DYNAMIC BEHAVIOUR OF A FRACTIONAL ORDER WITH A DISEASED PREDATOR-PREY MODEL WITH PREY REFUGE

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Abstract:
In this paper, a diseased fractional-order prey-predator model in one predator population and refuge in the other is modelled and investigated using a Holling type II functional response. Critical points are used to examine a model's stability using its eigenvalues. The solutions' boundness, uniqueness, existence, and positivity have also been analysed. Critical points have been used to study the model's locally asymptotically stable properties, and the Lyapunov function has been used to examine the model's globally asymptotically stable properties. Finally, numerical simulations are shown to verify the analytical solutions.

Keywords: Prey refuge, Fractional order, Boundedness, Existence and Uniqueness.

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1 Introduction

Fractional calculus is a popular field that aims to realize real-life phenomena using non-integer derivative models. In this field, non-integer series derivatives are used for differentiation and integration. The sequence is a derivative that meets the following additional criteria rather than being based on integers: The prime function is what we have when the order of the derivative is zero, and the first-order integer sequence derivative is what we have when the sequence is one [19]. Fractional derivatives have a memory impact and conserved physical properties [3], [7]. The theory of fractional calculus and its illustrative applications in this context are continuously growing in popularity on a worldwide level [14]. In order to simulate real-world issues, a large number of unique fractional operators have been created with various properties [8].Recently, there has been a rise in the demand for developing models for dynamical systems based on fractional-order differential equations [2]. The fractional order derivative is defined by a number of methods. The initial conditions for Caputo fractional differential equations are expressed similarly to how they are for integer-order differential equations, and the Caputo definition is more comprehensive [15], [16]. The main reason is that fractional-order differential equations are closely related to fractals and naturally related to memory-based systems, which are present in the majority of biological systems [6]. Studies on the stability of fractional-order predator-prey systems are still in their infancy because there are few theories available for examining their dynamics [4], [17]. In this study, we investigated the following fractional-order prey-predator model that incorporates a prey refuge and a Holling type II functional response: Numerous researchers have examined the dynamic behaviour of the typical predator-prey system below, which includes a prey refuge [1], [12]. In biology and ecology, a refuge is a place where an organism gets refuge from predators by hiding in a place that is distant, difficult to locate, or otherwise unfavourable to predators [9]. When refuges are present, populations of both predators and prey are much greater due to population dynamics, and an area can support a significant number of additional species [10]. It is obvious that the presence of prey refuges can have a significant impact on the coexistence of predators and their prey. If the proportion of prey in a refuge reduces as prey biomass rises, refuges have a stabilising effect [5], [13]. This is valid when there are a fixed number of prey refuges in the system, but the proportion of prey in refuges also rises when predator biomass and predation rates grow [6], [11]. Numerous authors have researched the issue of predator-prey interactions in the presence of a prey refuge [18]. This study's goal is to examine the dynamics of the proposed fractional-order eco-epidemiological system and demonstrate the global stability analysis of all biologically possible equilibrium points. In this paper, solutions to the proposed fractional-order eco-epidemiological system, in which prey are infected by a disease, are investigated for existence, uniqueness, non-negativity, and boundness.

The paper is followed as: A mathematical model has been formulated in the section 2. The Preliminary dynamics of the fractional order dynamical system have been studied in section 3. The Uniqueness and boundedness of the solution of the proposed model have been studied in section 4.In section 5 stability analysis of the proposed model have been examined. In section 6 numerical simulations are examined for the proposed model. In section 7 we give some main outcomes of our work.

2 Model Formulation

The nonlinear differential equation is

$$\left.\begin{aligned}\frac{dL}{dT} &= r_1L(1-\frac{L+M}{K})-\lambda ML-\frac{\alpha_1LN}{a_1+L},\\ \frac{dM}{dT} &= -d_1M+\lambda ML-\frac{b_1(1-g)MN}{a_1+(1-g)M},\\ \frac{dN}{dT} &= -d_2N+\frac{cb_1(1-g)MN}{a_1+(1-g)M}+\frac{c\alpha_1LN}{a_1+L}.\end{aligned}\right\}\tag{2.1}$$

and the positive conditions are described as $L(0) = L_0 \geq 0, M(0) = M_0 \geq 0, N(0) = N_0 \geq 0$.

A System (2.1) has all positive parameters.The detailed environmental meanings of the parameters are shown in the Table.

Parameters	Environmental representation	Units
r	Prey growth rate	per day (t^{-1})
L	Susceptible Prey	number per unit area (tons)
M	Infected Prey	number per unit area (tons)
N	Predator	number per unit area (tons)
K	Carrying capacity of environment	number per unit area (tons)
a_1	Constant of Half-saturation	m
$\zeta 1$	Predation rate of Susceptible prey	per day (t^{-1})
$b1$	Predation rate of Infected prey	per day (t^{-1})
c	Predator-to-prey conversion rate.	$0 \leq C \leq 1$
d_1	Death rate of infected prey	per day (t^{-1})
d_2	Death rate of Predator population	per day (t^{-1})
λ	Infection rate	per day (t^{-1})
g	Refuge constant of Prey	$m-1$

