THE KY FAN TYPE INEQUALITIES FOR THE MEANS WHOSE ARGUMENTS LYING ON LINEAR AND CURVED PATH

By

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ABSTRACT

In this paper, the mathematical proof for establishing some new Ky Fan type mean inequalities and strengthen of existing inequalities for the arguments lying on the paths of triangular wave function (linear) and new parabolic function (curved) over the interval (0,1) are discussed and justified using Taylor series.

Keywords: Ky Fan type inequality, linear and curved paths, means, Taylor series.

INTRODUCTION

The Hand Book of Means and their Inequalities, by Bullen [1], contains a tremendous work on mathematical means and the corresponding inequalities involving a huge number of means. The concept of Mathematical means is introduced and studied by Greek Mathematicians school [1, 4]. Lokesha et al. discovered the relationship between series and important means in recent years [5]. In [6-12, 15,16], Nagaraja et al. established good number of inequalities involving means. On the basis of proportion ten greek means were defined, the familiar ones are

$$A =\frac{a+b}{2}, G=\sqrt{ab}, C=\frac{a^2+b^2}{a+b}, H=\frac{2ab}{a+b}, H_e=\frac{1}{3}\left(a+\sqrt{ab}+b\right).$$

For any arbitrary non-negative real numbers $y \in \left]0,\frac{1}{2}\right]$ & $(1-y) \in \left[\frac{1}{2},1\right[$is represented as a function given below,

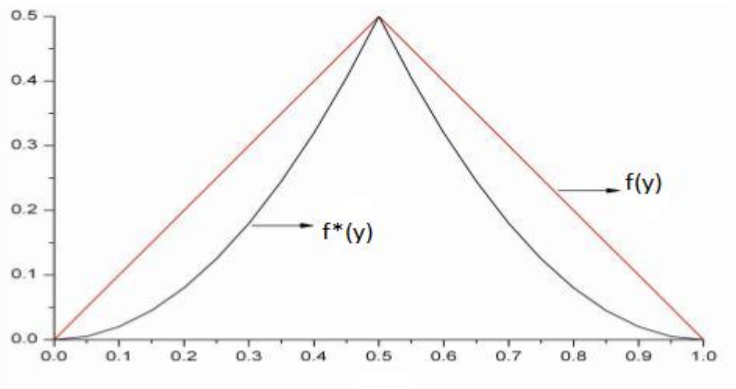
$$f(y)=\begin{cases} y, & 0<y\le\frac{1}{2} \\ (1-y), & \frac{1}{2}\le y<1 \end{cases}$$

The motivation of the work carried out by the eminent researchers [13,14] and discussion with experts, results in the study of a function that is parabolic in nature and is defined as follows:

For any arbitrary non-negative real numbers $y \in \left]0,\frac{1}{2}\right]$ & $(1-y) \in \left[\frac{1}{2},1\right[$is represented as a function given below,

$$f^*(y)=\begin{cases} 2y^2, & 0<y\le\frac{1}{2} \\ 2(1-y)^2, & \frac{1}{2}\le y<1 \end{cases}$$

The functions $f(y)$ and $f^*(y)$ are illustrated graphically below.



For two variables, let $a, b \in \left]0, \frac{1}{2}\right]$ then

$$CT(a,b) = \frac{2}{3}\left(\frac{a^2+ab+b^2}{a+b}\right) \text{ and } CT^{(i)}(a,b) = \frac{3}{2}\left(\frac{ab(a+b)}{a^2+ab+b^2}\right)$$

$$M_{\frac{1}{3}}(a,b) = \left(\frac{a^{\frac{1}{3}}+b^{\frac{1}{3}}}{2}\right)^3 \text{ and } M_{\frac{1}{3}}^{(i)}(a,b) = \frac{ab}{\left(\frac{a^{\frac{1}{3}}+b^{\frac{1}{3}}}{2}\right)^3}$$

$$M_{\frac{2}{3}}(a,b) = \left(\frac{a^{\frac{2}{3}}+b^{\frac{2}{3}}}{2}\right)^{\frac{3}{2}} \text{ and } M_{\frac{2}{3}}^{(i)}(a,b) = \frac{ab}{\left(\frac{a^{\frac{2}{3}}+b^{\frac{2}{3}}}{2}\right)^{\frac{3}{2}}}$$

$$L(a,b) = \frac{b-a}{logb - loga} \text{ and } L^{(i)}(a,b) = \frac{ab(logb-loga)}{b-a}$$

are respectively called Centroidal mean, invariant Centroidal mean, Power mean, invariant power mean, logarithmic mean and invariant logarithmic mean.

