\beta z}{(\lambda+\gamma-\lambda C(z))}\right]\left[\frac{-lQ\left\{1+\left[\frac{\alpha\bar{A}^{(1)}[k]}{1-\bar{A}^{(1)}[k]}\right]\langle 1-\bar{A}^{(2)}[k]\rangle\right\}\left(\frac{1-\bar{B}[l]}{l}\right)}{\left\{\begin{array}{c}\left[z-(1-\alpha)\bar{A}^{(1)}[k]-\alpha\bar{A}^{(1)}[k]\bar{A}^{(2)}[k]\right]\\ -\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)\left[\frac{\gamma\beta z}{(\lambda+\gamma-\lambda C(z))}\right]\bar{B}[l]\left[1+\frac{\alpha\bar{A}^{(1)}[k]\left(\frac{1-\bar{A}^{(2)}[k]}{k}\right)}{\left(\frac{1-\bar{A}^{(1)}[k]}{k}\right)}\right]\end{array}\right\}}\right] \tag{5.29}$$

Finally, all the generating functions found in (5.27), (5.28) and (5.29) are completely determined and the only unknown Q appearing in the numerators can be determined by the normalizing condition

$$P(1)=\sum_{j=1}^{2}W^{(j)}(1)+\ F(1)+D(1)+Q=1. \tag{5.30}$$

References

[1] R. Fadhil et al, “An M(X)/G/1 queue with general vacation times, random breakdowns, general delay times and general repair times”, Applied Mathematical Sciences, Vol. 5, No. 1, pp. 35-51, 2011.

[2] A. Federgruen and K. C. So, “Optimal maintenance policies for single-server queueing systems subject to breakdowns” Operations Research, Vol. 38, No. 2, pp.330-343, 1990.

[3] S. Hur and S. Ahn, “Batch arrival queues with vacations and server setup”, Applied Mathematical Modeling”, Vol. 29, pp.1164-1181, 2005.

[4] A. Jayawardene and O. Kella, “M/G/∞ with altering renewal breakdowns”, Queueing Systems”, Vol. 22, pp.79-95, 1996.

[5] J. C. Ke, “Operating characteristic analysis on the M[x] /G/1 system with a variant vacation policy and balking”, Applied Mathematical Modelling, Vol. 31, No. 7, pp.1321-1337, 2007.

[6] L. Tadj and J.C. Ke, “A single service and batch service queue with random breakdowns”, International Services Sciences, Vol.5(2), pp. 116-132. 2014.

[7] Madan, K. C., ‘An M/G/1 Queue with Time-Homogeneous Breakdowns and Deterministic Repair Times’, 29(1), pp 103-110, 2003.

[8] K. C. Madan and A. Z. Abu Al-Rub, “On a single server queue with optional phase type server vacations based on exhaustive deterministic service and a single vacation policy”, Applied Mathematics and Computation, Vol. 149, pp.723-734, 2004.

[9] Madan, K. C., Abu-Dayyeh, W. and Gharaibeh, M., "On two parallel servers with random breakdowns", Soochow Journal of Mathematics, Vol. 29, No. 4, pp.413-423,2003.

[10] K. C. Madan and E. Malalla, "On a batch arrival queue with second optional service, random breakdowns, delay time for repairs to start, and restricted admissibility of arrivals during breakdown periods", Journal of Mathematical and Computational Science, Vol. 7, No. 1, pp. 175-188, 2017.

[11] R. Nunez-Queija, "Sojourn times in a processor sharing queue with service interruptions", *Queueing Systems*, Vol. 34, pp.351-386, 2000.

[12] Takine, T. and Sengupta, B., A single server queue with service interruptions, Queueing Systems, Vol. 26, pp.285-300, 1997.

[13] B. Vinck and H. Bruneel, "System delay versus system content for discrete-time queueing systems subject to server interruptions", European Journal of Operational Research, Vol. 175, pp.362-375, 2006.