Strong Split Monophonic Number of a Graph

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Abstract

In this paper we introduce a new graph theoretic parameter, strong split monophonic number of a graph as follows. A set $S \subseteq V(G)$ is a strong split monophonic set of G, if S is a monophonic set and subgraph induced by $< V - S >$ is totally disconnected. The strong split monophonic number of a graph G is denoted by $m_{ss}(G)$, is the minimum cardinality of the strong split monophonic set of G. Here we investigate the strong split monophonic number for some special graphs. Also we have shown that, for integers r, p with $2 \leq r < p - 1$, there exists a connected graph G of order p such that $m_{ss}(G) = r$. For any three integers r, s, t with $r, s, t \geq 2$ there exist a connected graph G such that $m(G) = r$, $m_s(G) = s, m_{ss}(G) = t$.

Keywords: monophonic number, split monophonic number, strong split monophonic number.

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1.Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by $p\ and\ q$ respectively. For basic graph theoretic terminology we refer to Harary [5]. For vertices u and v in a connected graph G, the distance d(u, v) is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. It is known that d is a metric on the vertex set V of G. The neighborhood of a vertex v is the set $N(v)$ consisting of all points u which are adjacent with v. The closed neighborhood of a point v is the set $N[v] = N(v) \cup \{v\}$. A point v is an extreme point if the subgraph induced by its neighbors is complete. The closed interval $I[x, y]$ consists of all points lying on some $x - y$ geodesic of G, while for $S \subseteq V, I[S] = \cup_{x,y \in S} I[x, y]$. A set S of points is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set. The geodetic number of a graph was introduced in [1,6,8] and further studied in [2,4]. The detour distance $D(u, v)$ between two points u and v in G is the length of a longest $u - v$ path in G. An $u - v$ path of length $D(u, v)$ is called an $u - v$ detour[12]. It is known that D is a metric on the point set V of G. The concept of detour distance was introduced and studied in [3].

A chord of a path P is an edge joining two non-adjacent points of P. A path P is called monophonic if it is a chordless path[13,14]. For any two points u and v in a connected graph G, the monophonic distance $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G[13,14]. The monophonic eccentricity $e_m(v)$ of a vertex v in G is $e_m(v) =$ max $\{d_m(u, v) : u \in V(G)\}$. The monophonic radius, $rad_m G$ of G is $rad_m G = min\{e_m(v) : v \in V(G)\}$ and the monophonic diameter, $diam_m G$ of G is $diam_m G$ = max $\{e_m(v): v \in V(G)\}$. A point u in G is monophonic eccentric point of a point v in G if $e_m(v) = d_m(u, v)$ The usual distance d and the detour distance D are metrics on the point set V of a connected graph G, where as the monophonic distance d_m is not a metric on V . The monophonic distance was introduced and studied in [7].A monophonic set of G is a set S ⊆ V(G) such that every point of G is contained in a monophonic path joining some pair of points in S. The monophonic number $m(G)$ of G is the minimum cardinality of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set of G[9,10,15].

A set $S \subseteq V(G)$ is a split monophonic set of G, if S is a monophonic set and subgraph induced by $< V - S >$ is disconnected. The split monophonic number of a graph G is denoted by $m_s(G)$, is the minimum cardinality of the split monophonic set of G[11].

The following theorems are used in the next sections.

Theorem 1.1 ([6]). Each extreme point of a connected graph G belongs to every geodetic set of G.

Theorem 1.2 ([6]). For any tree T with k endpoints, $g(T) = k$.

Throughout this paper G denotes a connected graph with at least two points.

RESULTS AND DISCUSSIONS

In section 2 we discussed about strong split monophonic number of a graph and also explored some theorems based on it. In addition , we have shown strong split monophonic number for some special graphs with proof. In section 3 we discussed about the strong split monophonic number of a graph when adding end edges and in the last section realization results based on the topic where discussed.

2. Strong Split Monophonic Number of a Graph

In this Section we define the strong split monophonic number $m_{ss}(G)$ of a graph and initiate a study of this parameter.