For $a' = 1-a, b' = 1-b \in \left[\frac{1}{2}, 1\right[$, the above said means are given by

$$CT'(a,b) = \frac{2}{3}\left(\frac{(1-a)^2+(1-a)(1-b)+(1-b^{)2}}{2-(a+b)}\right)$$

$$CT^{(i)'}(a,b) = \frac{3}{2}\left(\frac{(1-a)(1-b)\big(2-(a+b)\big)}{(1-a)^2+(1-a)(1-b)+(1-b)^2}\right)$$

$$M'_{\frac{1}{3}}(a,b) = \left(\frac{(1-a)^{\frac{1}{3}}+(1-b)^{\frac{1}{3}}}{2}\right)^3$$

$$M_{\frac{1}{3}}^{(i)'}(a,b)=\frac{(1-a)(1-b)}{\left(\frac{(1-a)^{\frac{1}{3}}+(1-b)^{\frac{1}{3}}}{2}\right)^{3}}$$

$$M'_{\frac{2}{3}}(a,b)=\left(\frac{(1-a)^{\frac{2}{3}}+(1-b)^{\frac{2}{3}}}{2}\right)^{\frac{3}{2}}$$

$$M_{\frac{2}{3}}^{(i)'}(a,b)=\frac{(1-a)(1-b)}{\left(\frac{(1-a)^{\frac{2}{3}}+(1-b)^{\frac{2}{3}}}{2}\right)^{\frac{3}{2}}}$$

$$L'(a,b)=\frac{a-b}{\log(1-b)-\log(1-a)}$$

$$L^{(i)'}(a,b)=\frac{(1-a)(1-b)(\log(1-b)-\log(1-a))}{a-b}$$

Let

$$f_1(a,b)=\frac{CT^{(i)}(a,b)}{CT^{(i)'}(a,b)}=\left(\frac{ab(a+b)}{a^2+ab+b^2}\right)\left(\frac{(1-a)^2+(1-a)(1-b)+(1-b)^2}{(1-a)(1-b)\big(2-(a+b)\big)}\right)$$

$$f_2(a,b)=\frac{M_{\frac{2}{3}}^{(i)}(a,b)}{M_{\frac{2}{3}}^{(i)'}(a,b)}=\left(\frac{ab}{(1-a)(1-b)}\right)\left(\frac{(1-a)^{\frac{2}{3}}+(1-b)^{\frac{2}{3}}}{a^{\frac{2}{3}}+b^{\frac{2}{3}}}\right)^{\frac{3}{2}}$$

$$f_3(a,b)=\frac{M_{\frac{1}{3}}^{(i)}(a,b)}{M_{\frac{1}{3}}^{(i)'}(a,b)}=\left(\frac{ab}{(1-a)(1-b)}\right)\left(\frac{(1-a)^{\frac{1}{3}}+(1-b)^{\frac{1}{3}}}{a^{\frac{1}{3}}+b^{\frac{1}{3}}}\right)^{3}$$

$$f_4(a,b)=\frac{L^{(i)}(a,b)}{L^{(i)'}(a,b)}=\left(\frac{ab}{(1-a)(1-b)}\right)\left(\frac{logb-loga}{\log(1-a)-\log(1-b)}\right)$$

$$f_5(a,b)=\frac{L(a,b)}{L'(a,b)}=\left(\frac{\log(1-b)-\log(1-a)}{loga-logb}\right)$$

$$f_6(a,b)=\frac{M_{\frac{1}{3}}(a,b)}{M'_{\frac{1}{3}}(a,b)}=\left(\frac{a^{\frac{1}{3}}+b^{\frac{1}{3}}}{(1-a)^{\frac{1}{3}}+(1-b)^{\frac{1}{3}}}\right)^{3}$$

$$f_7(a,b)=\frac{M_{\frac{2}{3}}(a,b)}{M'_{\frac{2}{3}}(a,b)}=\left(\frac{a^{\frac{2}{3}}+b^{\frac{2}{3}}}{(1-a)^{\frac{2}{3}}+(1-b)^{\frac{2}{3}}}\right)^{\frac{3}{2}}$$

and

$$f_8(a,b) = \frac{CT(a,b)}{CT'(a,b)} = \left(\frac{(a^2+ab+b^2)\big(2-(a+b)\big)}{(a+b)((1-a)^2+(1-a)(1-b)+(1-b)^2)}\right)$$

