ON N-GROUP'S PRIME IF IDEAL

Pradip Saikia[1] and L.K. Barthakur[2]

[1]Assistan Professor, Department of Mathematics, Morigaon College, Assam, India[2]Principal,Morigaon College, Assam, India

Article History: **Received:** 19.05.2023 **Revised:**15.06.2023 **Accepted:** 01.07.2023

Abstract

Through the discussions of some distinguished properties of IF ideal of N-group an attempt has been made to introduce the notion of prime IF ideal. Thereafter, we try to establish some results of prime IF ideal in the light of f-invariant IFS and finally we have discussed about the IF coset and prime IF ideal of quotient N-group.

Keyword:

N-group, IF ideal, prime IF ideal, f-invariant, IF coset

AMS Subject Classification: 03E72,16Y30

1. Introduction and preliminaries

As soon as the importance of a non membership grade of an element is established by Atanassov in [4] and introduced a new notion called intuitionistic fuzzy set (IFS) as an extended view of Zadeh's fuzzy set [5], lots of the researchers took part in the journey of the study of various algebraic structures in the light of that notion newly or been already studied under fuzzy set. One of the most important algebraic structure is near-ring duiscussed by Pilz[1] and Clay[3] and its module structure called N-group. Saikia and Barthakur in[2],Ratnabala and Ranjit [6]contribute in the field of ideals of N-group in case of fuzzy set extension. Moreover Ratnabala in [7] introduced intuitionistic fuzzy extended notion of ideal and coset and quotient group of N-group. We too in [8] discussed various properties of IF ideal. We provide the concept of prime IF ideal and some related properties in the paper. Here IF means intuitionistic fuzzy and IFS means intuitionistic fuzzy set.

Now we recall some basic definitions and theorems which will be helpful during the study.

Definition1.1[7].For near-ring N, an N-group E's IFS $A =< \mu_A, \mathrm{P}_A >$ is said to be an E's IF ideal if the following holds:

(i) $\mu(u-v) \geq \mu_A(u) \wedge \mu_A(v)$; $\mathrm{P}_A(u-v) \leq \mathrm{P}_A(u) \vee \mathrm{P}_A(v)$

(ii) $\mu(v+u-v) \geq \mu_A(u)$; $\mathrm{P}_A(v+u-v) \leq \mathrm{P}_A(u)$

(iii) $\mu(nu) \geq \mu_A(u)$; $\mathrm{P}_A(nu) \leq \mathrm{P}_A(u)$

(iv) $\mu(n(v+u)-nv) \geq \mu_A(u)$; $\mathrm{P}_A(n(v+u)-nv) \leq \mathrm{P}_A(u)$ for all $u, v \in E, n \in N$.

Theorem 1.2.Consider the near-ring N with unity and E be its N-group and $A =< \mu_A, \mathrm{P}_A >$ be IFS of E such that $\mu(nx) \geq \mu_A(x); \mathrm{P}_A(nx) \leq \mathrm{P}_A(x), \forall x \in E$, then $\mu_A(nx) = \mu_A(x); \mathrm{P}_A(nx) = \mathrm{P}(x), \forall x \in E$ if n is left invertible for $0 \neq n \in N$.

Proof: Let n_1 be the left inverse of n. Then $n_1 n = 1$.

Now for $x \in E, \mu(x) = \mu_A(1.x) = \mu(n_1 nx) \geq \mu_A(nx)$ and $\mathrm{P}_A(x) = \mathrm{P}_A(1.x) = \mathrm{P}(n_1 nx) \leq \mathrm{P}_A(nx)$. Therefore $\mu(nx) = \mu_A(x); \mathrm{P}_A(nx) = \mathrm{P}_A(x)$.

Theorem 1.3.[7] Every IF ideal $A =< \mu_A, \mathrm{P}_A >$ of N-group E satisfies the following:

(i) $\mu(0) \geq \mu_P(u)$ and $\mathrm{P}_P(0) \leq \mathrm{P}_P(u), \forall u \in E$

(ii) $\mu(p+q) = \mu_P(q+p)$ and $\mathrm{P}_P(p+q) \leq \mathrm{P}_P(q+p), \forall p, q \in E$

Theorem 1.4. [8] For N-group E, its IFS $A =< \mu_A, \mathrm{P}_A >$ is an E's IF ideal $\Leftrightarrow {}_{(s,t)}$ is an E'sideal .

Theorem 1.5 For N-group E and $P \subseteq E$. Now for $s, t, s', t' \in [0,1]$, define

$\mu_P(p) = \begin{cases} s & ,p \in P \\ s', & otherwise \end{cases}$; $\mathrm{P}_P(p) = \begin{cases} t & ,p \in P \\ t', & otherwise \end{cases}$ .Then P is an E's ideal $\Leftrightarrow < \mu_P, \mathrm{P}_P >$ is an E's IF ideal, where $s > s', t < t'$.