Definition 2.1. A set $S \subseteq V(G)$ is a strong split monophonic set of G, if S is a monophonic set and subgraph induced by $< V - S >$ is totally disconnected. The strong split monophonic number of a graph G is denoted by $m_{ss}(G)$, is the minimum cardinality of the strong split monophonic set of G.

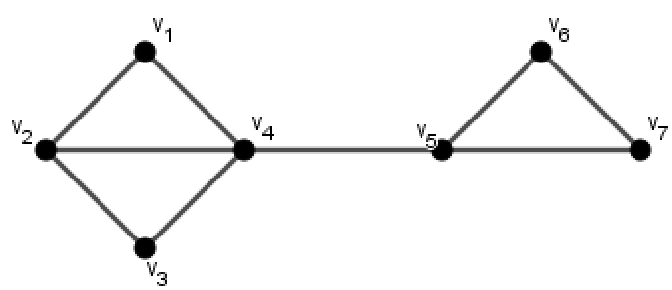

Figure 2.1: G

Example 2.2. Consider the graph G in Figure 2.1. Here the set $P = \{v_1, v_3, v_6, v_7\}$ is the minimum monophonic set, thus $m(G) = 4$ .The set $Q = \{v_1, v_3, v_4. v_6, v_7\}$ is the minimum split monophonic set as well as minimum strong split monophonic set of G, therefore $m_s(G) = m_{ss}(G) = 5$.

Theorem 2.3. *For any path* P_n ,$n \geq 3, m_{ss}(G) = 2 + \left\lceil\frac{n-3}{2}\right\rceil$.

Proof. Let $G = P_n$ and let $F = \{v_1, v_n\}$ be the end points in P_n , $|F| = 2$. Consider $S = F \cup H$ where $H \subseteq V(G) - F$, so that H contains the points with the largest degree and the smallest set of alternating points in $V - F$, $|H| = \left\lceil\frac{n-3}{2}\right\rceil$. Now, S is the minimum set of points covering all points in G. Obviously the set of points of the subgraph $< V - S >$ is totally disconnected, then by the above argument S is a minimum strong split monophonic set of G. Obviously $|S| = |F \cup H| = 2 + \left\lceil\frac{n-3}{2}\right\rceil$.

Theorem 2.5. *For cycle* C_n *of order* $n > 3$,

$$m_{ss}(C_n) = \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ \frac{n+1}{2} & if\ n\ is\ odd \end{cases}$$

Proof. Let $n > 3$, we have the following cases.

Case 1 : Let *n* be even .

Let $v_1, v_2, v_{3,} \dots v_n, v_1$ be a cycle with *n* points where n is even. Let $S = \{v_1, v_3, v_{5,} \dots v_{n-1}\}$ be the monophonic set of alternating points which covers all the points of C_n and for any $v_i \in V - S$, deg $v_i = 0$. Clearly S forms minimum strong split monophonic set of C_n, it follows that $|S| = \frac{n}{2}$.

Hence $m_{ss}(C_n) = \frac{n}{2}$

Case 2. Let *n* be odd.

Let $v_1, v_2, v_{3,} \dots v_n, v_1$ be a cycle with *n* points where *n* is odd. Let $S = (v_1, v_n) \cup (v_{3,}v_5 \dots v_{n-2})$ be the monophonic set of points which covers all the points of $C_n$and for any $v_i \in V - S$, deg $v_i = 0$. Clearly S forms minimal strong split monophonic set of C_n, it follows that $|S| = \frac{n+1}{2}$. Therefore $m_{ss}(C_n) = \frac{n+1}{2}$.

Corollary 2.6. *For cycle C_n of order $n > 3$, $m_{ss}(C_n)$=$\alpha_o(C_n)$.*

Proof. We have the following cases.

Case 1: Let *n* be even.

Let $n > 3$, where *n* is even. α_o be the point covering number of C_n. For an even cycle, the point covering number of an even cycle is $\alpha_o(C_n) = \frac{n}{2}$, which is the strong split monophonic number of a cycle graph with even number of points. Hence $m_{ss}(C_n)$=$\alpha_o(C_n)$.

