ABOUT ONE REMARKABLE IDENTITY

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Abstract

This article examines approximately one remarkable feature. In this, problems on identities, factors, equations, inequalities, discriminant calculations are considered and can be used to solve various problems. Let's start with the same changes.

Key words: identities, factors, equation, inequality, discriminant.

This is the well-known identity:

$$a^3+b^3+c^3-3abc=(a+b+c)\left(a^2+b^2+c^2-ab-ac-bc\right) \qquad (1)$$

Usually this identity is proved like this:

$$\begin{aligned}&a^3+b^3+c^3-3abc=a^3+3a^2b+3b^2a+b^3+c^3-3abc-3a^2b-3b^2a=\\&=(a+b)^3+c^3-3ab(a+b+c)=(a+b+c)\left((a+b)^2-(a+b)c+c^2\right)-\\&-3ab(a+b+c)=(a+b+c)\left(a^2+2ab+b^2-ac-bc+c^2-3ab\right)=\\&=(a+b+c)\left(a^2+b^2+c^2-ab-ac-bc\right).\end{aligned}$$

Identity (1) can be used to solve various problems. Let's start with the identical transformations:

Task 1. Factorize:

$$(b-c)^3+(c-a)^3+(a-b)^3$$

Solution. We use formula (1):

$$(b-c)^3+(c-a)^3+(a-b)^3=(b-c+c-a+a-b)\cdot A+3(b-c)(c-a)(a-b)=$$
$$=0\cdot A+3(b-c)(c-a)(a-b)=3(b-c)(c-a)(a-b).$$

$$(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b).$$

Task 2. Factorize:

$$\left(y^2-z^2\right)^3+\left(z^2-x^2\right)^3+\left(x^2-y^2\right)^3$$

Solution. We use formula (1):

$$\left(y^2-z^2\right)^3+\left(z^2-x^2\right)^3+\left(x^2-y^2\right)^3=\left(y^2-z^2+z^2-x^2+x^2-y^2\right)\cdot A+$$
$$+3\left(y^2-z^2\right)\left(z^2-x^2\right)\left(x^2-y^2\right)=0\cdot A+3\left(y^2-z^2\right)\left(z^2-x^2\right)\left(x^2-y^2\right)=$$
$$=3\left(y^2-z^2\right)\left(z^2-x^2\right)\left(x^2-y^2\right).$$

$$\left(y^2-z^2\right)^3+\left(z^2-x^2\right)^3+\left(x^2-y^2\right)^3=3\left(y^2-z^2\right)\left(z^2-x^2\right)\left(x^2-y^2\right).$$

2. We use formula (1) and solve the equations

Task 3 . solve the equation $3x^3-9x+10=0$

Solution.

$$3x^3-9x+10=0\,\Big|:3\Leftrightarrow x^3-3x+\frac{10}{3}=0\Leftrightarrow x^3-3x+3+\frac{1}{3}=0\Leftrightarrow$$
$$\Leftrightarrow x^3+3+\frac{1}{3}-3x=0\Leftrightarrow x^3+\left(\sqrt[3]{3}\right)^3+\left(\sqrt[3]{\frac{1}{3}}\right)^3-3x\cdot\sqrt[3]{3}\cdot\frac{1}{\sqrt[3]{3}}=0\Leftrightarrow$$
$$\Leftrightarrow\left(x+\sqrt[3]{3}+\frac{1}{\sqrt[3]{3}}\right)\left(x^2+\sqrt[3]{9}+\frac{1}{\sqrt[3]{9}}-x\sqrt[3]{3}-x\frac{1}{\sqrt[3]{3}}-\sqrt[3]{3}\cdot\frac{1}{\sqrt[3]{3}}\right)=0\Leftrightarrow$$
$$\Leftrightarrow\left(x+\sqrt[3]{3}+\frac{1}{\sqrt[3]{3}}\right)\left(x^2-\left(\sqrt[3]{3}+\frac{1}{\sqrt[3]{3}}\right)x+\frac{1}{\sqrt[3]{9}}+\sqrt[3]{9}-1\right)=0\Leftrightarrow$$
$$\Leftrightarrow\left[\begin{array}{l}x+\sqrt[3]{3}+\dfrac{1}{\sqrt[3]{3}}=0\\ x^2-\left(\dfrac{\sqrt[3]{9}+1}{\sqrt[3]{3}}\right)x+\dfrac{1+\sqrt[3]{81}-\sqrt[3]{9}}{\sqrt[3]{9}}=0\end{array}\right.$$