Proof: Suppose P be an E's ideal. Then for $p, q \in E$ if $p, q \in P$ then $p - q \in P$. Now $\mu(p - q) = s \geq s \wedge s = \mu(p) \wedge \mu_P(q)$ and $\mathrm{P}_P(p-q) = t \leq t \vee t = \mathrm{P}_P(p) \vee \mathrm{P}_P(q)$. For $p \in P, q \notin P$, $since\ p - q \notin P\ so\ \mu(p-q) = s' \geq s \wedge s' = \mu_P(p) \wedge \mu_P(q)$ and $\mathrm{P}_P(p-q) = t' \leq t \vee t' = \mathrm{P}_P(p) \vee \mathrm{P}_P(q)$. Also for $p, q \notin P,\ since\ p - q \notin P\ \ so\ \mu(p-q) = s' \geq s' \wedge s' = \mu_P(p) \wedge \mu(q)$ and $\mathrm{P}_P(p-q) = t' \leq t' \vee t' = \mathrm{P}_P(p) \vee \mathrm{P}_P(q)$. Let $p \in P$. Since P is an ideal so $q + p - q \in P$ for $q \in E$. Hence $\mu(q+p-q) = s = \mu_P(p)$ and $\mathrm{P}_P(q+p-q) = t = \mathrm{P}_P(p)$. If $\mu(y+x-y) = s'$ and $\mathrm{P}_P(y+x-y) = t'$ then $y + x - y \notin P$ and hence $\mu_P(x) = s', \mathrm{P}_P(p) = t'$. Again for $n \in N, p \in P$ since P is an ideal so $np \in P$ and hence $\mu(np) = s = \mu_P(p), \mathrm{P}_P(np) = t = \mathrm{P}_P(p)$. If $\mu(np) = s', \mathrm{P}_P(np) = t'$ then $np \notin P$ which implies $p \notin P$ so that $\mu_P(p) = s', \mathrm{P}_P(p) = t'$. Also for $p \in P, n \in N, q \in E$ since P is an ideal so $(q+p) - nq \in P$ so that $\mu(n(q+p) - nq) = s = \mu_P(p)$ and $\mathrm{P}_P(n(q+p) - nq) = t = \mathrm{P}_P(p)$. If $p \notin P$ then $\mu_P(n(q+p) - nq) \geq s' = \mu_P(p)$ and $\mathrm{P}_P(n(q+p) - nq) \leq t' = \mathrm{P}_P(p)$.

Hence $< \mu_P, \mathrm{P}_P >$ is IF ideal of E.

Conversely, suppose $< \mu_P, \mathrm{P}_P >$ is an IF ideal of E. Let $x, y \in P$ so that $\mu(p) = s, \mathrm{P}_P(p) = t$; $\mu_P(q) = s', \mathrm{P}_P(q) = t$. Then $\mu(p-q) \geq \mu_P(p) \wedge \mu_P(q) = s \wedge s = s$ and $\mathrm{P}_P(p-q) \leq \mathrm{P}(p) \vee \mathrm{P}_P(q) = t \vee t = t$ gives $p - q \in P$.Let $p \in P$. Since $< \mu_P, \mathrm{P}_P >$ is an IF ideal so for all $q \in E, \mu(q+p-q) \geq \mu_P(p) = s$ and $\mathrm{P}_P(q+p-q) \leq \mathrm{P}_P(p) = t$. Therefore $q + p - q \in P$. Again for $n \in N, p \in P$ since $< \mu_P, \mathrm{P}_P >$ is an IF ideal of E, so $\mu(np) \geq \mu_P(p) = s$ and $\mathrm{P}(np) \leq \mathrm{P}_P(p) = t$ gives $np \in P$. Also for $n \in N, p \in P, q \in E, \mu(n(q+p) - nq) \geq$

$\mu(p) = s$ and $\text{Ꝑ}_P(n(q+p) - nq) \leq \text{Ꝑ}_P(p) = t$. Therefore $(q+p) - nq \in P$. Hence P isideal of E.

In the above result the condition "$s > s'; t < t'$" is necessary. For this let us see the following remarks.

Remark: If we replace the "$s > s'; t < t'$" by "$s = s'; t = t'$" then the result will not be true. For justification, let $E = Z_5$ and N=Z, the ring of integers. The E is an N-group. Take $P = \{1,2,3\} \subseteq E$. Define $\mu_P(p) = \begin{cases} s & ,p \in P \\ s', & otherwise \end{cases}$; $\text{Ꝑ}_P(p) = \begin{cases} t & ,p \in P \\ t', & otherwise \end{cases}$

Now if $s = s'; t = t'$ then $< \mu_P, \text{Ꝑ}_P >$ is E's IF ideal whereas P is not.