It is convenient to scale variables to minimize the amount of the system (2.1) parameters as $l = \frac{L}{K}, m = \frac{M}{K}, n = \frac{n}{K}$ and to consider dimensionless time $t = \lambda KT$. Transformation leads to dimensional equations

$$\left.\begin{aligned} \frac{dl}{dt} &= s_1 l(1-l-m) - lm - \frac{s_2 ln}{s_3 + l}, \\ \frac{dm}{dt} &= -s_4 m + ml - \frac{s_5(1-g)mn}{s_3 + (1-g)m}, \\ \frac{dn}{dt} &= -s_6 n + \frac{cs_5(1-g)mn}{s_3 + (1-g)m} + \frac{cs_2 ln}{s_3 + l}. \end{aligned}\right\} \tag{2.2}$$

subject to the positive conditions $l(0) = l_0 \geq 0$, $m(0) = m_0 \geq 0$, $n(0) = n_0 \geq 0$. The fractional system is

$$\left.\begin{aligned} \frac{d^\alpha l}{dt^\alpha} &= s_1 l(1-l-m) - lm - \frac{s_2 ln}{s_3 + l}, \\ \frac{d^\alpha m}{dt^\alpha} &= -s_4 m + ml - \frac{s_5(1-g)mn}{s_3 + (1-g)m}, \\ \frac{d^\alpha z}{dt^\alpha} &= -s_6 n + \frac{cs_5(1-g)mn}{s_3 + (1-g)m} + \frac{cs_2 ln}{s_3 + l}. \end{aligned}\right\} \tag{2.3}$$

subject to the positive conditions $l(0) = l_0 \geq 0$, $m(0) = m_0 \geq 0$, $n(0) = n_0 \geq 0$.

3 Preliminaries

In this section, we provide basic definitions, significant results, and characteristics of fractional differential equations that are useful in the proof of theorems.

DEFINITION 3.1 The Caputo fractional derivative of order α is defined as

$$^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds$$

where $t \geq 0, f \in C^n([0,+\infty), R)$ and Γ is a Gamma function.

LEMMA 3.1 Consider a system of fractional order caputo derivatives

$^C D_t^\alpha x(t) = f(t, x(t)), t > 0, x(0) \geq 0$, $\alpha \in (0,1]$,

where $f : (0,\infty) \times \Omega \rightarrow R^n$. If $f(t,x(t))$ satisfies the locally Lipschitz condition with respect to x, then the equation on $(0,\infty) \times \Omega$ has a unique solution.

THEOREM 3.1 Consider the N-dimensional fractional differential equation system

$$\frac{d^\alpha (x)}{dt} = f(x); x(0) = x_0$$

Where A is the arbitrary constant $N * N$ is the matrix and $\alpha \in (0,1)$.

(i) The solution $x = 0$ is asymptotically stable if and only if all eigenvalues $\lambda_{ij}, j = 1,2,3,...N$ of A satisfies $|arg(\lambda_j)| > \frac{\alpha\pi}{2}$.

(ii) The solution $x = 0$ is stable if and only if all the eigenvalues with $|arg(\lambda_j)| = \frac{\alpha\pi}{2}$ have same geometric multiplicity and algebraic multiplicity. THEOREM 3.2 Consider the fractional-order system

$$\frac{d^\alpha(x)}{dt} = f(x); x(0) = x_0$$

with $x \in R^n$ and $\alpha \in (0,1)$.The above system's equilibrium points are the solutions to the equation $f(x) = 0$. If all of the eigenvalues of the Jacobian matrix $J = \frac{df}{dx}$ evaluated at equilibrium satisfy $|arg(\lambda_j)| > \frac{\alpha\pi}{2}$, then the equilibrium point is considered to be locally asymptotically stable.

4 Uniqueness of Solutions

In this section,the boundedness of the system (2.3) solution has been examined. The system of fractional orders is as follows:

$$\frac{d^\alpha X(t)}{dt^\alpha} = f(t, X(t)), \quad \alpha \in (0,1].$$

Theorem 1 For the non-negative initial conditions, the fractional order system (2.3) has an unique solution.
Proof. A sufficient condition of system (2.3) in the region $\chi \times (0,T]$ where $\chi = (L,M,N) \in R^3 : max(|L|,|M|,|N|) \leq \eta$. Now, let us define a mapping $V(X) = (V_1(X), V_2(X), V_3(X))$ where

$$\mathcal{V}_1(X) = s_1 l(1-l-m) - lm - \frac{s_2 ln}{s_3 + l},$$

$$\mathcal{V}_2(X) = ml - s_4 m - \frac{s_5(1-g)mn}{s_3 + (1-g)m},$$

$$\mathcal{V}_3(X) = -s_6 n + \frac{cs_5(1-g)mn}{s_3 + (1-g)m} + \frac{cs_2 ln}{s_3 + l}.$$