Put $b = \frac{1}{2}$ and $a = t \in \left]0, \frac{1}{2}\right]$, then by Taylor's series expansion

$$f_1\left(t,\frac{1}{2}\right) = 1 + 2\left(t-\frac{1}{2}\right) + 2\left(t-\frac{1}{2}\right)^2 + 7.3333\left(t-\frac{1}{2}\right)^3 + 12.6666\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_2\left(t,\frac{1}{2}\right) = 1 + 2\left(t-\frac{1}{2}\right) + 2\left(t-\frac{1}{2}\right)^2 + 5.3333\left(t-\frac{1}{2}\right)^3 + 8.6666\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_3\left(t,\frac{1}{2}\right) = 1 + 2\left(t-\frac{1}{2}\right) + 2\left(t-\frac{1}{2}\right)^2 + 4.6666\left(t-\frac{1}{2}\right)^3 + 7.6666\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_4\left(t,\frac{1}{2}\right) = 1 + 2\left(t-\frac{1}{2}\right) + 2\left(t-\frac{1}{2}\right)^2 + 4.6666\left(t-\frac{1}{2}\right)^3 + 7.6666\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_5\left(t,\frac{1}{2}\right) = 1 + 2\left(t-\frac{1}{2}\right) + 2\left(t-\frac{1}{2}\right)^2 + 3.3333\left(t-\frac{1}{2}\right)^3 + 4.6666\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_6\left(t,\frac{1}{2}\right) = 1 + 2\left(t-\frac{1}{2}\right) + 2\left(t-\frac{1}{2}\right)^2 + 3.3333\left(t-\frac{1}{2}\right)^3 + 4.6666\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_7\left(t,\frac{1}{2}\right) = 1 + 2\left(t-\frac{1}{2}\right) + 2\left(t-\frac{1}{2}\right)^2 + 2.6666\left(t-\frac{1}{2}\right)^3 + 3.3333\left(t-\frac{1}{2}\right)^4 + \cdots$$

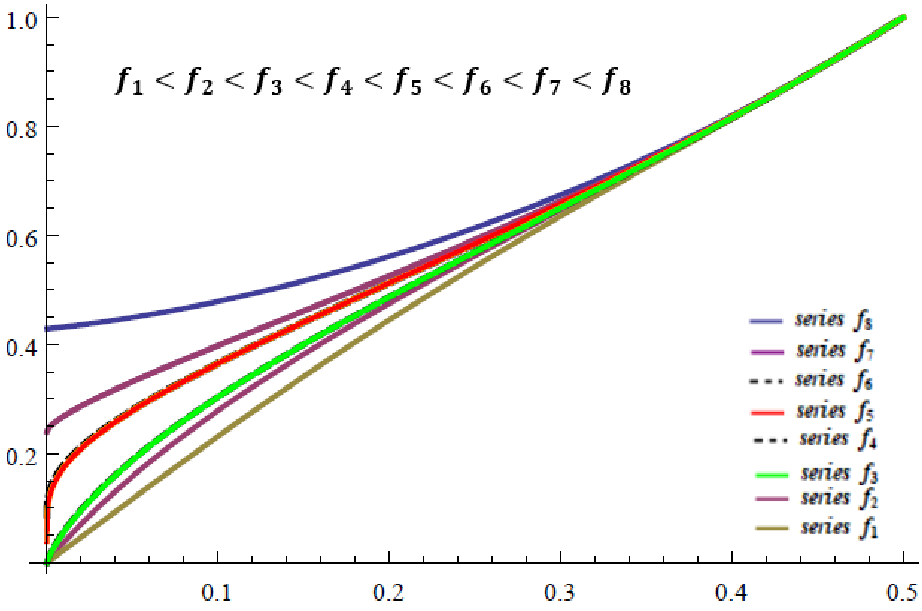
$$f_8\left(t,\frac{1}{2}\right) = 1 + 2\left(t-\frac{1}{2}\right) + 2\left(t-\frac{1}{2}\right)^2 + 0.6666\left(t-\frac{1}{2}\right)^3 - 0.6666\left(t-\frac{1}{2}\right)^4 + \cdots$$