Remark: If "$s > s'; t < t'$" is replaced by "$s < s'; t > t'$" then also the result will not be true. For justification, let P be an N-group E's ideal and define
$\mu_P(x) = \begin{cases} s & ,x \in P \\ s', & otherwise \end{cases}$; $\text{Ꝑ}_P(x) = \begin{cases} t & ,x \in P \\ t', & otherwise \end{cases}$ such that $s < s'; t > t'$. Now if possible assume $< \mu_P, \text{Ꝑ}_P >$ be an IF ideal of E. Then by theorem 1.3 , $\mu_P(0) \geq \mu_P(x)$ and $\text{Ꝑ}(0) \leq \text{Ꝑ}_P(x), \forall x \in E$. Now for $y \in E - P$ we have since $0 \in P$ so $s = \mu(0) \geq \mu_P(y) = s'$ and $t = \text{Ꝑ}_P(0) \leq \text{Ꝑ}_P(y) = t'$ since $y \notin P$ gives $s \geq s'$ and $t \leq t'$, which is a contradiction. Hence the result is not true for the condition "$s < s'; t > t'$".

Theorem1.6.[7] For N-groups E_1 and E_2 and N-homomorphism $f: E_1 \to E_2$, $f(A)$ is an IF ideal of E_2 when $A =< \mu_A, \text{Ꝑ}_A >$ is E_1's IF ideal.

Theorem1.7.[8] For N-groups E_1 and E_2 and N-homomorphism $f: E_1 \to E_2$, $f^{-1}(A)$ is an IF ideal of E_1 whenever $A =< \mu_A, \text{Ꝑ}_A >$ is an IF ideal of E_2.

Defintion1.8. An N-group E's proper ideal A is called E's prime ideal if for $0 \neq n \in N, u \notin A$ we have $nu \notin A$. If the ideal (0) is E's prime ideal then E is named as prime N-group.

Theorem1.9. E/A is non zero prime N-group ⇔ A is an E's prime ideal, for E's any ideal A.

Proof: Consider E/A as non zero prime N-group. Thus {0+A},the zero ideal is an E/A's prime ideal. Since E/A is non zero N-group so A is a proper ideal. Let $0 \neq n \in N, e \in E$ such that $ne \in A$. Now since {0+A} is E's prime ideal so, $(e + A) = ne + A = 0 + A \in \{0 + A\}$ which implies $e + A \in \{0 + A\}$ or $e \in A$. Hence A is E's prime ideal.

Conversely suppose that A is E's prime ideal. Since $A \neq E$, so E/A is a non zero N-group. Clearly {0+A} is proper ideal of E/A. Let $0 \neq n \in N, e + A \in E/P$ such that $(e + A) \in \{0 + A\}$. Then $ne + A = 0 + A$ and $ne \in A \Rightarrow e \in$ (since A is prime ideal and $0 \neq n$) $\Rightarrow x + A \in \{0 + A\}$. Therefore the zero ideal {0+A} is E/A's prime ideal and so E/A is a non zero prime N-group.

2. Prime IF ideal of N-group E:

Definition2.1. A non constant IF ideal $A =< \mu_A, \text{P}_A >$ of N-group E is called an E's prime IF ideal if $\forall\ 0 \neq n \in N, x \in E, \mu(nx) \leq \mu_A(x)$ and $\text{P}_A(nx) \geq \text{P}_A(x)$

Note: For a zero symmetric near-ring N, since for any $n \in N, x \in E, \mu(nx) \geq \mu_A(x)$ and $\text{P}(nx) \leq \text{P}_A(x)$, where $A =< \mu_A, \text{P}_A >$ is IF ideal of E so we can conclude that A is prime ideal - $\mu_A(nx) = \mu_A(x)$; $\text{P}_A(nx) = \text{P}_A(x)$.

Example: Every non constant IF ideal of E considered for division ring R is a prime IF ideal since every non zero element of R possesses inverse, it follows by Theorem1.2.

Remark: For N-group E, let IFS $A =< \mu_A, \text{P}_A >$ be such that $\mu(x) = 0.5$; $\text{P}_A(x) = 0.4, \forall x \in E$. Then A is a constant IF ideal and hence not a prime IF ideal.

Theorem 2.2.Any E's ideal P is E's prime ideal ⇔ the generalized characteristics function $\mu_P(x) = \{\begin{matrix} s & ,x \in P \\ s', & otherwise \end{matrix}$; $\text{P}_P(x) = \{\begin{matrix} t & ,x \in P \\ t', & otherwise \end{matrix}$ with $s > s', t < t'$ is an E's prime IF ideal.