$$\begin{aligned}
||\mathcal{V}(X) - \mathcal{V}(\bar{X})|| &= |\mathcal{V}_1(X) - \mathcal{V}_1(\bar{X})| + |\mathcal{V}_2(X) - \mathcal{V}_2(\bar{X})| + |\mathcal{V}_3(X) - \mathcal{V}_3(\bar{X})|, \\
&= |s_1 l(1-l-m) - lm - \frac{s_2 ln}{s_3+l} - s_1\bar{l}(1-\bar{l}-\bar{m}) + \bar{l}\bar{m} - \frac{s_2\bar{l}\bar{n}}{s_3+\bar{l}}| \\
&+ |ml - s_4 m - \frac{s_5(1-g)mn}{s_3+(1-g)m} - \bar{m}\bar{l} + s_4\bar{m} + \frac{s_5(1-m)\bar{m}\bar{n}}{s_3+(1-g)\bar{m}}| \\
&+ |-s_6 n + \frac{cs_5(1-g)mn}{s_3+(1-g)m} + \frac{cs_2 ln}{s_3+l} + s_6\bar{n} - \frac{cs_5(1-g)\bar{m}\bar{n}}{s_3+(1-g)\bar{m}}| - \frac{cs_2\bar{l}\bar{n}}{s_3+\bar{l}}|, \\
&= |s_1(l-\bar{l}) - s_1(l+\bar{l})(l-\bar{l}) - s_1 l(m-\bar{m}) + s_1\bar{m}(l-\bar{l}) - l(m-\bar{m}) + \bar{m}(l-\bar{l}) \\
&- \frac{s_2 s_3 l(n-\bar{n})}{(s_3+l)(s_3+\bar{l})} + \frac{s_2\bar{n}s_3(l-\bar{l})}{(s_3+l)(s_3+\bar{l})} - \frac{s_2 l\bar{l}(n-\bar{n})}{(s_3+l)(s_3+\bar{l})}| \\
&+ |l(m-\bar{m}) + \bar{m}(l-\bar{l}) - s_4(m-\bar{m}) - \frac{s_5(1-g)n(m-\bar{m})}{s_3+(1-g)m} + \frac{s_5(1-g)\bar{m}(n-\bar{n})}{s_3+(1-g)m} \\
&+ |\frac{cs_2 ls_3(n-\bar{n})}{(s_3+l)(s_3+\bar{l})} + \frac{cs_2\bar{n}s_3(l-\bar{l})}{(s_3+l)(s_3+\bar{l})} + \frac{cs_2 ls_3\bar{l}l(n-\bar{n})}{(s_3+l)(s_3+\bar{l})} \\
&+ \frac{cs_5(1-g)n(m-\bar{m})}{s_3+(1-g)m} + \frac{cs_5(1-g)m(n-\bar{n})}{s_3+(1-g)m} - s_6(n-\bar{n})| \\
&= |l-\bar{l}|\left\{s_1 - s_1 l - s_1\bar{l} + s_1\bar{m} + \bar{m} + \frac{s_2\bar{n}s_3}{(s_3+l)(s_3+\bar{l})}\right\} \\
&+ |m-\bar{m}|\left\{-s_1 l - s_4 - \frac{s_5(1-g)n}{s_3+(1-g)m} + \frac{cs_5(1-g)n}{s_3+(1-g)m}\right\} \\
&+ |n-\bar{n}|\left\{\frac{-s_2 ls_3}{(s_3+l)(s_3+\bar{l})} - \frac{s_2 l\bar{l}}{(s_3+l)(s_3+\bar{l})} + \frac{s_5(1-g)\bar{m}}{s_3+(1-g)m} + \frac{cs_2 ls_3}{(s_3+l)(s_3+\bar{l})}\right. \\
&\left.+ \frac{cs_2 l\bar{l}}{(s_3+l)(s_3+\bar{l})} + \frac{cs_5(1-g)m}{s_3+(1-g)m} - s_6\right\} \\
&\leq |l-\bar{l}|\{s_1 + 2s_1\chi + s_1\chi + \chi + (s_2 s_3 + cs_2 s_3)\eta\} \\
&+ |m-\bar{m}|\{s_1\chi + s_4 + [s_5(1-g)n + cs_5(1-g)n]\chi\}
\end{aligned}$$

$$+ |n-\bar{n}|\{(s_2 s_3 + cs_2 s_3)\chi + (s_2 + cs_2)\chi + [s_5(1-g) + cs_5(1-g)]\chi + s_6\}$$

$$\leq \mathcal{M}|X - \bar{X}|$$

Where,

$$\mathcal{V} = max\left\{s_1 + 2s_1\eta + (2+s_1)\eta + \frac{(1+c)s_2s_3\eta}{(s_2+\eta^2}, (s_1+s_5+cs_5)\eta + s_4 \right.$$
$$\left. \frac{(1+c)s_2s_3\eta}{(s_3+\eta)^2} + \frac{(1+c)s_2s_3\eta^2}{(s_3+\eta)^2} + (1+c)s_5\eta + s_6\right\},$$

Thus V(X) satisfies the Lipschitz condition.
So, the solution of system (2.3) exist and has unique.

□

4.1 Boundedness of solutions

Theorem 2 All of the system (2.3) solutions beginning at R_3^+ are positive and bounded.
Proof. Let $l(t)$,$m(t)$,$n(t)$ be any solution of the system with positive initial conditions.

$$\frac{dl}{dt} \leq s_1 l(1-l),$$

we have $limsup_t \rightarrow l(t) \leq 1$.

Defining a function
$W(t) = l(t) + m(t) + n(t)$.
Taking the Caputo time derivative of W and adding it to the system's solutions gives

$$\frac{dw}{dt} = s_1 l(1-l) - (1+s_1)lm - \frac{(1-c)s_5 ln}{s_3+l} + ml - s_4 m - \frac{(1-c)s_5(1-g)mn}{s_3+(1-g)m} - s_6 n$$
$$\leq s_1 l(1-l) - s_4 m - s_6 n (since c < 1)$$
$$\leq \frac{s_1}{4} - s_4 m - s_6 n.$$
$$max s_1 l(1-l) = \frac{s_1}{4} \leq \frac{s_1}{4} - \gamma W$$

Since where $\gamma = min\{s_4, s_6\}$
So, we have

$$\frac{dW}{dt} + s_1 W \leq \frac{s_1}{4}.$$

Using the theorem of differential inequality, we obtain at

$$0 < W < \frac{s_1}{4\gamma}(1 - exp^{-\gamma t}) + W(l_0, m_0, n_0)exp^{-\gamma t}$$

$$0 < W < \frac{s_1}{4\gamma}.$$

For $t \rightarrow \infty$, we have
As a result, all solutions of the system starting at R_3^+ are bounded in the region for any $\epsilon > 0$.

$$\Omega = \left\{(L, M, N)\epsilon R_+^3 : L + M + N \leq \frac{r}{4\gamma} + \epsilon\right\}.$$