1. $x+\sqrt[3]{3}+\sqrt[3]{\frac{1}{3}}=0\Leftrightarrow x_1=-\sqrt[3]{3}-\sqrt[3]{\frac{1}{3}}$

2.

$$x^2-\left(\frac{\sqrt[3]{9}+1}{\sqrt[3]{3}}\right)x+\frac{1+\sqrt[3]{81}-\sqrt[3]{9}}{\sqrt[3]{9}}=0\Leftrightarrow$$

$$\Leftrightarrow x_{2,3}=\frac{\frac{\sqrt[3]{9}+1}{\sqrt[3]{3}}\pm\sqrt{\left(\frac{\sqrt[3]{9}+1}{\sqrt[3]{3}}\right)^2-\frac{4\left(1+\sqrt[3]{81}-\sqrt[3]{9}\right)}{\sqrt[3]{9}}}}{2}=$$

$$=\frac{\frac{\sqrt[3]{9}+1}{\sqrt[3]{3}}\pm\sqrt{\frac{\sqrt[3]{81}+2\sqrt[3]{9}+1-4-4\sqrt[3]{81}+4\sqrt[3]{9}}{\sqrt[3]{9}}}}{2}=$$

$$=\frac{\frac{\sqrt[3]{9}+1}{\sqrt[3]{3}}\pm\sqrt{\frac{-3\sqrt[3]{81}+6\sqrt[3]{9}-3}{\sqrt[3]{9}}}}{2}=\frac{\frac{\sqrt[3]{9}+1}{\sqrt[3]{3}}\pm\sqrt{\frac{-3\left(\sqrt[3]{81}-2\sqrt[3]{9}+1\right)}{\sqrt[3]{9}}}}{2}=$$

$$=\frac{\frac{\sqrt[3]{9}+1}{\sqrt[3]{3}}\pm\sqrt{\frac{-3\left(\sqrt[3]{9}-1\right)^2}{\sqrt[3]{9}}}}{2}=\frac{\left(\sqrt[3]{9}+1\right)\pm3\left(\sqrt[3]{9}-1\right)i}{2\sqrt[3]{3}}$$

$$x_2=\frac{\left(\sqrt[3]{9}+1\right)-3\left(\sqrt[3]{9}-1\right)i}{2\sqrt[3]{3}},\ x_3=\frac{\left(\sqrt[3]{9}+1\right)+3\left(\sqrt[3]{9}-1\right)i}{2\sqrt[3]{3}}$$

Answer:

$$x_1=-\sqrt[3]{3}-\sqrt[3]{\frac{1}{3}},\ x_2=\frac{\left(\sqrt[3]{9}+1\right)-3\left(\sqrt[3]{9}-1\right)i}{2\sqrt[3]{3}},\ x_3=\frac{\left(\sqrt[3]{9}+1\right)+3\left(\sqrt[3]{9}-1\right)i}{2\sqrt[3]{3}}$$

Task 4 . Solve the equation in integers: $x^3+y^3-3xy=3$

Solution.

$$x^3+y^3-3xy=3\Leftrightarrow x^3+y^3+1^3-3xy=4\Leftrightarrow$$
$$\Leftrightarrow\left(x+y+1\right)\left(x^2+y^2+1-xy-x-y\right)=4\Leftrightarrow$$
$$\Leftrightarrow\left(x+y+1\right)\cdot\frac{1}{2}\left(2x^2+2y^2-2xy-2x-2y+2\right)=4\Leftrightarrow$$
$$\Leftrightarrow\left(x+y+1\right)\cdot\frac{1}{2}\left(x^2-2xy+y^2+x^2-2x+1+y^2-2y+1\right)=4\Leftrightarrow$$
$$\Leftrightarrow\left(x+y+1\right)\cdot\frac{1}{2}\left(\left(x-y\right)^2+\left(x-1\right)^2+\left(y-1\right)^2\right)=4$$

Thus, the following systems are possible:

1.

$$\begin{cases} x+y+1=2 \\ x^2+y^2+1-xy-x-y=2 \end{cases} \Leftrightarrow \begin{cases} y=1-x \\ x^2+(1-x)^2+1-x(1-x)-1=2 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} y=1-x \\ x^2+1-2x+x^2+1-x+x^2-3=0 \end{cases} \Leftrightarrow \begin{cases} y=1-x \\ 3x^2-3x-1=0 \end{cases}$$

the system has no solution in integers:

2.

$$\begin{cases} x+y+1=4 \\ x^2+y^2+1-xy-x-y=1 \end{cases} \Leftrightarrow \begin{cases} y=3-x \\ x^2+(3-x)^2+1-x(3-x)-3=1 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} y=3-x \\ x^2+9-6x+x^2+1-3x+x^2-4=0 \end{cases} \Leftrightarrow \begin{cases} y=3-x \\ 3x^2-9x+6=0 \,|\, :3 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} y=3-x \\ x^2-3x+2=0 \end{cases} \Leftrightarrow \begin{cases} y=3-x \\ \left[\begin{array}{l} x_1=2 \\ x_2=1 \end{array} \right. \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x_1=2 \\ y_1=1 \end{cases} \\ \begin{cases} x_2=1 \\ y_2=2 \end{cases} \end{array} \right.$$

3.

$$\begin{cases} x+y+1=1 \\ x^2+y^2+1-xy-x-y=4 \end{cases} \Leftrightarrow \begin{cases} y=-x \\ x^2+(-x)^2+1-x(-x)=1 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} y=-x \\ x^2+x^2+1+x^2-4=0 \end{cases} \Leftrightarrow \begin{cases} y=-x \\ 3x^2-3=0 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} y=-x \\ x^2=1 \end{cases} \Leftrightarrow \begin{cases} y=-x \\ \left[\begin{array}{l} x_3=1 \\ x_4=-1 \end{array} \right. \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x_3=1 \\ y_3=-1 \end{cases} \\ \begin{cases} x_4=-1 \\ y_4=1 \end{cases} \end{array} \right.$$

Answer: $\{(1;\,2),(2;\,1),(1;-1),(-1;\,1)\}$.

Task 5 . Solve the equation:

$$8(1-2x)^3-27(x-1)^3=125-343x^3$$

Solution:

$$8(1-2x)^3-27(x-1)^3=125-343x^3 \Leftrightarrow 343x^3+8(1-2x)^3+27(1-x)^3=125$$

We use formula (1)

$$343x^3+8(1-2x)^3+27(1-x)^3=125\Leftrightarrow$$
$$\Leftrightarrow(7x)^3+(2(1-2x))^3+(3(1-x))^3=125\Leftrightarrow$$
$$\Leftrightarrow(7x+2(1-2x)+3(1-x))((7x)^2+(2(1-2x))^2+(3(1-x))^2-$$
$$-7\cdot 2x(1-2x)-7\cdot 3x(1-x)-2\cdot 3(1-2x)(1-x))+$$
$$+3\cdot 7\cdot 2\cdot 3x(1-2x)(1-x)=125\Leftrightarrow$$
$$\Leftrightarrow(7x+2-4x+3-3x)(49x^2+4(1-2x)^2+9(1-x)^2-14x(1-2x)-$$
$$-21x(1-x)-6(1-3x+2x^2))+126x(1-3x+2x^2)=125\Leftrightarrow$$
$$\Leftrightarrow 5(49x^2+4(1-4x+4x^2)+9(1-2x+x^2)-14x(1-2x)-21x(1-x)-$$
$$-6(1-3x+2x^2))+126x(1-3x+2x^2)=125\Leftrightarrow$$

$$\Leftrightarrow 5(49x^2+4-16x+16x^2+9-18x+9x^2-14x+28x^2-21x+21x^2-6+$$
$$+18x-12x^2)+126x-378x^2+252x^3-125=0\Leftrightarrow 5(111x^2-51x+7)+$$
$$+126x-378x^2+252x^3-125=0\Leftrightarrow 252x^3+555x^2-255x+35+126x-$$
$$-378x^2-125=0\Leftrightarrow 252x^3+177x^2-129x-90=0\Leftrightarrow$$

$$\Leftrightarrow(7x-5)(36x^2+51x+18)=0\Leftrightarrow\left[\begin{array}{l}7x-5=0\\36x^2+51x+18=0\end{array}\right.\Leftrightarrow\left[\begin{array}{l}x_1=-\frac{2}{3}\\x_2=-\frac{3}{4}\\x_3=\frac{5}{7}\end{array}\right.$$

Answer : $x_1=-\frac{2}{3};\quad x_2=-\frac{3}{4};\ x_3=\frac{5}{7}$.