Proof: Suppose P be prime ideal. Therefore P is proper and by Theorem1.5, $< \mu_P, \text{P}_P >$ is an E's non constant IF ideal. Let $0 \neq n \in N, \in E$. Suppose $x \notin P$. Therefore $\mu(x) = s', \text{P}_P(x) = t'$. Now being P as a prime ideal $nx \notin P$, which implies $\mu(nx) = s', \text{P}_P(nx) = t'$ so that $\mu(nx) = \mu_P(x), \text{P}_P(nx) = \text{P}_P(x)$.Suppose $x \in P$. Then $\mu(x) = s, \text{P}_P(x) = t$, so $\mu_P(nx) \geq \mu(x) = s, \text{P}_P(nx) \leq \text{P}_P(x) = t$ which shows $\mu_P(nx) = \mu_P(x), \text{P}_P(nx) = \text{P}_P(x)$. Therefore $< \mu_P, \text{P}_P >$ is E's prime IF ideal.

Conversely assume that $< \mu_P, \text{P}_P >$ is E's prime IF ideal. Therefore $< \mu_P, \text{P}_P >$ is non constant IF ideal and hence P is proper ideal. Since $x \notin P$ so $\mu(x) = s', \text{P}_P(x) = t'$. Now as $< \mu_P, \text{P}_P >$ is prime IF ideal so $\mu(nx) = \mu_P(x) = s', \text{P}_P(nx) = \text{P}_P(x) = t'$ and which implies $nx \notin P$. Thus P is E's prime ideal.

Corrollary2.3. P is E's prime ideal $\Longleftrightarrow$ characteristic function $< \varphi_P, \psi >$ is E's prime IFid

Proof: Proof is straight forward by putting $s = 1, s' = 0; t = 0, t' = 1$.

Theorem2.4.Let $A =< \mu_A, \text{P}_A >$ be an E's non constant IF ideal. Then A is an E's prime IF ideal - for any $s \in [0, \mu_A(0)], t \in [0, \text{P}_A(0)]$, either $A_{(s,t)} = E$ or $A_{(s,t)}$ is an E's prime ideal.

Proof: Assume $A =< \mu_A, \text{P}_A >$ be E's prime IF ideal and $s \in [0, \mu(0)], t \in [0, \text{P}_A(0)]$. Now if $_{(s,t)} = E$ then there is nothing to prove. So let $_{(s,t)} \neq E$. Then since A is E's IF ideal so by theorem1.4, $_{(s,t)}$ is E's proper ideal. Let $x \notin {}_{(s,t)}$ and $0 \neq n \in N$. Then $\mu(x) < s, \text{P}_A(x) > t$. Now since A is prime IF ideal so $\mu(nx) = \mu_A(x) < s; \text{P}_A(nx) = \text{P}_A(x) > t$ and which implies $nx \notin A_{(s,t)}$. Hence $_{(s,t)}$ is E's prime ideal.

Conversely, suppose that for any $s \in [0, \mu(0)], t \in [0, Ρ_A(0)]$, either $A_{(s,t)} = E$ or $A_{(s,t)}$ is an E's prime ideal. Let $0 \neq n \in N$ and $x \in E$. Suppose $\mu(nx) = s, Ρ_A(nx) = t$.So $nx \in A_{(s,t)}$. Now $_{(s,t)} = E$ implies $x \in A_{(s,t)}$ and if $A_{(s,t)}$ is assumed as E's prime ideal then $nx \in A_{(s,t)} \Rightarrow x \in A_{(s,t)}$. Therefore in each case $x \in _{(s,t)} \Rightarrow \mu_A(x) \geq s = \mu_A(nx)$ and $Ρ_A(x) \leq t = Ρ_A(nx)$. So $\mu(nx) = \mu_A(x)$; $Ρ_A(nx) = Ρ_A(x), \forall\ 0 \neq n \in N, x \in E$.

Hence $A =< \mu_A, Ρ_A >$ is E's prime IF ideal.

Remark: In theorem2.4, $A =< \mu_A, Ρ_A >$ must be non constant. Since for IFS $A =< \mu_A, Ρ_A >$ ofE such that $\mu(x) = 0.4, Ρ_A(x) = 0.4, \forall x \in E$,we have $A_{(s,t)} = E, \forall s \in [0, \mu_A(0)], t \in [0, Ρ(0)]$,whereas $A =< \mu_A, Ρ_A >$ is not a prime IF ideal.

3. f-invariant IFS and prime IF ideal:

The definition of f-invariant IFS are recalled and on the basis of these some related results for prime IF ideal are introduced in this part.

Definition3.1. Consider any $A =< \mu_A, Ρ_A >$ be IFS of a set Y. Then for function $f: Y \to Y$, IFS $A =< \mu_A, Ρ_A >$ is called f-invariant if $\forall u, v \in E, (u) = f(v) \Rightarrow \mu_A(u) = \mu_A(v);\ Ρ_A(u) = Ρ_A(v)$.