It was observed that the series expansions are same up to 2^{nd} degree term. By considering 3^{rd} and 4^{th} degree terms, the following interpolating Ky-fan type inequality chain holds.

$$\frac{CT^{(i)}(a,b)}{CT^{(i)'}(a,b)} < \frac{M_{\frac{2}{3}}^{(i)}(a,b)}{M_{\frac{2}{3}}^{(i)'}(a,b)} < \frac{M_{\frac{1}{3}}^{(i)}(a,b)}{M_{\frac{1}{3}}^{(i)'}(a,b)} < \frac{L^{(i)}(a,b)}{L^{(i)'}(a,b)} <$$

$$\frac{L(a,b)}{L'^{(a,b)}} < \frac{M_{\frac{1}{3}}(a,b)}{M'_{\frac{1}{3}}(a,b)} < \frac{M_{\frac{2}{3}}(a,b)}{M'_{\frac{2}{3}}(a,b)} < \frac{CT(a,b)}{CT'(a,b)}$$

The graphical representation of the series from f_1 to f_8 is illustrated in the following graph: $[f_1 < f_2 < f_3 < f_4 < f_5 < f_6 < f_7 < f_8]$.



Let $f(a) = 2a^2$ for $a \in \left]0, \frac{1}{2}\right]$ and $2(1-a)^2$ for $a \in \left[\frac{1}{2}, 1\right[$ then

$$CT^*(a,b) = \frac{4}{3}\left(\frac{a^4 + a^2b^2 + b^4}{a^2 + b^2}\right) \; and \; (CT^*)^{(i)}(a,b) = \left(\frac{3a^2b^2(a^2+b^2)}{a^4 + a^2b^2 + b^4}\right)$$

$$M^*_{\frac{1}{3}}(a,b) = \left(\frac{(2a^2)^{\frac{1}{3}} + (2b^2)^{\frac{1}{3}}}{2}\right)^3 \; and \; (M^*)^{(i)}_{\frac{1}{3}}(a,b) = \frac{a^2b^2}{\left(\frac{(2a^2)^{\frac{1}{3}} + (2b^2)^{\frac{1}{3}}}{2}\right)^3}$$

$$(M^*)_{\frac{2}{3}}(a,b) = \left(\frac{(2a^2)^{\frac{2}{3}} + (2b^2)^{\frac{2}{3}}}{2}\right)^{\frac{3}{2}} \; and \; (M^*)^{(i)}_{\frac{2}{3}}(a,b) = \frac{a^2b^2}{\left(\frac{(2a^2)^{\frac{2}{3}} + (2b^2)^{\frac{2}{3}}}{2}\right)^{\frac{3}{2}}}$$

$$L^*(a,b) = \frac{b^2 - a^2}{logb - loga} \; and \; (L^*)^{(i)}(a,b) = \frac{a^2b^2(logb - loga)}{b^2 - a^2}$$

For $a' = 1-a, b' = 1-b \in \left[\frac{1}{2}, 1\right[$, the above said means are given by

$$(CT^*)'(a,b) = \frac{4}{3}\left(\frac{(1-a)^4 + (1-a)^2(1-b)^2 + (1-b)^4}{(1-a)^2 + (1-b)^2}\right)$$

$$(CT^*)^{(i)'}(a,b) = \left(\frac{3(1-a)^2(1-b)^2((1-a)^2 + (1-b)^2)}{(1-a)^4 + (1-a)^2(1-b)^2 + (1-b)^4}\right)$$

$$(M^*_{\frac{1}{3}})'(a,b) = \left(\frac{(2(1-a)^2)^{\frac{1}{3}} + (2(1-b)^2)^{\frac{1}{3}}}{2}\right)^3$$

$$(M^*)_{\frac{1}{3}}^{(i)'}(a,b) = \frac{(1-a)^2(1-b)^2}{\left(\frac{(2(1-a)^2)^{\frac{1}{3}} + (2(1-b)^2)^{\frac{1}{3}}}{2}\right)^3}$$

$$(M^*_{\frac{1}{3}})''(a,b) = \left(\frac{(2(1-a)^2)^{\frac{2}{3}} + (2(1-b)^2)^{\frac{2}{3}}}{2}\right)^{\frac{3}{2}}$$

