ON THE FORCING SIGNAL NUMBER OF A GRAPH

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ABSTRACT

For two vertices x and y of a graph G, the set $L[x,y]$ consist of x and y and all vertices lying on some $x-y$ goosing of G and for a non-empty set $S \subseteq V(G)$, $L[S] = \bigcup_{x,y \in S} L[x,y]$. A set $S \subseteq V(G)$ is said to be a signal set of G if $L[S] = V(G)$. The minimum cardinality of a signal set is known as signal number and is denoted by $sn(G)$. A subset T of a minimum signal set S is called a forcing subset for S if S is the unique minimum signal set containing T. The forcing signal number $f_G s(S)$ of S is the minimum cardinality among the forcing subsets of S, and the forcing signal number $fs(G)$ of G is the minimum forcing signal number among all minimum signal set of G. In this paper, the forcing signal number of several classes of graphs are determined some of its general properties also studied.

Keywords: Signal set, Signal number, Forcing Signal number.

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1 Introduction

By a graph $G = (V, G)$ we mean a finite, undirected, connected graph without loops or multiple edges. The order and size of G are denoted by m and n respectively.

For basic definitions and terminologies, we refer to [1]. For any vertex v in $V(G)$, the open neighbourhood $N(v)$ is the set of all vertices adjacent to that v and $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of v. Let $\Delta = \Delta(G)$ and $\delta = \delta(G)$ denote the maximum and minimum degree of G, respectively. A vertex v is said to be an extreme vertex of G, if its neighbourhood $N(v)$ induces a complete subgraph of G. If G is a connected graph, then the length of a shortest $x - y$ path in G.

A set $S \subseteq V(G)$ is said to be a signal set of G if $L[S] = V(G)$. The minimum cardinality of a signal set is known as signal number and is denoted by $snG)$.

We present some basic information in this area that help in the creation of the paper. In section 2, we defined and demonstrated the forcing signal number of a graph. Section 3, contains the paper's conclusion. In the sequel, the following results are used.

Theorem 1.1 [5] Each extreme vertex of G belongs to every signal set of G.

Theorem 1.2[5] $sn(G) = 2$ if and only if there exist vertices u, v such that v is an u −signal vertex of G.

Theorem 1.3 [2] For any connected graph G, $2 \leq sn(G) \leq n$.

2. Forcing Signal number of a graph.

In this section, we define the forcing signal number $fs(G)$ of a graph and initiate a study of this parameter.

Definition 2.1 Let G be a connected graph and S be a minimum signal set of G. A subset $T \subseteq S$ is called a forcing subset of S, if S is the unique minimum signal set containing T. The forcing subset of S of minimum cardinality is the minimum forcing subset of S. The forcing signal number of S denoted by $fs(G)$ is the cardinality of the minimum forcing subset of S and is given by $fs(G) = \min\{fs(G)\}$, where the minimum is taken over all minimum signal sets of G.

Example 2.2 For the graph G given in Figure 2.1, $S = \{v_1, v_4\}$ is the unique minimum signal set of G and thus $fs(G) = 0$. Also, for the graph G given in Figure 2.2, $S_1 = \{v_1, v_4, v_5\}$, $S_2 = \{v_1, v_3, v_6\}$, $S_3 = \{v_1, v_3, v_5\}$, $S_4 = \{v_1, v_4, v_6\}$, $S_5 = \{v_1, v_4, v_7\}$, and $S_6 = \{v_1, v_2, v_5\}$ are the minimum signal sets of G.

Since S_5 and S_6 is the only minimum signal set containing v_7 and v_5, respectively. It follows that $fs(S_5) = fs(S_6) = 1$. No other vertex of G belongs to only one minimum signal set and so $fs(S_i) \geq 2$ for $i = 1,2,3,4$. Therefore $fs(G) = 1$.

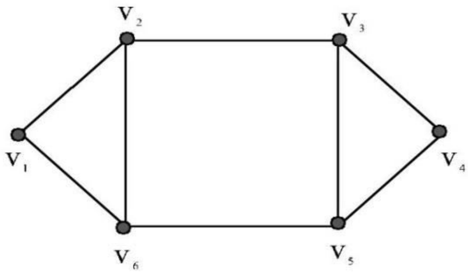

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Figure 2.1

A graph with forcing signal number 0.