Example: Let us define $f: Z \to Z$ by $(u) = \begin{cases} 1, u \in 2Z \\ 0.5, u \notin 2Z, u > 0 \\ 0, u \notin 2Z, u < 0 \end{cases}$

Define IFS $A =< \mu_A, Ρ_A >$ of Z by $\mu_A(u) = \begin{cases} 0.2, u \in 2Z \\ 0.4, u \notin 2Z \end{cases}$ and $Ρ_A(u) = \begin{cases} 0.4, u \in 2Z \\ 0.5, u \notin 2Z \end{cases}$

Then $A =< \mu_A, Ρ_A >$ is f-invariant.

Example: Let $E_1 = Z$, $E_2 = Z_6$ be two N-groups of the ring of integers Z and $f: E_1 \to E_2$ be a canonical N-epimorphism. Let us define IFS $A =< \mu_A, Ρ_A >$ of E_1 such that

$A =< \varphi_{2Z}, \psi_{2Z} >$, characeristic function on 2Z. Then by Theorem2.2, A is IF ideal of E_1.Again for any $x, y \in E_1$ such that $f(x) = f(y)$, since $x - y$ is divisible by 6 gives $x - y$ is divisible by 2 so for both x, y are even $\Rightarrow \mu_A(x) = \mu_A(y) = 1;\ Ρ_A(x) = 0, Ρ_A(y) = 0$ and for both x, y are odd $\Rightarrow \mu_A(x) = \mu_A(y) = 0;\ Ρ_A(x) = 1, Ρ_A(y) = 1$. Hence $A =< \mu_A, Ρ_A >$ is f-invariant.

Note: There exists a non constant IF ideal $A =< \mu_A, Ρ_A >$ and a non constant N-epimorphism f such that f(A) is a constant IF ideal. For justification we check the following example.

Example: Consider $E_1 = Z, E_2 = Z_6$ as two N-groups of the near-ring N=Z of integers. Also let $f: E_1 \to E_2$ be natural epimorphism of N-groups. Then clearly f is a non constant N-homomorphism.

Let $s, s', t, t' \in [0,1]$ such that $s > s', t < t'$ and consider IFS $A =< \mu_A, \mathrm{P}_A >$ on E_1 such that by $\mu_A(x) = \begin{cases} s, x \in 5Z \\ s', otherwise \end{cases}$ and $\mathrm{P}_A(x) = \begin{cases} t, x \in 5Z \\ t', otherwise \end{cases}$

Then $A =< \mu_A, \mathrm{P}_A >$ is a generalized characteristics function and it is clear that it is a non constant IFS of E_1 .Then by Theorem1.5, A is an IF ideal of E. Now we verify that f(A) is a constant IF ideal of E_2.

$\mu_{(A)}(0) = \vee \{\mu_A(x): x \in f^{-1}(0)\} \geq \mu_A(0)$ ($\because$ $0 \in f^{-1}(0)$)

$= s$ ($\because 0 \in 5Z \Rightarrow \mu(0) = s$)

and $\mathrm{P}_{f(A)}(0) = \wedge \{\mathrm{P}_A(x): x \in f^{-1}(0)\} \leq \mathrm{P}_A(0)$ ($\because$ $0 \in f^{-1}(0)$)

$= t$ ($\because 0 \in 5Z \Rightarrow \mathrm{P}(0) = t$)

Therefore $\mu_{(A)}(0) = s, \mathrm{P}_{f(A)}(0) = t$

Again, $\mu_{(A)}(1) = \vee \{\mu_A(x): x \in f^{-1}(1)\} \geq \mu_A(25)$ ($\because$ $25 \in f^{-1}(1)$)

$= s$ ($\because 25 \in 5Z$)

and $\mathrm{P}_{f(A)}(1) = \wedge \{\mathrm{P}_A(x): x \in f^{-1}(1)\} \leq \mathrm{P}_A(25)$ ($\because$ $25 \in f^{-1}(1)$)

$= t$ ($\because 25 \in 5Z$)

Therefore $\mu_{(A)}(1) = s, \mathrm{P}_{f(A)}(1) = t$. We can verify similarly that $\mu_{(A)}(x) = s, \mathrm{P}_{f(A)}(x) = t, \forall x$. Hence $(A) =< \mu_{f(A)}, \mathrm{P}_{f(A)} >$ is constant IF ideal of E_2.

Theorem3.2. Suppose E_1 and E_2 be two N-groups and f: $E_1 \rightarrow E_2$ is a non constant N-epimorphism and $A =< \mu_A, \mathrm{P}_A >$ be an f-invariant IF ideal of E. If $A =< \mu_A, \mathrm{P}_A >$ is a E_1 's prime IF ideal then $f(A)$ is either constant or a E_2's prime IF ideal .