Case2: Let *n* be odd.

Let $n > 3$, where *n* is odd . α_o be the point covering number of C_n. For an odd cycle , the point covering number of an odd cycle is $\alpha_o(C_n) = \frac{n+1}{2}$,which is the strong split monophonic number of a cycle graph with odd number of points.

Hence $m_{ss}(C_n)$=$\alpha_o(C_n)$.

Theorem 2.8 . *For any integers m,n ≥ 2 , $m_{ss}(k_{m,n}) = min\ \{m, n\}$.*

Proof. Let $G = k_{m,n}$, such that $U = \{u_1, u_2, u_{3,} \dots u_m\}$, $W = \{w_1, w_2, w_{3,} \dots w_n\}$ are the partite sets of *G* , where $m \leq n$ and also $V = U \cup W$.Consider $S = U$ for every w_k, $1 \leq k \leq n$ lies on the $u_i - u_j$ monophonic path for $1 \leq i \neq j \leq m$. Since *V-S* is totally disconnected, we have *S* is a strong split monophonic set of *G*. Let $X = \{u_1, u_2, u_{3,} \dots u_{m-1}\}$ be any set of points such that $|X| < |S|$, then *X* is not a monophonic set of *G* , since $u_m \notin I[X]$. It is clear that *S* is a minimum strong split monophonic set of *G.*

Hence, $m_{ss}(k_{m,n}) = min\ \{m, n\}$.

Theorem 2.9. *For a windmill graph $Wd(p, n)$, $p \geq 2$, $n \geq 2$ $m_{ss}(Wd(p, n)) = (p - 1) + n$.*

Proof. Let $Wd(p, n)$ be a windmill graph obtained by joining n copies of complete graph K_p at a shared universal vertex. Thus $V(Wd(p, n)) = (p - 1)n + 1$. All vertices except the universal vertex forms the minimum monophonic set, also the removal of such points results in totally disconnected graph. Hence from the above argument $m_{ss}(Wd(p, n)) = (p - 1) + n$.

Theorem 2.10. *For a complete tripartite graph $k_{l,m,n}$ with l,m,n ≥ 2 then, $m_{ss}(k_{l,m,n}) =$ $\min\ \{l, m\} + \min\{m, n\}$.*

Proof. Let $G = k_{l,m,n}$, such that $U = \{u_1, u_2, u_{3,} \dots u_l\}$, $V = \{v_1, v_2, v_{3,} \dots v_m\}$, $W = \{w_1, w_2, w_{3,} \dots w_n\}$ are the partite sets of *G* , where $l \leq m \leq n$ and also $Z = U \cup V \cup W$.Consider

$S = U$ for every v_i, $1 \le i \le m$ lies on the $u_i - u_j$ monophonic path for $1 \le i \neq j \le l$..Let $X = \{u_1, u_2, u_{3,} \dots u_{l-1}\}$ be any set of points such that $|X| < |S|$, then X is not a monophonic set of G , since $u_r \notin I[X]$. It is clear that S is a minimum strong split monophonic set of G. But $Z - S$ is not totally disconnected . Consider $S' = U \cup V$ .Clearly $Z - S'$ is totally disconnected. From the above argument it is clear that S' is a minimum strong split monophonic set of G.

Hence , $m_{ss}(k_{l,m,n}) = \min\{l, m\} + \min\{m, n\}$.

Theorem 2.11. *For the wheel* $W_{1,n-1}$, $n \ge 5$

$$m_{ss}(W_{1,n-1}) = \begin{cases} \frac{n+2}{2} & if\ n\ is\ even \\ \frac{n+1}{2} & if\ n\ is\ odd \end{cases}$$

Proof. Let $W_{1,n-1}$, $n \ge 5$ be a wheel graph and let $V(W_{1,n-1}) = \{x, u_1, u_2, u_{3,} \dots u_{n-1}\}$ where $deg(x) = n - 1 > 3$. We have the following cases