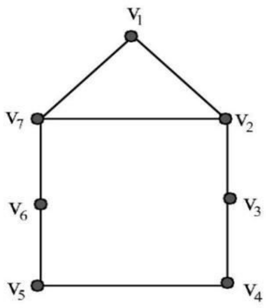

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Figure 2.2

A graph with forcing signal number 1.

Theorem 2.3 For any connected graph G, $0 \leq fs(G) \leq sn(G) \leq n$.

Proof.

It is clear from the definition of forcing signal set of G that, $fs(G) \geq 0$. Let S be a minimum signal set of G. Since $fs(S) \leq sn(G)$ and $fs(G) = \min\{fs(S)\}$, it follows that $fs(G) = sn(G)$. Also, from Theorem 1.2, $sn(G) \leq n$. Hence $0 \leq fs(G) \leq sn(G) \leq n$.

Remark 2.4 The bounds in Theorem 2.3 are strict. For the graph G given in Figure 2.1, $fs(G) = 0$. For the graph G given in Figure 2.2, $V(G) = 7$, $sn(G) = 3$ and $fs(G) = 1$. Thus, $0 \leq fs(G) \leq sn(G) < n$.

Theorem 2.5 Let G be any connected graph. Then,

(i) $fs(G) = 0$ if and only if G has a unique minimum signal set.

(ii) $fs(G) = 1$ if and only if G has atleast two minimum signal sets, one of which is the unique minimum signal set containing one of its element.

(iii) $fs(G) = sn(G)$ if and only if no minimum signal set of G is the unique minimum signal set containing any of its proper subsets.

Proof.

(i) Assume $fs(G) = 0$. Then by the definition 2.1, $fs(G) = 0$ for some signal set S of G and so the empty set ϕ is the minimum forcing subset for S. Since the empty set ϕ is a subset for every set, it follows S is the unique minimum signal set of G. Conversely, assume that S is the unique signal set. It is so clear that ϕ is a forcing subset of S. Hence, $fs(G) = 0$.

(ii) Assume $fs(G) = 1$. Then by Theorem 2.5 (i), G has at least two minimum signal sets. Also since $fs(G) = 1$, there is a singleton subset S_1 of a minimum signal set S of G such that S_1 is not a subset of any other minimum signal set of G. Thus that S is the unique minimum signal set containing are of its element.

Conversely, assume that G has at least two signal sets, there exists an element in one of the signal sets which is not in any other signal set. Hence $fs(G) = 1$.

(iii) Let $fs(G) = sn(G)$. Then $fs(G) = sn(G)$ for every minimum signal set S in G. Also, by Theorem 1.2, $sn(G) \geq 2$ and so $fs(G) \geq 2$. Then by Theorem 2.5(i), G has at least two minimum signal sets. So that the empty set is not a forcing subset for any minimum signal set of G. Since $fs(G) = sn(G)$, no proper subset of S is a forcing subset for S. Hence no signal set of G is the unique signal set containing any of its proper subsets.

Conversely assume that there is no minimum signal set of G is the unique minimum signal set containing any of its proper subsets. We prove that $fs(G) = sn(G)$. By our assumption G contains more than one minimum signal set and no subset of any minimum signal set S other than S is a forcing subset for S. Hence, $fs(G) = sn(G)$.

Definition 2.6 A vertex x of a connected graph G is said to be a signal vertex of G of x belongs to every minimum signal set of G.

Example 2.7 For the graph G given in Figure 2.2 $S_1 = \{v_1, v_4, v_5\}$, $S_2 = \{v_1, v_3, v_6\}$, $S_3 = \{v_1, v_3, v_5\}$, $S_4 = \{v_1, v_4, v_6\}$, $S_5 = \{v_1, v_4, v_7\}$, and $S_6 = \{v_1, v_2, v_5\}$ are the minimum signal sets of G. Here it is clear that v_1 is the unique signal vertex of G.

Theorem 2.8 Let G be a connected graph and let S be a minimum signal set of G. Then no signal vertex of G belongs to any minimum forcing set of S.