Proof: Suppose $A =< \mu_A, \mathrm{P}_A >$ be prime IF ideal of E. Then by theorem1.6, f(A) is an IF ideal of E_2. Now if f(A) is constant then we have nothing to prove. Suppose f(A) is non constant. Then to show f(A) is prime IF ideal of E_2. For that let $0 \neq n \in N$ and $x_1 \in E_1$ .Since f is an epimorphism there exists some $x \in E$ such that $f(x) = x_1$. Again $(nx) = nf(x) = nx_1 \Rightarrow nx \in f^{-1}(nx_1)$. If $p, q \in f^{-1}(nx_1)$ then since $f(p) = nx_1 = f(q)$ so $\mu_A(p) = \mu_A(q)$ and $\mathrm{P}_A(p) = \mathrm{P}_A(q)$ as $A =< \mu_A, \mathrm{P}_A >$ is f-invariant. Which shows

$\{(\mu(x), \mathrm{P}_A(x)): x \in f^{-1}(nx_1)\} = \{(\mu(nx), \mathrm{P}_A(nx))\}$--------------------(i)

Since $(x) = x_1$ we have $x \in f^{-1}(x_1)$. Now if $p_1, q_1 \in f^{-1}(x_1)$, then $f(p_1) = f(q_1)$ and so by invariant nature $\mu_A(p_1) = \mu_A(q_1)$ and $\mathrm{P}_A(p_1) = \mathrm{P}_A(q_1)$, which shows

$\{(\mu(y), P_A(y)): y \in f^{-1}(x_1)\} = \{(\mu(x), P_A(x))\}$-------------------(ii)

Now $\mu_{(A)}(nx_1) = \vee \{\mu_A(x): x \in f^{-1}(nx_1)\}$

$= \vee \{\mu(nx)\}$ (by (i))

$= \mu(nx) = \mu_A(x)$ ($\because A =< \mu_A, P_A >$ is prime IF ideal)

$= \vee \{\mu_A(x)\}$

$= \vee \{\mu(x): x \in f^{-1}(x_1)\}$ (by (ii))

$= \mu_{(A)}(x_1)$

and $P_{f(A)}(nx_1) = \wedge \{P_A(x): x \in f^{-1}(nx_1)\}$

$= \wedge \{P(nx)\}$ (by (i))

$= P(nx) = P_A(x)$ ($\because A =< \mu_A, P_A >$ is prime IF ideal)

$= \wedge \{P_A(x)\}$

$= \wedge \{P(x): x \in f^{-1}(x_1)\}$ (by (ii))

$= P_{(A)}(x_1)$

Hence $(A) =< \mu_{f(A)}, P_{f(A)} >$ is constant IF ideal of E_2.

Note: There exists a non constant IF ideal $A =< \mu_A, P_A >$ and non constant N-homomorphism f such that $f^{-1}(A) =< \mu_{f^{-1}(A)}, P_{f^{-1}(A)} >$ is a constant IF ideal. For this we observe the example mentioned below.

Example: Let $E_1 = Z, E_2 = Z$ as two N-groups of the near-ring N=Z of integers. Let us define $f: E_1 \to E_2$ such that $(x) = 2x, \forall x \in E_1$. Then f is a non constant N-homomorphism. Let $s, s', t, t' \in [0,1]$ be such that $s > s', t < t'$. Also define IFS $A =< \mu_A, P_A >$ of E_2 such that

by $\mu_A(x) = \begin{cases} , & x \in 2Z \\ s', & otherwise \end{cases}$ and $P_A(x) = \begin{cases} , & x \in 2Z \\ t', & otherwise \end{cases}$

Then $A =< \mu_A, P_A >$ is a generalized characteristics function and it is clear that it is non constant IFS of $E_2 = Z$. Then by theorem1.5, $A =< \mu_A, P_A >$ is an IF ideal of E_2. Now we check that $f^{-1}(A)$ is a constant IF ideal of E_1.

For any $x \in E_1$, $f^{-1}(\mu_A)(x) = \mu_A(f(x)) = \mu_A(2x) = s$ and $f^{-1}(P_A(x)) = P_A(f(x)) = P(2x) = t$;which shows $f^{-1}(A)$ is constant. Moreover the only level set of $f^{-1}(A)$ is

$f^{-1}(A)_{(s,t)} = 2Z$ is an ideal of $E_1 = Z$. Hence by theorem 1.7, $f^{-1}(A)$ is an IF ideal of E_1. Hence $f^{-1}(A)$ is a constant IF ideal of E_1.

Theorem3.3. For N-groups E_1, E_2 and non constant N-homomorphism $f: E_1 \to E_2$ and E_2 's If ideal $A =< \mu_A, \mathrm{P}_A >$ if A is E_2's prime IF ideal then $f^{-1}(A)$ is either constant or E_1's prime IF ideal .