$$(M^*)_{\frac{2}{3}}^{(i)'}(a,b) = \frac{(1-a)^2(1-b)^2}{\left(\frac{(2(1-a)^2)^{\frac{2}{3}} + (2(1-b)^2)^{\frac{2}{3}}}{2}\right)^{\frac{3}{2}}}$$

$$(L^*)'(a,b) = \frac{(1-b)^2 - (1-a)^2}{\log(1-b) - \log(1-a)}$$

$$(L^*)^{(i)'}(a,b) = \frac{(1-a)^2(1-b)^2(\log(1-b) - \log(1-a))}{(1-b)^2 - (1-a)^2}$$

Let,

$$\begin{aligned} f_1^*(a,b) &= \frac{(CT^*)^{(i)}(a,b)}{(CT^*)^{(i)'}(a,b)} \\ &= \left(\frac{3a^2b^2(a^2+b^2)}{a^4 + a^2b^2 + b^4}\right)\left(\frac{(1-a)^4 + (1-a)^2(1-b)^2 + (1-b)^4}{(1-a)^2(1-b)^2((1-a)^2 + (1-b)^2)}\right) \end{aligned}$$

$$f_2^*(a,b) = \frac{(M^*)_{\frac{2}{3}}^{(i)}(a,b)}{(M^*)_{\frac{2}{3}}^{(i)'}(a,b)} = \left(\frac{a^2b^2}{(1-a)^2(1-b)^2}\right)\left(\frac{(2(1-a)^2)^{\frac{2}{3}} + (2(1-b)^2)^{\frac{2}{3}}}{(2a^2)^{\frac{2}{3}} + (2b^2)^{\frac{2}{3}}}\right)^{\frac{3}{2}}$$

$$f_3^*(a,b) = \frac{(M^*)_{\frac{1}{3}}^{(i)}(a,b)}{(M^*)_{\frac{1}{3}}^{(i)'}(a,b)} = \left(\frac{a^2b^2}{(1-a)^2(1-b)^2}\right)\left(\frac{(2(1-a)^2)^{\frac{1}{3}} + (2(1-b)^2)^{\frac{1}{3}}}{(2a^2)^{\frac{1}{3}} + (2b^2)^{\frac{1}{3}}}\right)^3$$

$$f_4^*(a,b) = \frac{(L^*)^{(i)}(a,b)}{(L^*)^{(i)'}(a,b)} = \left(\frac{a^2b^2}{(1-a)^2(1-b)^2}\right)\frac{((1-b)^2 - (1-a)^2)(\log b - \log a)}{(b^2 - a^2)(\log(1-b) - \log(1-a))}$$

$$f_5^*(a,b) = \frac{L^*(a,b)}{(L^*)'(a,b)} = \frac{(b^2 - a^2)(\log(1-b) - \log(1-a)}{((1-b)^2 - (1-a)^2)(\log b - \log a)}$$

$$f_6^*(a,b) = \frac{M^*_{\frac{1}{3}}(a,b)}{(M^*_{\frac{1}{3}})'(a,b)} = \left(\frac{(2a^2)^{\frac{1}{3}} + (2b^2)^{\frac{1}{3}}}{(2(1-a)^2)^{\frac{1}{3}} + (2(1-b)^2)^{\frac{1}{3}}}\right)^3$$

$$f_7^*(a,b) = \frac{M_{\frac{2}{3}}^*(a,b)}{\left(M_{\frac{2}{3}}^*\right)'(a,b)} = \left(\frac{(2a^2)^{\frac{2}{3}} + (2b^2)^{\frac{2}{3}}}{(2(1-a)^2)^{\frac{2}{3}} + (2(1-b)^2)^{\frac{2}{3}}}\right)^{\frac{3}{2}}$$

and

$$f_8^*(a,b) = \frac{CT^*(a,b)}{(CT^*)'(a,b)} = \left(\frac{(a^4+a^2b^2+b^4)((1-a)^2+(1-b)^2)}{((1-a)^4+(1-a)^2(1-b)^2+(1-b)^4)(a^2+b^2)}\right) -$$