Case 1.Let n be even. Consider $= \{u_1, u_3\}$, which is a minimum monophonic set of G . But $< V - S >$ is not totally disconnected. Now consider $S_1 = \{x, u_1, u_3, u_{5,} \dots u_{n-1}\}$. Clearly $< V - S_1 >$ is totally disconnected and it has $\frac{n}{2} + 1$ vertices. Hence $m_{ss}(W_{1,n-1}) = \frac{n+2}{2}$

Case 2.Let n be odd. Consider $S = \{u_1, u_3\}$,which is a minimum monophonic set of G . But $< V - S >$ is not totally disconnected. Now consider $S_1 = \{x, u_1, u_3, u_{5,} \dots u_{n-2}\}$. Clearly $< V - S_1 >$ is totally disconnected and it has $\frac{n-1}{2} + 1$ vertices. Hence $m_{ss}(W_{1,n-1}) = \frac{n+1}{2}$.

3. ADDING AN END EDGE

Theorem 3.1. Let G' be the graph obtained by adding end edges $(u_i, v_j), i = 1,2, \dots, n , j = 1,2, \dots, k$ to each point of $G = C_n$ of order $n > 3$ such that $u_i \in G , v_j \notin G$. Then

$$m_{ss}(G) = \begin{cases} \frac{2k+n}{2} & if\ n\ is\ even \\ \frac{2k+n+1}{2} & if\ n\ is\ odd \end{cases}$$

Proof. **Case 1**: Let *n* be even.

Let $v_1, v_2, v_{3,} \dots v_n, v_1$ be a cycle with *n* points where n is even and $\{u_1, u_2, u_{3,} \dots u_k\}$ be the end points incident with the points of the cycle C_n. Let $H = \{u_1, u_2, u_{3,} \dots u_k\}$. Clearly H forms the monophonic set of G. Therefore $|H| = k$. Let $F = \{v_1, v_3, v_{5,} \dots v_{n-1}\}$ where $F \subseteq V(G) - H$, is the set of alternating points of C_n such that for any $v_i \in V - S$, deg $v_i = 0$. It follows that $|F| = \frac{n}{2}$. Let $S = H \cup F$. Clearly S forms the minimum strong split monophonic set of G.

Hence $|S| = |H \cup F| = k + \frac{n}{2} = \frac{2k+n}{2}$. Therefore, $m_{ss}(G) = \frac{2k+n}{2}$, n is even .

Case 2: Let n be odd.

Let $v_1, v_2, v_{3,} \dots v_n, v_1$ be a cycle with n points where n is odd and $\{u_1, u_2, u_{3,} \dots u_k\}$ be the end vertices incident with the points of the cycle C_n. Let H $= \{u_1, u_2, u_{3,} \dots u_k\}$. Clearly H forms the monophonic set of G. Therefore $|H| = k$. Let $F = (v_1, v_n) \cup (v_{3,} v_5 \dots v_{n-2})$ where F $\subseteq$ V(G) $-$ H, is the set of alternating points of C_n such that for any $v_i \in V - S$, deg $v_i = 0$. It follows that $|F| = \frac{n+1}{2}$. Let $S = H \cup F$. Clearly S forms the minimum strong split monohonic set of G.

Hence $|S| = |H \cup F| = k + \frac{n+1}{2} = \frac{2k+n+1}{2}$. Therefore, $m_{ss}(G) = \frac{2k+n+1}{2}$, n is odd.

4. REALIZATION RESULTS

Theorem 4.1. For integers r, p with $2 \leq r < p - 1$, there exists a connected graph G of order p such that $m_{ss}(G) = r$.

Proof. Let $C_7: v_1, v_2, v_{3,} \dots v_7, v_1$ be a cycle of order 7. Let G be the graph obtained from C_7 by adding the points $u_1, u_2 \dots, u_r$ and join each $u_i (1 \leq i \leq r - 1)$ to v_4 and u_r to v_1. The resulting graph G is shown in Figure 4.1. Let $S = \{u_1, u_2 \dots, u_{r-1}, u_r\}$.Clearly S forms the minimum strong split monophonic set of G . Hence $m_{ss}(G) = r$.

