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PROBABILISTIC STRUCTURE BY A FUNCTION THROUGH SOME CONDITION

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Abstract

Using F-type contraction Suzuki define new contraction known as F-Suzuki contraction and published a paper in which they proved some fixed-point theorems using the sequence converging method via F-contraction., In this paper we define Suzuki contraction in probabilistic spaces and prove fixed point result in Probabilistic spaces.

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Introduction [Wardowski, D 2012].

Fixed point theory is a crucial topic in mathematics, particularly in analysis. Because of its applications in various disciplines such as engineering, computer science, biological sciences, and economics, many researchers have become interested in fixed point theory and its applications. The Banach contraction principle (S. Banach, 1922.) is the foundation of fixed-point theory. Because of the importance and simplicity of the Banach contraction principle, several authors have obtained many interesting extensions and generalizations in various directions over the last century. The concept of best proximity point results is one possible direction. (I. Altun, M. Aslantas, and H. Sahin, 2021.) obtained intriguing, best proximity point results and derived fixed results, one of which is the Banach fixed point, because of their efforts. Furthermore, assuming the concept's primary role. The set of all real numbers is denoted by R throughout the article, the set of all positive real numbers is denoted by R +, and the set of all natural numbers is denoted by N. (X, d) is the metric space with the metric d.

There are numerous extensions to the well-known Banach Contraction principle in the literature, which states that any self-mapping H defined on a complete metric space (X, d) satisfying.

$\forall\ x, y \epsilon\ d(H_x, H_y)\ \leq\ \varphi d(x, y)$,
where $\varphi \epsilon\ (0, 1)$,

Has a unique fixed point and for every $x_0\ \in X\ a\ sequence\ \{H^n\ x_0\}_{n \in N}$ is a convergent to the fixed point.

In this article, we introduce a new type of contraction called F-contraction and prove a new fixed point theorem concerning F-contraction using a mapping $F: R^+ \to R$. We obtain contractions of the type known from the literature, as well as Banach contractions, for the concrete mappings F. The article includes examples of F-contractions as well as an example demonstrating that the obtained extension is significant. The nature of F-contractions is illustrated by theoretical considerations that are supported by computational data.

Contraction Mapping [Wardowski, D 2012].

A contraction mapping, also known as a contraction or contractor mapping, on a Metric space (S, d) is a function f from S to itself that has the property that there is some real number $0 \leq K \leq 1$ such that for all x and y in S,

$$\boldsymbol{d(f(x), f(y)) \leq kd(x, y)}.$$

The Lipschitz constant of f is the smallest such value of k. Contractive maps are sometimes called **Lipschitzian maps**. If the preceding condition is satisfied for $k \leq 1$, the mapping is said to be non-expansive.

A contractive mapping can be defined more broadly for maps between metric Space. Thus, if (S, d) and (P, d') are two metric spaces, then $f: M \to N$ is a contractive mapping if there is a constant $0 \leq k < 1$ such that

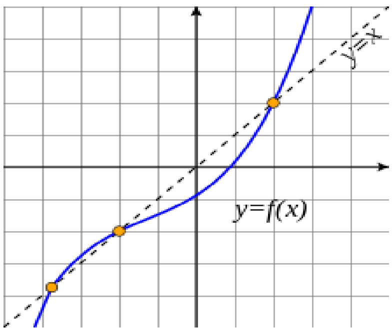
$$\boldsymbol{d'(f(x), f(y)) \leq kd(x, y)}$$

For all x and y in S.

Every contraction mapping is **Lipschitz continuous** and hence **uniformly continuous** (for a Lipschitz continuous function, the constant *k* is no longer necessarily less than 1). A contraction mapping has at most one **fixed point**. Moreover, the **Banach fixed-point theorem** states that every contraction mapping on a **non-empty complete metric space** has a unique fixed point, and that for any *x* in *M* the **iterated function** sequence $x, f(x), f(f(x)), f(f(f(x)))$, ... converges to the fixed point. This concept is very useful for iterated function system where contraction mappings are often used. Banach's fixed-point theorem is also applied in proving the existence of solutions of ordinary differential equations, and is used in one proof of the inverse function theorem. Contraction mappings play an important role in dynamic programming problems.

FIXED POINT [Wardowski, D 2012].

A fixed point (also known as an invariant point) is a value that remains constant when subjected to a given transformation. A fixed point of a function is an element that the function maps to itself in mathematics.



Fixed point of a function

Formally, c is a fixed point of a function f if and only if c belongs to both domain and codomain such that $f(c) = c$.

Fixed point iteration

Fixed-point iteration is a method of computing the fixed points of a function in numerical analysis. The fixed-point iteration is given a function f with the same domain and codomain as well as a point x_0 in f's domain.

$x_{n+1}=f(x_n)$, n=0,1,2, …

Which gives rise to the sequence $x_0, x_1, x_2 \dots\dots\dots$ of iterated functions applications.

$x_0, f(x_0), f(f(x_0))$, ………………

Which is hoped to converge to a point x. If f is continuous, then the obtained x must be a fixed point of f.

Periodic points are points that return to the same value after a finite number of iterations of the functions. A periodic point with period one is referred to as a fixed point.