Put $b = \frac{1}{2}$ and $a = t \in \left]0, \frac{1}{2}\right]$, then by Taylor's series expansion

$$f_1^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 29.3333\left(t-\frac{1}{2}\right)^3 + 85.3333\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_2^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 21.3333\left(t-\frac{1}{2}\right)^3 + 53.3333\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_3^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 18.6666\left(t-\frac{1}{2}\right)^3 + 42.6666\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_4^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 18.6666\left(t-\frac{1}{2}\right)^3 + 42.6666\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_5^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 13.3333\left(t-\frac{1}{2}\right)^3 + 21.3333\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_6^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 13.3333\left(t-\frac{1}{2}\right)^3 + 21.3333\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_7^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 10.6666\left(t-\frac{1}{2}\right)^3 + 10.6666\left(t-\frac{1}{2}\right)^4 + \cdots$$

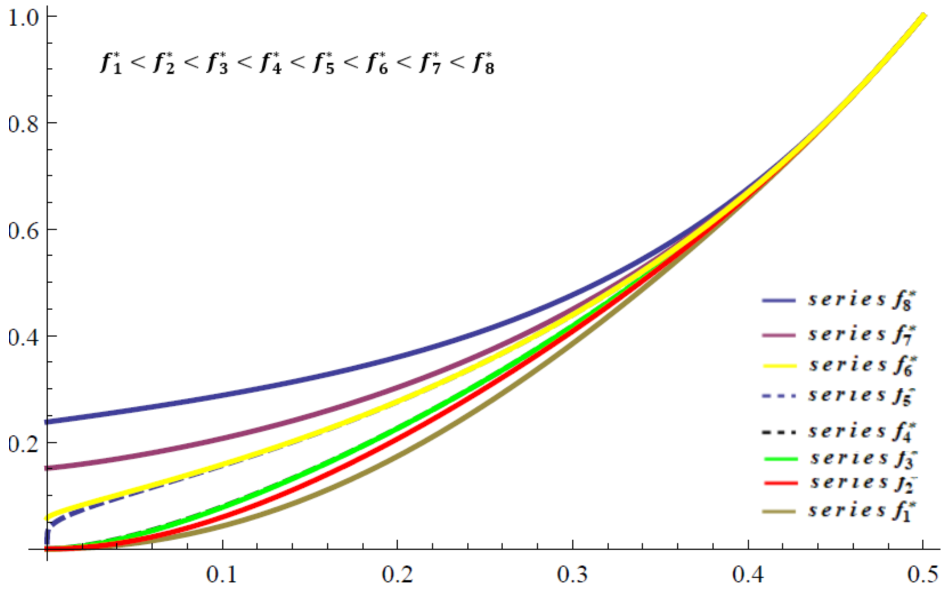
$$f_8^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 2.6666\left(t-\frac{1}{2}\right)^3 - 21.3333\left(t-\frac{1}{2}\right)^4 + \cdots$$

It was observed that the series expansions are same up to 2nd degree term. By considering 3rd and 4th degree terms, the following interpolating Ky-fan type inequality chain holds.

$$\frac{(CT^*)^{(i)}(a,b)}{(CT^*)^{(i)'}(a,b)} < \frac{(M^*)_{\frac{2}{3}}^{(i)}(a,b)}{(M^*)_{\frac{2}{3}}^{(i)'}(a,b)} < \frac{(M^*)_{\frac{1}{3}}^{(i)}(a,b)}{(M^*)_{\frac{1}{3}}^{(i)'}(a,b)} < \frac{(L^*)^{(i)}(a,b)}{(L^*)^{(i)'}(a,b)} <$$

$$\frac{L^*(a,b)}{(L^*)'(a,b)} < \frac{M_{\frac{1}{3}}^*(a,b)}{(M_{\frac{1}{3}}^*)'(a,b)} < \frac{M_{\frac{2}{3}}^*(a,b)}{(M_{\frac{2}{3}}^*)'(a,b)} < \frac{CT^*(a,b)}{(CT^*)'(a,b)}$$

The graphical representation of the series from f_1 to f_8 is illustarated in the following graph:$[f_1^* < f_2^* < f_3^* < f_4^* < f_5^* < f_6^* < f_7^* < f_8^*]$



CONCLUSION

New inequality chains involving centroidal mean, power mean, logarithmic mean and their invariant means are established by considering the arguments lying in the triangular wave function $f(y)$ and new parabolic function $f^*(y)$ in $(0,1)$. Further investigation to be carried out on more parabolic functions based on their nature and verifying the properties of means.

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