5 Equilibrium Points and Stability analysis

(i) The trivial equilibrium point is $E_0(0,0,0)$.
(ii) The infected prey and predator-free equilibrium point is $E_1(1,0,0)$.
(iii) The infected-free equilibrium point $E_2(\bar{l},0,\bar{n})$ where $\bar{l} = \frac{s_3s_6}{cs_2 - s_6}$ and $\bar{n} = \frac{s_3c((cs_2-s_6)(s_1) - s_3s_1s_6)}{(cs_2-s_6)^2}$
(iv) The Predator-free equilibrium point $E_3(\hat{l},\hat{m},0)$ where $\hat{l} = s_4, \hat{m} = \frac{s_1(1-s_4)}{s_1+1}$.
(v) The interior equilibrium point $E^*(L^*,M^*,N^*)$ where

$$m^* = \frac{s_3(s_3s_6 + (s_6 - cs_2)\ \ l^*)}{(1-g)(cs_2l^* + (cs_5 - s_6)(s_3 + l^*))}, n^* = \frac{s_3c(\ \ l^* - s_4)(s_3 + \ \ l^*)}{(1-g)(cs_2l^* + (cs_5 - s_6)(s_3 + l^*))}$$

and l^* is a unique positive root of a quadratic equation $PS^2 + QS + R = 0$, with
$P = s_1(1-g)(cs_2+cs_5-s_6)$, $Q = (1-g)(cs_5-s_6)(-s_1+s_3s_1)+s_2c(-s_1))+s_3(s_6+(s_6-cs_2)s_1)$, $R = -s_3((1-g)(s_1)(cs_5 - s_6) + (cs_2(s_4) - s_3s_6(1+s_1)))$.

5.1Stability Analysis

In order to analyse local stability around various equilibrium points we compute the Jacobian matrix.At each given point (*L,M,N*), the Jacobian matrix is given by

$$J(L,M,N) = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}.$$

where

$$n_{11} = s_1(1-2l) - m(s_1+1) - \frac{s_2 s_3 n}{(s_3+l)^2}, n_{12} = -l(s_1+1), n_{13} = -displaystyle\frac{s_2 l}{s_3+l}$$
$$n_{21} = m, n_{22} = l - s_4 - \frac{s_3 s_5 (1-g)n}{(s_3+(1-g)m)^2}, n_{23} = -\frac{s_5(1-g)m}{(s_3+(1-g)m)},$$
$$n_{31} = \frac{s_3 c s_2 n}{(s_3+l)^2}, n_{32} = \frac{s_3 c s_5 (1-g)n}{(s_3+(1-g)m)^2}, n_{33} = -s_6 + \frac{c s_5 (1-g)m}{s_3+(1-g)m} + \frac{s_2 c l}{s_3+l},$$

Theorem 3 $E_0(0,0,0)$ is the trivial equilibrium point which is unstable.
Proof. The Jacobian matrix at an equilibrium point E_0 is given by

$$J(E_0) = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & -s_4 & 0 \\ 0 & 0 & -s_6 \end{pmatrix}$$

The characteristic equation that represents the Jacobian matrix J at the point $E_0(0,0,0)$ is
$(\lambda_1 - s_1)(\lambda_2 + s_4)(\lambda_3 + s_6) = 0$

The eigenvalues at $E_(0,0,0)$ are $\lambda_1=-s_4, \lambda_2=s_1, \lambda_3=-s_6$.

Thus, $|arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$,$|arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$ and $|arg(\lambda_3)| = \pi > \frac{\alpha\pi}{2}$.

Therefore, $E_0(0,0,0)$ is unstable. □

Theorem 4 The infected free and predator free equilibrium point $E_1(1,0,0)$ is stable if $cs_2 < s_6$ and $1 < s_4$.
Proof. The Jacobian matrix at an equilibrium point $J(E_1)$ is given by

$$J(E_1) = \begin{pmatrix} -s_1 & -(s_1+1) & \frac{-s_2}{s_3+l} \\ 0 & 1-s_4 & 0 \\ 0 & 0 & -s_6 + \frac{cs_2}{s_3+1} \end{pmatrix}.$$

The characteristic equation of the above Jacobian matrix is

$$(\lambda_{11} - (-s_1))(\lambda_2 - (1-s_4))(\lambda_3 - (-s_6 + \frac{s_2 c}{s_3+1}))$$

$\lambda_1 = -s_1$,$\lambda_2 = (1-s_4)$,$\lambda_3 = (-s_6 + \frac{s_2 c}{s_3+1})$.

Here

Thus,$|arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$,$|arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$ and $|arg(\lambda_3)| = \pi > \frac{\alpha\pi}{2}$.

E_1 is stable if $cs_2 < s_6$ and $1 < s_4$. □

Theorem 5 The infected-free equilibrium point $E_2(\bar{l},0,\bar{n})$ is locally asymptotically stable if $P,R,PQ-R$ are positive.
Proof. The Jacobian matrix at an equilibrium point $J(E_2)$ is given by

$$J(E_2) = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & 0 \\ u_{31} & u_{32} & 0 \end{pmatrix}.$$

where

$$u_{11} = s_1 - \frac{2s_3s_1s_6}{(cs_5 - s_6)}, u_{12} = -\frac{s_3(1+s_1)s_6}{cs_5 - s_6}, u_{13} = -\frac{s_6}{c}$$
$$u_{21} = 0, u_{22} = -s_4 - \frac{s_3s_6}{cs_2 - s_6} - \frac{(1-g)s_5\bar{n}}{s_3}, u_{23} = 0,$$
$$u_{31} = \frac{(cs_2 - s_6)^2\bar{n}}{s_3cs_2}, u_{32} = \frac{c(1-g)s_5\bar{n}}{s_3}, u_{33} = 0.$$
,

The characteristic equation for $J(E2)$ is
$\lambda^3 + P\lambda^2 + Q\lambda + R = 0,$

where
$P = -u11 - u22,$
$Q = -u31u13 + u22u11,\ R = u13u22u31.$

According to the Routh-Hurwitz criteria, if and only if P, R, and $PQ-R$ are all positive, then all of the roots of the characteristic equation have negative real parts.