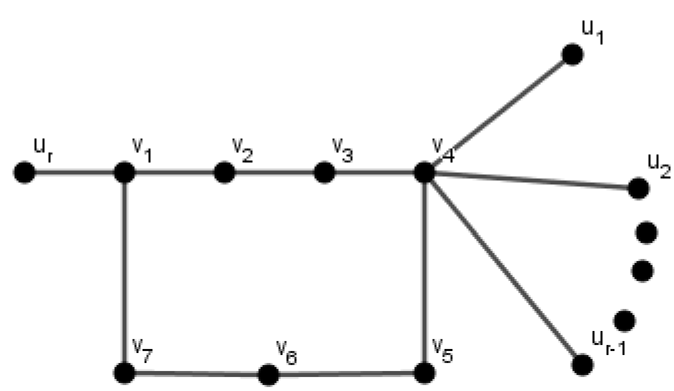

Figure 4.1:G

Theorem 4.2. For any three integers r, s, t with $r, s, t \geq 2$ there exist a connected graph G such that $m(G) = r, m_s(G) = s, m_{ss}(G) = t$.

Proof. Case1. $r = s = t$

Let K_4 be a complete graph with point set $\{v_1, v_2, v_{3,} v_4\}$.The graph G is obtained from K_4 by adding new points $\{u_1, u_2 \ldots.., u_{r-3}\}$ and by joining each $u_i (1 \leq i \leq r-3)$ to v_4.The resulting graph G is shown in Figure 4.2. Let $S = \{u_1, u_2 \ldots.., u_{r-3}, v_1, v_3, v_3\}$. Clearly S forms the minimum monophonic set, minimum split monophonic set, minimum strong split monophonic set of G. Thus $m(G) = m_s(G)$= $m_{ss}(G) = r$.