Proof: Let $A =< \mu_A, \mathrm{P}_A >$ be E_2's IF ideal. Then by theorem1.7, $f^{-1}(A)$ is an IF ideal of E_1. If $f^{-1}(A)$ is constant then the proof is over. So let $f^{-1}(A)$ be non constant. Now for $x \in E_1$ and $0 \neq n \in N$,we have $f^{-1}(\mu_A)(nx) = \mu_A(f(nx))$

$$= \mu(nf(x)) \quad (\because f \text{ is N-homomorphism})$$

$$= \mu(f(x)) \quad (\because A =< \mu_A, \mathrm{P}_A > \text{is prime IF ideal })$$

$$= f^{-1}(\mu_A)(x)$$

and $f^{-1}(\mathrm{P}_A)(nx) = \mathrm{P}_A(f(nx)) = \mathrm{P}_A(nf(x))$ $(\because f$ is N-homomorphism)

$$= \mathrm{P}(f(x)) \quad (\because A =< \mu_A, \mathrm{P}_A > \text{is prime IF ideal })$$

$$= f^{-1}(\mathrm{P}_A)(x)$$

Hence $f^{-1}(A)$ is an E_1's prime IF ideal.

4. IF coset and prime IF ideal:

In this part we recall some notions of IF cosets and related results and then will discuss some results related to prime IF ideal.

Definition4.1.[7] For IF ideal $A =< \mu_A, \mathrm{P}_A >$ of an N-group E and $e \in E$, the IFS

$e + A =< e + \mu_A, e + \mathrm{P}_A >$ defined as $(e + \mu_A)(e') = \mu_A(e' - e); (e + \mathrm{P}_A)(e') = \mathrm{P}(e' - e), e' \in E$ is called IF coset of $A =< \mu_A, \mathrm{P}_A >$.

Theorem4.2. .[7] If $A =< \mu_A, \mathrm{P}_A >$ is an IF coset of E then

$< e + \mu_A, e + \mathrm{P}_A >=< e' + \mu_A, e' + \mathrm{P}_A > \Longleftrightarrow \mu(e' - e) = \mu_A(0)$ and $\mathrm{P}_A(e' - e) = \mathrm{P}_A(0)$

Corrollary4.3. .[7] If $< e + \mu_A, e + \mathrm{P}_A >=< e' + \mu_A, e' + \mathrm{P}_A >$ then $\mu_A(e) = \mu_A(e'); \mathrm{P}_A(e) = \mathrm{P}_A(e')$

Definition4.4. .[7] Let $A =< \mu_A, \mathrm{P}_A >$ be IF ideal of N-group E. Then the set of all IF cosets of $A =< \mu_A, \mathrm{P}_A >$ with respect to the operations defined as

$(e+< \mu_A, \mathrm{P}_A >) + (e'+< \mu_A, \mathrm{P}_A >) = (e + e')+< \mu_A, \mathrm{P}_A >$ and

$(e+<\mu_A, \mathsf{P}_A>) = ne+<\mu_A, \mathsf{P}_A>$ for all $e, e' \in E, n \in N$

forms an N-group and is called quotient N-group and is denoted by E/A.

Theorem4.5.[7] Let $<\mu_A, \mathsf{P}_A>$ be an IF ideal of E. Then the IFS $<\theta_\mu, \theta_{\mathsf{P}}>$ defined as

$\theta(e+<\mu_A, \mathsf{P}_A>) = \mu_A(e); \theta_{\mathsf{P}}(e+<\mu_A, \mathsf{P}_A) = \mathsf{P}_A(e)$ is an IF ideal of E.

Theorem4.6. Let $A =<\mu_A, \mathsf{P}_A>$ and $B =<\mu_B, \mathsf{P}_B>$ be two IF ideal of E such that $A \subseteq B$ with $\mu(0) = \mu_A(0)$; $\mathsf{P}_B(0) = \mathsf{P}_A(0)$ and B is prime IF ideal of E. Then IFS $B' =<\mu_{B'}, \mathsf{P}_{B'}>$ defined as $\mu_B(e + A) = \mu_B(e)$ and $\mathsf{P}_{B^F}(e + A) = \mathsf{P}_B(e)$ for all $e + A \in E/A$, is an IF ideal ofE/A.

Proof: Let $e + A, e' + A \in E/A$ be such that $e + A = e' + A$ which implies $\mu(0) = \mu_A(e - e')$ and $\mathsf{P}_A(0) = \mathsf{P}_A(e - e')$ (by theorem4.2).