Task 6 . Solve the equation: $tg^6x+ctg^6x=2$

Solution : Let's simplify the left side of the equation, given that $tgx\cdot ctgx=1$

$$tg^6x+ctg^6x=2\Leftrightarrow(tg^2x)^3+(ctg^2x)^3-1^3=1\Leftrightarrow$$
$$\Leftrightarrow(tg^2x+ctg^2x-1)(tg^4x+ctg^4x+1-tg^2x\cdot ctg^2x+tg^2x+ctg^2x)-$$
$$-3tg^2x\cdot ctg^2x-1=0\Leftrightarrow$$
$$\Leftrightarrow(tg^2x+ctg^2x-1)((tg^2x+ctg^2x)^2-2+1-1+tg^2x+ctg^2x)-3-1=0\Leftrightarrow$$
$$\Leftrightarrow(tg^2x+ctg^2x-1)((tg^2x+ctg^2x)^2-2+tg^2x+ctg^2x)-4=0$$

Let $tg^2x+ctg^2x=y$, where $y>0$, then (1) will take the form:

$$\begin{aligned}&(y-1)(y^2+y-2)-4=0 \Leftrightarrow y^3+y^2-2y-y^2-y+2-4=0 \Leftrightarrow\\ &\Leftrightarrow y^3-3y-2=0 \Leftrightarrow y^3-4y+y-2=0 \Leftrightarrow y(y^2-4)+(y-2)=0 \Leftrightarrow\\ &\Leftrightarrow y(y-2)(y+2)+(y-2)=0 \Leftrightarrow (y-2)(y^2+2y+1)=0 \Leftrightarrow\\ &\Leftrightarrow \left[\begin{array}{l} y-2=0\\ y^2+2y+1=0\end{array}\right. \Leftrightarrow \left[\begin{array}{l} y_1=2\\ y_2=-1\end{array}\right.\end{aligned}$$

So $y_1=2$ or

$$\begin{aligned}&tg^2x+ctg^2x=2 \Leftrightarrow tg^2x+\frac{1}{tg^2x}=2 \Leftrightarrow tg^4x-2tg^2x+1=0 \Leftrightarrow\\ &\Leftrightarrow (tg^2x-1)^2=0 \Leftrightarrow tg^2x-1=0 \Leftrightarrow tg^2x=1 \Leftrightarrow tgx=\pm 1 \Leftrightarrow\\ &\Leftrightarrow x=\pm\frac{\pi}{4}+\pi n,\ n\in Z.\end{aligned}$$

Since $y=tg^2x+ctg^2x>0$, then the root $y_2=-1$ does not fit.

Answer. $x=\pm\frac{\pi}{4}+\pi n,\ n\in Z.$

Task 7 . Prove the inequality

$$\left(a^3+b^3+c^3-3abc\right)^2 \le \left(a^2+b^2+c^2\right)^3$$

Proof.

$$\begin{aligned}&\left(a^3+b^3+c^3-3abc\right)^2 \le \left(a^2+b^2+c^2\right)^3 \Rightarrow\\ &\Rightarrow \left((a+b+c)\left(a^2+b^2+c^2-ab-ac-bc\right)\right)^2 \le \left(a^2+b^2+c^2\right)^3 \Rightarrow\\ &\Rightarrow (a+b+c)^2\left(a^2+b^2+c^2-ab-ac-bc\right)^2 \le \left(a^2+b^2+c^2\right)^3 \Rightarrow\\ &\Rightarrow \left(a^2+b^2+c^2+2ab+2ac+2bc\right)\left(a^2+b^2+c^2-ab-ac-bc\right)^2 \le\\ &\le \left(a^2+b^2+c^2\right)^3\end{aligned}$$

Let $a^2+b^2+c^2=A;\ ab+ac+bc=B$ where $A\ge B$, then the last inequality has the form:

$$\begin{aligned}&(A+2B)(A-B)^2 \le A^3 \Rightarrow (A+2B)\left(A^2-2AB+B^2\right) \le A^3 \Rightarrow\\ &\Rightarrow A^3-2A^2B+AB^2+2A^2B-4AB^2+2B^3 \le A^3 \Rightarrow -3AB^2 \le -2B^3 \Rightarrow\\ &\Rightarrow 2B \le 3A.\end{aligned}$$

The inequality has been proven.

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