Thus, $|arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$, $|arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$ and $|arg(\lambda_3)| = \pi > \frac{\alpha\pi}{2}$.

The infecte-free equilibrium point E_2 is locally asymptotically stable. □

Theorem 6 The predator-free equilibrium point $E_3(\hat{l}, \hat{m}, 0)$ is locally asymptotically stable if $s_6 > c(s_2 + s_5)$.
Proof. The Jacobian matrix at an equilibrium point E_3 is given by

$$J(E_3) = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & 0 & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}.$$

where

$$u_{11} = -s_4s_1, u_{12} = (-1 - s_1)\bar{l}, u_{13} = -\frac{s_2\bar{l}}{s_3 + \bar{l}},$$
$$u_{21} = \bar{m}, u_{22} = 0, u_{23} = -\frac{(1-g)s_5\bar{m}}{s_3 + (1-g)\bar{m}},$$
$$u_{31} = 0, a_{32} = 0, u_{33} = \frac{cs_2\bar{l}}{s_3 + \bar{l}} - s_6 + \frac{c(1-g)s_5\bar{m}}{s_3 + (1-g)\bar{m}}.$$

The characteristic equation corresponding to $J(E_3)$ is
$\lambda^3 + P\lambda^2 + Q\lambda + R = 0.$

where
$P = -u11 - u33,$
$Q = -u21u12 + u33u11,\ R = u12u21u33.$

According to the Routh-Hurwitz criteria, if and only if P, R, and $PQ-R$ are all positive, then all of the roots of the characteristic equation have negative real parts.
Therefore, the predator-free equilibrium point E_3 is locally asymptotically stable.

Theorem 7 The interior equilibrium point E^* is locally asymptotically stable.
Proof.

$$J(E^*) = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}.$$

where

$$n_{11} = \frac{-l^*(-s_1 + s_3 s_1 + (1+s_1)m^* + 2s_1 l^*)}{s_3 + l^*}, n_{12} = -(s_1+1)l^*, n_{13} = \frac{-s_2 l^*}{s_3 + \bar{l}}$$
$$n_{21} = m^*, n_{22} = \frac{s_3 s_5 (1-g)^2 m^* n^*}{(s_3 + (1-g)m^*)^2}, n_{23} = \frac{-(1-g)s_5 m^*}{s_3 + (1-g)m^*},$$
$$n_{31} = \frac{s_3 c s_2 n^*}{(s_3 + l^*)^2}, n_{32} = \frac{s_3 c(1-g)s_5 n^*}{(s_3 + (1-g)m^*)^2}, n_{33} = 0$$
,

where
$P = -n11 - n22, Q = -n21n12 + n22n11 - n13n31 + n23n32,$
$R = n13(-n22n31 + n21n32) + n23(n12n31 - n11n32)$

The characteristic equation is given by
$\lambda^3 + P\lambda^2 + Q\lambda + R = 0$
According to Routh-Hurwitz criterion, P,R and $PQ - R$ must all be positive, the characteristic of all the roots be negative.
Hence, E^* is hence locally asymptotically stable.

6 Global Stability Analysis

THEOREM 6.1 The equilibrium point E_1 is globally asymptotically stable when $1 < s_4$ and $s_6(s_3+l) < cs_2$.
Proof. Consider a Lyapunov function

$$V(l,m,n) = [l - 1 - lnl] + m + \left(\frac{1}{c}\right)n.$$

Applying the caputo fractional derivative, we obtain

$$\begin{aligned} {}^C D_t^\alpha V(l,m,n) &\leq \left(\frac{l-1}{l}\right) {}^C D_t^\alpha l + {}^C D_t^\alpha m + \frac{1}{c} {}^C D_t^\alpha n \\ &= (l-1)\left[s_1(1-l-m) - m - \frac{s_2 n}{s_3 + l}\right] + \left[lm - s_4 m - \frac{s_5(1-g)mn}{s_3 + (1-g)m}\right] \\ &+ \frac{1}{c}\left[-s_6 n + \frac{cs_5(1-g)mn}{s_3 + (1-g)m} + \frac{cs_2 ln}{s_3 + l}\right] \\ &\leq -s_1(l-1)^2 - s_1 m(l-1) + m(1-s_4) + n\left(\frac{s_6}{c} - \frac{s_2}{s_3 + l}\right) \end{aligned}$$

Thus, $1 < s_4$ and $s_6(s_3 + l) < cs_2$.

Therefore, E_1 is globally asymptotically stable. □

THEOREM 6.2 The infected prey equilibrium point E_2 is globally asymptotically stable if $\hat{l} < min\left[\frac{s_4}{s_1}, \frac{((1+s_1)s_5cs_2 - cs_5s_1)}{cs_5s_1}\right]$ and $\hat{n} < s_3^2$.

Proof. Define a Lyapunov function

$$V(l,m,n) = \left[l - \hat{l} - \hat{l}ln\frac{l}{\hat{l}}\right] + m + \left[\frac{1}{c}\right]\left[n - \hat{n} - \hat{n}ln\frac{n}{\hat{n}}\right]$$

Applying the caputo fractional derivative, we obtain

$$
\begin{aligned}
{}^{C}D_t^{\alpha}V(l,m,n) &\leq \left[\frac{l-\hat{l}}{l}\right]{}^{C}D_t^{\alpha}l + {}^{C}D_t^{\alpha}m + \left[\frac{1}{c}\right]\left[\frac{n-\hat{n}}{n}\right]{}^{C}D_t^{\alpha}n \\
&= (l-\hat{l})\left[s_1(1-l-m) - m - \frac{s_2 n}{s_3+l}\right] + \left[lm - s_4 m - \frac{s_5(1-g)mn}{s_3+(1-g)m}\right]m \\
&+ \left[\frac{1}{c}\right](n-\hat{n})\left[-s_6 - \frac{cs_5(1-g)m}{s_3+(1-g)m} + \frac{cs_2 l}{s_3+l}\right] \\
&\leq -\left(1-\frac{\hat{n}}{s_3^2}\right)(l-\hat{l})^2 - \left(\frac{s_4}{s_1} - \zeta\right)(1+s_1)m - \left(\frac{(1+s_1)s_5}{s_1} - \frac{(s_3+\hat{l}cs_5)}{cs_2 s_3}\right)\frac{mn}{s_3+m}.
\end{aligned}
$$