Metric fixed-point property

Let (X, d) be a metric space and H: X→X be mapping. Then a point s$\in X$ is called a fixed point of H if s is mapped into itself i.e.,

$$\boldsymbol{H(s) = s.}$$

F-contraction [Secelean NA,2013].

In 2012, Wardowski introduce a new type of contractions called F-contraction and prove a new fixed point theorem concerning F-contractions. In this way, Wardowski [Wardowski D, 2012] generalized the Banach contraction principle in a different manner from the well-known results from the literature. Wardowski defined the F-contraction as follows.

Let (X, d) be a metric space. A mapping $H: X \to X$ is said to be an F-contraction if there exists. $\omega > 0$ such that

$$\forall x, y \in X, [d(Hx, Hy) > 0 \Rightarrow \omega + S(d(Hx, Hy)) \leq S(d(x, y))], \quad (1)$$

where $S: R_+ \to R$ is a mapping satisfying the following conditions:

1. (S1)

F is strictly increasing, i.e., for all $p, q \in R_+$ such that , $S(p) < S(q)$;

2. (S2)

   For each sequence $\{p_n\}_{n=1}^{\infty}$ of positive numbers $\lim_{n\to\infty} p_n = 0$ if and only if $\lim_{n\to\infty} S(p_n) = -\infty$;

3. (S3)

   There exists $k \in (0,1)$ such that $\lim_{x\to 0^+} p^k S(p) = 0$.

We denote by $, the set of all functions satisfying the conditions (S1) -(S3). For examples of the function S the reader is referred to [Secelean NA,2013] and [Wardowski D,2012].

Remark 1.

From (S1) and (1) it is easy to conclude that every F-contraction is necessarily continuous.

Wardowski [Wardowski D,2012] stated a modified version of the Banach contraction principle as follows.

Theorem

Let (X, d) be a complete metric space and let $H: X \to X$ be an F-contraction. Then H has a unique fixed-point $x * \in X$ and for every $x \in X$ the sequence $\{H^n x\}_{n\in N}$ converges to $x *$.

Very recently, Secelean [NA,2013] proved the following lemma.

Lemma 1. [Secelean NA,2013].

Let $S: R_+ \to R$ be an increasing mapping and $\{p_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following assertions hold:

(a) if $\lim_{n\to\infty} S(p_n) = -\infty$; then $\lim_{n\to\infty} p_n = 0$;

(b) if $inf S = -\infty$, and $\lim_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} S(p_n) = -\infty$;

By proving Lemma 1, Secelean showed that the condition (S2) in Definition (2) can be replaced by an equivalent but a more -simple condition, (S2′) $infF = -\infty$

or, also, by

(S2″) there exists a sequence$\{p_n\}_{n=1}^{\infty}$ of positive real numbers such that.

$$\lim_{n\to\infty} S(p_n) = -\infty;$$

Remark 2.

Define $S_B: R_+ \to R$ by $S_B(p) = lnp$, then $S_B \in \$$.

Note that with $S = S_B$ the F-contraction reduces to a Banach contraction.

Therefore, the Banach contractions are a particular case of F-contractions. Meanwhile there exist F-contractions which are not Banach contractions (SEE [Wardowski D,2012, Secelean NA,2013]).

In this chapter, we use the following condition instead of the condition (S3) in Definition 2:

(S3′) S is continuous on (0, ∞) (0, ∞).

We denote by $\Im$ the set of all functions satisfying the conditions (S1), (S2′), and (S3′).

Remark 3.

Note that the conditions (S3) and (S3′) are independent of each other. Indeed, for

$$p \geq 1, S(p) = \frac{1}{p^{\alpha}}$$

satisfies the conditions (S1) and (S2) but it does not satisfy (S3), while it satisfies the condition (S3′). Therefore,$\Im \in \$$.

Again, for $p > 1$,

$$t \in (0,1/a),\ S(p) = \frac{1}{(p+|p|)^{t}},$$

where [p] denotes an integral part of p, satisfies the conditions (S1) and (S2) but it does not satisfy (S3′), while it satisfies the condition (S3) for any $k \in (1/p, 1)$.

Therefore, $\$ \not\subseteq \mathfrak{I}$.

Also, if $S(p) = lnp$,

then $S \in \$$ and $S \in \$$.

Therefore,

$$\$ \cap \mathfrak{I} \neq \emptyset.$$

In view of Remark 3, it is meaningful to consider the result of Wardowski [Wardowski D,2012] with the mappings $S \in \mathfrak{I}$ instead $S \in \$$. Suzuki define the F-Suzuki contraction and proved fixed point result as follows,

Definition.

Let (X, d) be a metric space. A mapping $H: X \to X$ is said to be an F-Suzuki contraction if there exists $\omega > 0$ such that for all $x, y \in X$ with $Hx \neq Hy$

$$\frac{1}{2} d(x, Hx) < d(x, y) \Rightarrow \omega + S(d(Hx, Hy)) \leq S(d(x, y)),$$

where $S \in \mathfrak{I}$

Theorem.