Proof.

Let G be a connected graph and let S be a minimum signal set of G. Let x be a signal vertex of G. Then by definition x belongs to every minimum signal set S of G. Let $T \subseteq S$ be any minimum forcing subset for any minimum signal set S of G. We claim that $x \notin T$. Suppose $x \in T$. Then $T' = T - \{x\}$ is a proper subset of T such that T' is the unique minimum signal set containing T' so that T' is a forcing subset for S with $|T'| < |T|$, which is a contradiction to T is a minimum forcing subset for S. Thus, $x \notin T$. Therefore no signal vertex of G belongs to any minimum forcing set of S.

Theorem 2.9 Let G be a connected graph and T be the set of all signal vertices of G. Then $fs(G) \leq sn(G) - T$.

Proof.

Let S be a minimum signal set of G. Then $sn(G) = |S|$, $T \subseteq S$ and S is the unique minimum signal set containing $S - T$. Hence, $fs(G) \leq |S - T| = |S| - |T| = sn(G) - T$.

Corollary 2.10 If G is a connected graph with k extreme points, then $fs(G) \leq sn(G) - k$.

Theorem 2.11 For any complete graph $G = K_n (n \geq 2)$ or any non-trivial tree G, $fs(G) = 0$.

Proof.

For $G = K_n$, it is clear that the set of all vertices of G is the unique minimum single set. Therefore by Theorem 2.5(i), it follows that $fs(G) = 0$.

If G is a non-trivial tree, then the set of all end vertices of G is the unique minimum signal set if G and so by Theorem 2.5(i), $fs(G) = 0$.

Theorem 2.12 For any cycle graph $G = C_n (n \geq 4)$, a set $S \subseteq V(G)$ is a minimum signal set of G if and only if S consists of two antipodal vertices.

Proof.

If S contains only the two antipodal vertices, then it is clear that S is a minimum signal set of G. Conversely, let S be any minimum signal set of G. Then $sn(G) = |S|$. Let S' be any set of two antipodal vertices of G. Then as in the first part of this theorem, S' is a minimum signal set of G. Thus $|S| = |S'|$. So S contains two vertices, say $S = \{x, y\}$. If x and y are not antipodal, then any part of vertices that is not on the $x - y$ geosig. Hence S is not a minimum signal set, which is a contradiction. Thus S consists of two antipodal vertices.

Theorem 2.13 For any cycle C_n $(n \geq 4)$, $fs(C_n) = \begin{cases} 1\ if\ n\ is\ even \\ 2\ if\ otherwise \end{cases}$.

Proof.

If n is even, then $sn(C_n) = 2$ and by Theorem 2.12, every minimum signal set of C_n consists of pair of antipodal vertices. But C_n does not have a unique minimum signal set because C_n has $\frac{n}{2}$ minimum signal sets of C_n. Moreover, every vertex of C_n has a unique vertex which is antipodal to it and so by Theorem 2.5(ii), $fs(C_n) = 1$.

If n is odd, then $sn(C_n) = 3$. Also it is clear that C_n contains more than one minimum signal set. Again every vertex of C_n belongs to at least two distinct minimum signal sets and so $fs(G) \geq 2$. Other hand, for every pair x, y of adjacent vertices in C_n, there is a unique vertex z in C_n such that $d(x, z) = d(y, z)$. Therefore, it follows that $\{x, y, z\}$ is the unique minimum signal set of G containing $\{x, y\}$. This shows that $fs(C_n) = 2$.

Theorem 2.14 If G is a connected graph with $sn(G) = 2$, then $fs(G) < 2$.

Proof.

Let $S = \{x, y\}$ be a signal set of G. Then $d(x, y) = diam(G)$ and every vertex of G lies on some $x - y$ geosig of G. To prove $fs(G) < 2$. By Theorem 2.3, we have $fs(G) = 2$. Suppose $fs(G) = 2$. Then there exists a vertex $z \neq y$ such that $\{x, z\}$ is also a minimum signal set of G. It follows that z lies on $x - y$ geosig of G. This shows that $d(x, z) < d(x, y) = diam(G)$, which is a contradiction. Hence $fs(G) < 2$.

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