Hence, $\hat{l} < min\left\{\frac{s_4}{s_1}, \frac{((1+s_1)s_5cs_2 - cs_5s_1)}{cs_5s_1}\right\}$ Therefore, the infected prey equilibrium point E_3 is globally asymptotically stable. □

THEOREM 6.3 The predator free equilibrium point E_3 is globally asymptotically stable if $s_6 > cs_5 + \frac{cs_2 l}{s_3} + \frac{(1+s_1)cs_2s_5\bar{m}}{s_1s_3}$.

Proof. Define a Lyapunov function

$$V(l,m,n) = \left[l - \bar{l} - \bar{l}ln\frac{l}{\bar{l}}\right] + \left[m - \bar{m} - \bar{m}ln\frac{m}{\bar{m}}\right] + \frac{1}{c}n$$

Applying the caputo fractional derivative, we obtain

$$
\begin{aligned}
{}^{C}D_t^{\alpha}V(l,m,n) &\leq \left(\frac{l-\bar{l}}{l}\right){}^{C}D^{\alpha} + \left(\frac{m-\bar{m}}{m}\right){}^{C}D^{\alpha}m + \frac{1}{c}{}^{C}D^{\alpha}n \\
&= (l-\bar{l})\left[s_1(1-l-m) - m - \frac{s_2 n}{s_3+l}\right] + (m-\bar{m})\left[l - s_4 - \frac{s_5(1-g)n}{s_3+(1-g)m}\right] \\
&+ \frac{1}{c}\left[-s_6 n - \frac{cs_5(1-g)mn}{s_3+m} + \frac{cs_2 ln}{s_3+l}\right] \\
&\leq \frac{s_6}{c}\left[\frac{c\left(\frac{s_2\bar{l}}{s_3+l} + \frac{s_5(1-g)\bar{m}}{s_3+\bar{m}}\right)}{s_6} - 1\right]n
\end{aligned}
$$

Hence.

$$s_6 > cs_5 + \frac{cs_2\bar{l}}{s} + \frac{(1+s_1)cs_2s_5\bar{m}}{s\ \ s}$$

Therefore, E_3 is globally asymptotically stable.

THEOREM 6.4 The interior equilibrium point E^* is globally asymptotically stable if $cs_2s_5 > cs_5$ and $\frac{s_6 n^*}{cs_2} + \frac{(1+s_1)s_4m^*}{s_1} < l^* < min\left\{m^* + s_1m^* - 1, \frac{s_4}{s_1}, \frac{s_6s_1s_3 - (1+s_1)s_5s_3m^*}{s_1s_3}\right\}$.

Proof Consider a positive Lyapunov function

$$V_4 = \left[l - l^* - l^*ln\frac{l}{l^*}\right] + \left[m - m^* - m^*ln\frac{m}{m^*}\right] + \frac{1}{c}\left[n - n^* - n^*ln\frac{n}{n^*}\right]$$

Applying the caputo fractional derivative, we obtain

$$\leq \left(\frac{l-l^*}{l}\right) {}^C D^\alpha l + \left[\frac{m-m^*}{m}\right] + \frac{1}{c}\left(\frac{n-n^*}{n}\right)$$
$$= (l-l^*)\left[s_1(1-l-m) - m - \frac{s_2 n}{s_3+l}\right] + \frac{1+s_1}{s_1}(m-m^*)\left[l - s_4 - \frac{s_5(1-g)n}{s_3+(1-g)m}\right]$$
$$+ \frac{1}{c}(n-n^*)\left[-s_6 + \frac{cs_5(1-g)m}{s_3+(1-g)m} + \frac{cs_2 l}{s_3+l}\right]$$
$$\leq -((1+s_1)m^* - (1+l^*))l - \left(\frac{s_4}{s_1} - l^*\right)(1+s_1)m - \left(s_5 - \frac{cs_5(1-g)}{cs_2}\right)\frac{mn}{s_3+(1-g)m}$$
$$- \left(\frac{s_6}{cs_2} - \frac{l^*}{s_3} - \frac{(1+s_1)s_5 m^*}{s_1 s_3}\right)n - \left(l^* - \frac{s_6 n^*}{cs_2} - \frac{(1+s_1)s_4 m^*}{s_1}\right)$$

Obviously, $cs_2 > \frac{cs}{\frac{cs_5}{s_5}}$ and

$$\frac{s_6 n^*}{cs_2} + \frac{(1+s_1)s_4 m^*}{s_1} < l^* < min\left\{m^* + s_1 m^* - 1, \frac{s_4}{s_1}, \frac{s_6 s_1 s_3 - (1+s_1)s_5 s_3 m^*}{s_1 s_3}\right\}.$$

we conclude that E^* is globally asymptotically stable.

7 Numerical Analysis

In this section, we present some numerical simulation results for Caputo-sense fractional-order ecoepidemic models. To accomplish this, we use Diethelm et al.'s predictor-corrector approach to solve the defined model. Since there are no field data available, the simulations are carried out with the following assumed parameter values:
The parameter values are $s1 = 0.5; s_2 = 0.25; S3 = 0.3; S4 = 0.1; S5 = 0.4; S6 = 0.1 c = 0.5; m = 0.3$. Then the positive equilibrium point $E^*(0.61561;0.0325119;0.525293)$ for the derivative of $\alpha = 1$ becomes locally asymptotically stable is shown in figure (1).The positive equilibrium point $E^*(0.61561;0.0325119;0.525293)$ for the derivative of $\alpha = 0.94$ also becomes locally asymptotically stable is shown in figure (2).