Let H be a self-mapping of a complete metric space X into itself. Suppose $S \in \mathfrak{I}$ and there exists.

$\omega > 0$ such that

$$\forall x, y \in X, [d(Hx, Hy) > 0 \Rightarrow \omega + S(d(Hx, Hy)) \leq S(d(x, y))].$$

Then H has a unique fixed point $x* \in X$ and for every $x_0 \in X$ the sequence $\{H^n x\}_{n \in N}$ converges to x*.

Probabilistic Theory [Tripathi Piyush 2016]:

Definition. A mapping $f : R \to R^{+}$ is called a distribution function if it is

non decreasing, left continuous and inf *f(x) = 0* , sup *f(x) = 1*.

We shall denote by *L* the set of all distribution functions. The specific distribution

function $H \in L$ is defined by

$$\left.\begin{aligned} H(x) &= 0, \quad x \le 0 \\ &= 1, \quad x > 0 \end{aligned}\right\}$$

Definition. A probabilistic metric space (PM space) is an ordered pair

(*X*,*F*), *X* is a nonempty set and $F : X \times X \to L$ is mapping such that, by denoting

$F(p,q)$ by $F_{p,q}$ for all *p*, *q* in *X*, we have

(I) $F_{p,q}(x) = 1 \ \forall\, x > 0$ iff $p = q$

(II) $F_{p,q}(0) = 0$

(III) $F_{p,q} = F_{q,p}$

(IV) $F_{p,q}(x) = 1$, $F_{q,r}(y) = 1 \Rightarrow F_{p,r}(x+y) = 1.$

We note that $F_{p,q}(x)$ is value of the distribution function $F_{p,q} = F(p,q) \in L$ at $x \in R$.

Definition. A mapping $t : [01] \times [01] \to [01]$ is called t-norm if it is non-decreasing (by non-decreasing, we mean $a \le c, b \le d \Rightarrow t(a,b) \le t(c,d)$), commutative, associative and $t(a,1) = a$ for all *a* in [0, 1], *t*(0,0) = 0.

Main Result.

Now we define Suzuki contraction in probabilistic space as:

Definition. Let $(\boldsymbol{X},\boldsymbol{F})$ be a probabilistic metric space. A mapping $h: X \to X$ is said to be F-contraction in PM – space if there exists $w > 0$ such that

$$\frac{1}{2} F_{p,hp}(x) < F_{p,q}(x) \Rightarrow w + S\left(F_{hp,hq}(x)\right) < S\left(F_{p,q}(x)\right) \forall\, p,q \in X$$

Theorem. Let $(\boldsymbol{X},\boldsymbol{F})$ be a complete probabilistic metric space. Let $h: X \to X$ be a mapping satisfying the contraction condition

$$F_{hp,hq}(x) > 0 \Rightarrow w + S\left(F_{hp,hq}(x)\right) < S\left(F_{p,q}(x)\right) \forall\, p,q \in X$$

Then h has a unique fixed point in X.

Proof. Choose $p_0 \in X$ and define a sequence (p_n) such that

$$p_1 = hp_0\,,\, p_2 = hp_1 = h^2\, p_0\,, \dots, p_{n+1} = hp_n = h^{n+1}\, p_0 \;\; \forall\, n \in \mathbb{N}$$

If there exists $n \in \mathbb{N}$ such that $F_{p_n.hp_n}(x) = 0$ then nothing to do. Suppose

$$0 < F_{p_n,hp_n}(x) = F_{hp_{n-1},hp_n}(x)$$

Now

$$w + S\left(F_{hp_{n-1},hp_n}(x)\right) \le S\left(F_{p_{n-1},p_n}(x)\right)$$

$$S\left(F_{hp_{n-1},hp_n}(x)\right) \le S\left(F_{p_{n-1},p_n}(x)\right) - w$$

$$= S\left(F_{hp_{n-2},hp_{n-1}}(x)\right) - w$$

$$\le S\left(F_{p_{n-2},p_{n-1}}(x)\right) - 2w$$

$$\dots\dots\dots\dots$$

$$\le S\left(F_{p_0,p_1}(x)\right) - nw$$

Taking $n \to \infty$ we get $S\left(F_{hp_{n-1},hp_n}(x)\right) = 0 \implies F_{p_n,p_{n+1}}(x) = 0$

So (p_n) is a Cauchy sequence then by completeness of PM – space, (p_n) convereges to some point $p \in X$.

Again, $F_{hp,p}(x) = \lim\limits_{n \to \infty} F_{hp_n,p_n}(x) = \lim\limits_{n \to \infty} F_{p_{n+1},p_n}(x) = F_{p,p}(x) = 0$

$$F_{hp,p}(x) = 0 \implies hp = p.$$

Therefore p is a fixed point of h.

For uniqueness, suppose q is another fixed point of h in X.

Consider $F_{hp,hq}(x) = F_{p,q}(x)$ so we get,

$$S\left(F_{p,q}(x)\right) = S\left(F_{hp,hq}(x)\right) < w + S\left(F_{hp,hq}(x)\right) \le S\left(F_{p,q}(x)\right)$$

Which is a contradiction, therefore the fixed point is unique.

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