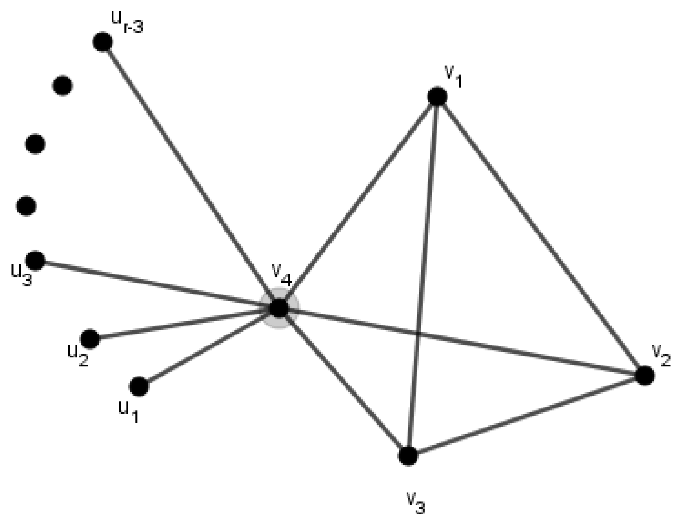

Figure 4.2 :G

Case2. $r < r + 3 = s = t$

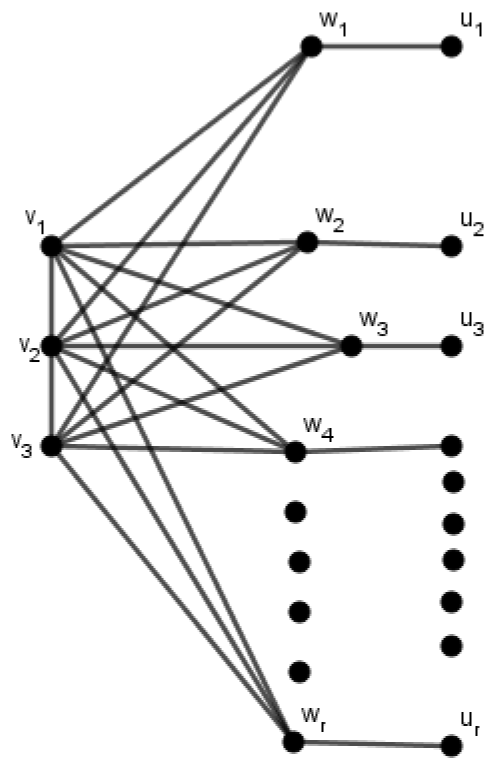

Figure 4.3 :G

Consider $K_{3,r}$ with partite sets $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3 \dots, w_r\}$. Let G be the graph obtained from $K_{3,r}$ by adding new points $u_1, u_2 \dots., u_r$ and adding the edges $w_1u_1, w_2u_2 \dots \dots, w_ru_r$ also v_1v_2 *and* v_2v_3. The resulting graph G is shown in Figure 4.3. Let $S = \{u_1, u_2 \dots., u_r\}$. Clearly S forms the minimum monophonic set of G.Thus $m(G) = r$. Now Consider $S_1 = S \cup \{v_1, v_2, v_3\}$. Clearly S_1 forms the minimum split monophonic set as well as minimum strong split monophonic set of G. Therefore $m_s(G) = r + 1 = s = t = m_{ss}(G)$.

Case3. $r < r + 1 = s < t$

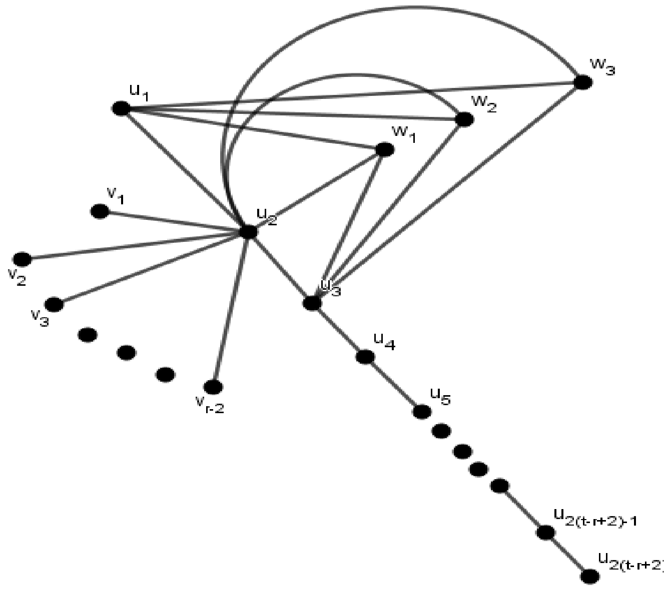

Figure 4.4 :G

Let $P_{2(t-r+2)} : u_1, u_2 \ldots., u_{2(t-r+2)}$ be a path of length $t-r+2$. Add $r+1$ new points, $v_1, v_2 \ldots., v_{r-2}$ and w_1, w_2, w_3 *and* join each $w_i (1 \leq i \leq 3)$to $u_1, u_2$and u_3 also join each vj(1 ≤ j ≤ r-2) to u2, thereby producing the graph G of Figure 4.4. Let $S = \{v_1, v_2 \ldots., v_{r-2}, u_1, u_{2(t-r+2)}\}$. Clearly S forms the minimum monophonic set of G.Hence $m(G) = r$ .Consider $S_1 = S \cup \{u_3\}$. Clearly S_1 forms the minimum split monophonic set of G.Therefore $m_s(G) = r + 1 = s$. Let us consider $S_2 = S_1 \cup \{u_2, u_4, u_6, \ldots., u_{2(t-r+2)-2}, u_{2(t-r+2)}\}$.Clearly S_2 forms the minimum strong split monophonic set of G.Thus $m_{ss}(G) = t$.

5..CONCLUION

In this paper we have examined strong split monophonic number of graphs and also explored strong split monophonic number for some special graphs. We evaluated the comparison between strong split monophonic number and covering number of a graph,as well as realization results pertaining to the topic.The ideas of strong split monophonic number can be extended to study forcing strong split monophonic number of a graph and strong split monophonic domination number of a graph

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