The parameter values are $s_1 = 0.5, s_2 = 0.2, s_3 = 0.3, s_4 = 0.3 = 2, s_5 = 0.4, s_6 = 0.2, c = 0.5, g$=variable.

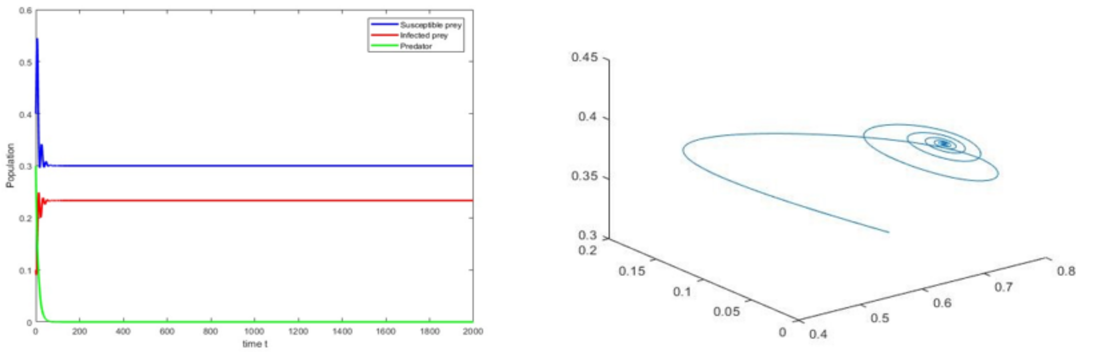

Figure 1: Time series and Phase portrait for the equilibrium point E^* for system (2.3) for $\alpha = 1$

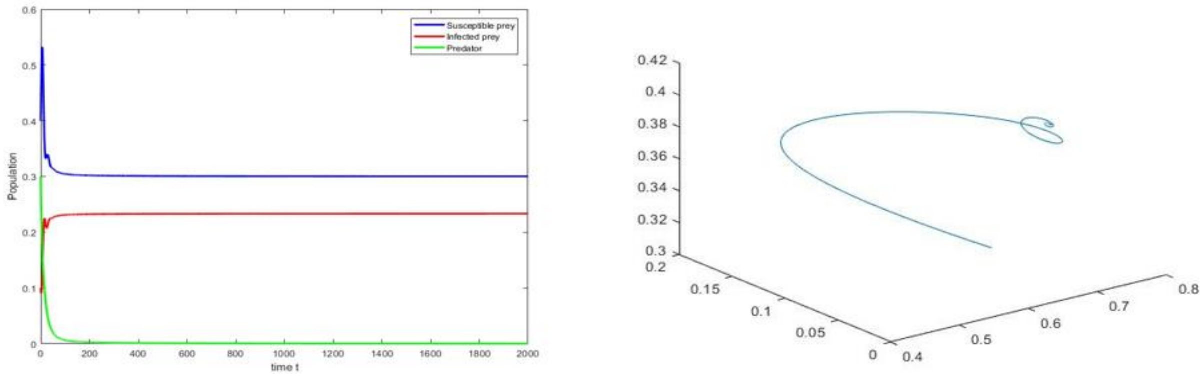

Figure 2: Time series and Phase portrait for the equilibrium point E^* for system (2.3) for $\alpha = 0.94$

Let us fix the value of refuge g as variable.From figure (1), it is observed that the system (2.3) undergoes an unstable solution for the derivative value of $\alpha = 1$ for the equilibirum point E_2 is shown in figure (3).The fractional-order derivative allows the system's solution (2.3) to become stable at 0.94 for the equilibirum point E_2, as shown in figure (4).Therefore, it can be concluded from figures (3) and (4) that the equilibrium point E_2 of the system might change from unstable to stable due to the fractional order derivative.Therefore, it may be stated that the fractional-order derivative may improve system stability. From Figure (5), we can observe that the density of the susceptible prey population decreases as the refuge constant increases. Figure (5) shows an increase in infected prey population as the refuge constant m increases from 0.2 to 0.6.Therefore, it can be concluded from Figure (5) that the stability of our suggested system is significantly affected by the fractional-order derivative.

8 Conclusion

In this study, we investigated a condition when a predator hunts on both susceptible and ill prey, which is known as a refuge in a prey.While the diseased prey density decreases, the susceptible prey density rises as the diseased prey refuge increases. A decrease in the population of diseased prey and an increase

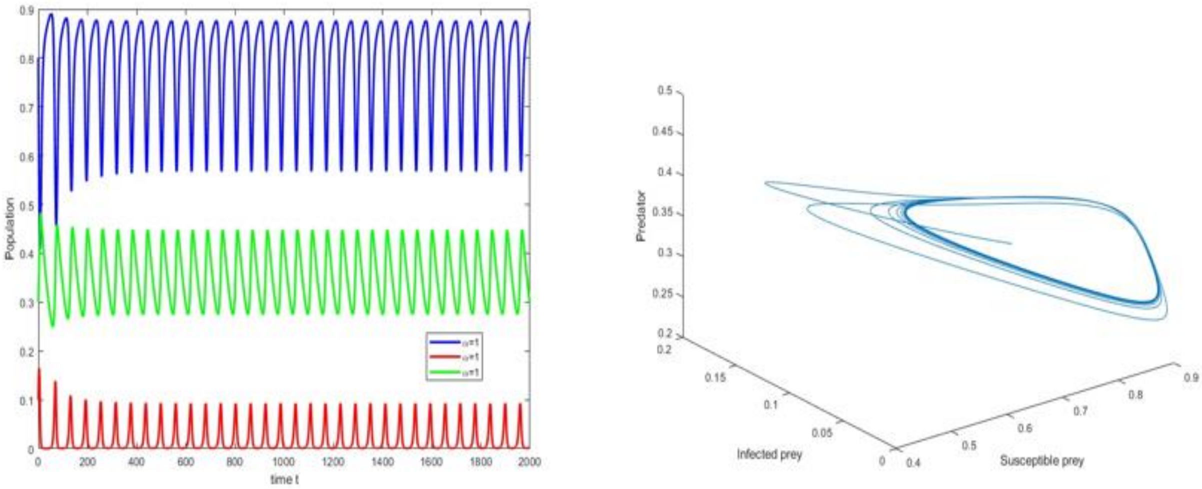

Figure 3: Unstable solution for equilibrium point E_2 of system (2.3) for $\alpha = 1$

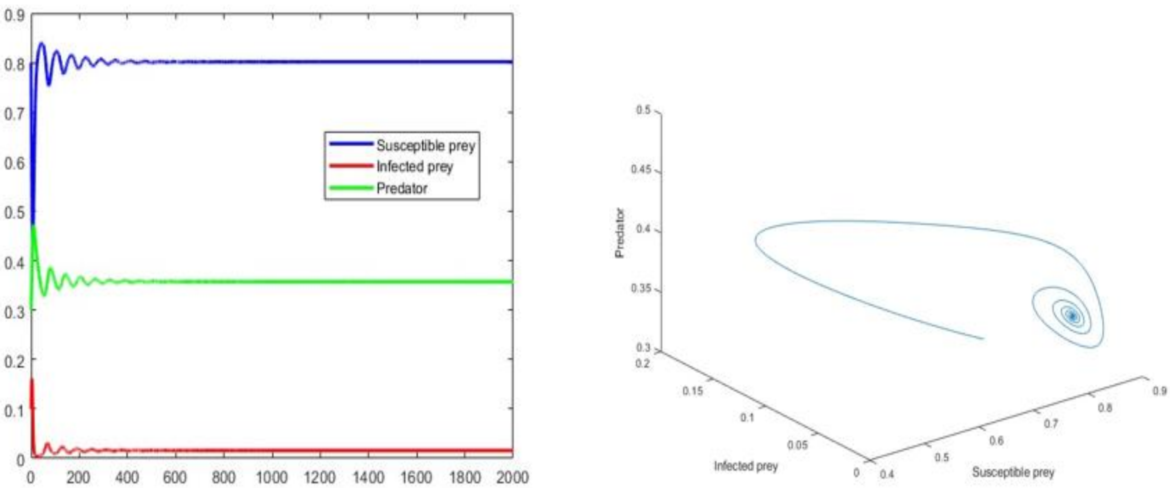

Figure 4: Stable solution for equilibrium point E_2 of system (2.3) for $\alpha = 0.94$

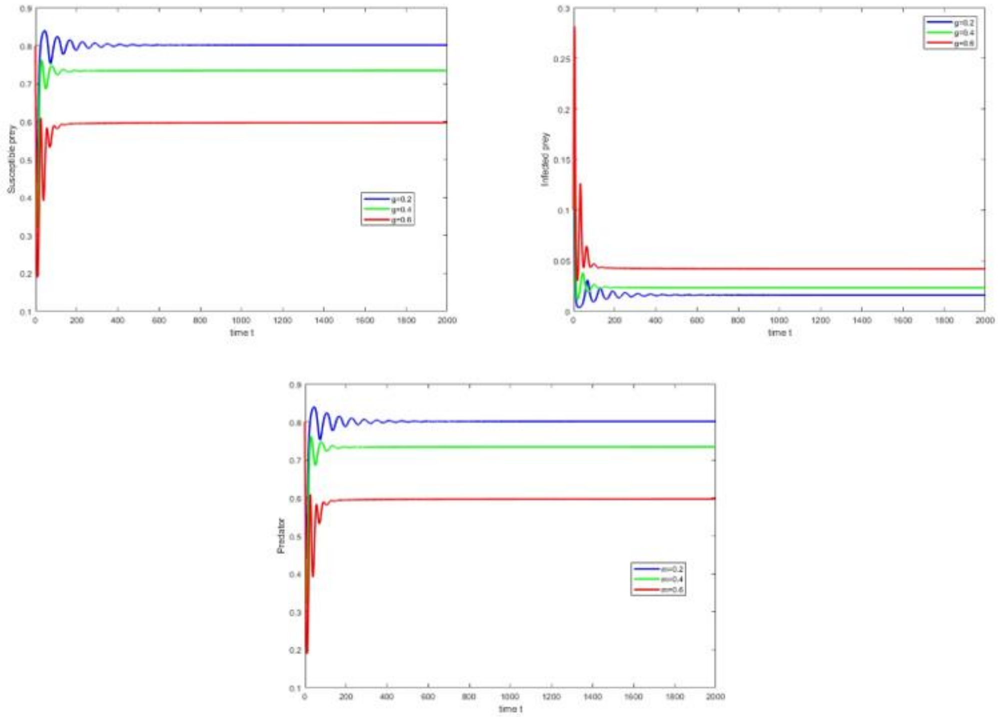

Figure 5: Different values of refuge $g = 0.2, 0.4, 0.6$ for the derivative of $\alpha = 0.94$ in the number of predators and prey are two effects of raising the susceptible predation rate. This study illustrates the complex behaviour of the suggested model.The infected-free and endemic equilibrium points emerge and become stable, particularly when the infected refuge and susceptible prey predation rate fall within a specified range. The infected prey refuge in the model generates complex dynamics.

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