Then we have, $\mu(0) \geq \mu_B(e - e')$

$\geq \mu_A(e - e')$ $(\because A \subseteq B)$

$= \mu(0)$

$= \mu(0)$

Thus $\mu(0) = \mu_B(e - e') \Rightarrow e + \mu_B = e' + \mu_B$

$\Rightarrow (e + \mu_B)(0) = (e' + \mu_B)(0)$

$\Rightarrow \mu(0 - e) = \mu_B(0 - e')$

$\Rightarrow \mu_B(-e) = \mu_B(-e')$

$\Rightarrow \mu(e) = \mu_B(e')$ $(\because B$ is IF ideal $)$

Also, $\mathsf{P}(0) \leq \mathsf{P}_B(e - e')$

$\leq \mathsf{P}_A(e - e')$ $(\because A \subseteq B)$

$= \mathsf{P}(0)$

$= \mathsf{P}(0)$

Thus $\mathsf{P}(0) = \mathsf{P}_B(a - b) \Rightarrow e + \mathsf{P}_B = e' + \mathsf{P}_B$

$\Rightarrow (e + \mathsf{P}_B)(0) = (e' + \mathsf{P}_B)(0)$

$\Rightarrow \mathsf{P}(0 - e) = \mathsf{P}_B(0 - e')$

$$\Rightarrow \mathsf{P}_B(-e) = \mathsf{P}_B(-e')$$

$$\Rightarrow \mathsf{P}(e) = \mathsf{P}_B e') \qquad (\because B \text{ is IF ideal })$$

Then $\mu(e) = \mu_B(e')$; $\mathsf{P}_B(e) = \mathsf{P}_B(e')$ implies that $\mu_{B^F}(e + A) = \mu_{B^F}(e' + A)$; $\mathsf{P}_{B^F}(e + A) = \mathsf{P}_B(e' + A)$ and consequently $B' =< \mu_{B'}, \mathsf{P}_{B'} >$ is well defined IFS.

Again for $e + A, e' + A \in E/A$,

$$\begin{aligned}
\mu_{B^F}((e + A) - (e' + A)) &= \mu_{B^F}((e - e') + A) \\
&= \mu_A(e - e') \\
&\geq \mu(e) \wedge \mu_A(e') \; (\because A \text{ is IF ideal }) \\
&= \mu_{B^F}(e + A) \wedge \mu_{B^F}(e' + A)
\end{aligned}$$

$$\begin{aligned}
\mathsf{P}_{B^F}((e + A) - (e' + A)) &= \mathsf{P}_{B^F}((e - e') + A) \\
&= \mathsf{P}_A(e - e') \\
&\leq \mathsf{P}(e) \vee \mathsf{P}_A(e') \; (\because A \text{ is IF ideal }) \\
&= \mathsf{P}_{B^F}(e + A) \vee \mathsf{P}_{B^F}(e' + A)
\end{aligned}$$

Also for $e + A, e' + A \in E/A$, and $n \in N$

$$\begin{aligned}
\mu_{B^F}(n((e' + A) + (e + A)) - n(e' + A)) &= \mu_{B^F}(n((e' + e) + A) - (ne' + A)) \\
&= \mu_{B^F}((n(e' + e) + A) - (ne' + A)) \\
&= \mu_{B^F}((n(e' + e) - ne') + A) \\
&= \mu_A(n(e' + e) - ne') \\
&\geq \mu(e) \qquad (\because A \text{ is IF ideal }) \\
&= \mu_{B^F}(e + A)
\end{aligned}$$

$$\begin{aligned}
\mathsf{P}_{B^F}(n((e' + A) + (e + A)) - n(e' + A)) &= \mathsf{P}_{B^F}(n((e' + e) + A) - (ne' + A)) \\
&= \mathsf{P}_{B^F}((n(e' + e) + A) - (ne' + A)) \\
&= \mathsf{P}_{B^F}((n(e' + e) - ne') + A) \\
&= \mathsf{P}_A(n(e' + e) - ne')
\end{aligned}$$

$$\leq \mathrm{P}(e) \qquad (\because A \text{ is IF ideal })$$

$$= \mathrm{P}_{B^F}(e+A)$$

Similarly we can show that $\mu_B((e'+A)+(e+A)-(e'+A)) \geq \mu_{B^F}(e+A)$; $\mathrm{P}_{B^F}((e'+A)+(e+A)-(e'+A)) \leq \mathrm{P}_{B^F}(e+A)$ and $\mu_{B^F}(n(e+A)) \geq \mu_{B^F}(e+A)$; $\mathrm{P}_{B^F}(n(e+A)) \leq \mathrm{P}_{B^F}(e+A))$. Hence $B' =< \mu_{B'}, \mathrm{P}_{B'} >$ is IF ideal of E/A.

Theorem4.7. If $P =< \mu_P, \mathrm{P}_P >$ is E's prime IF ideal, then E/P is prime N-group.

Proof: Suppose $P =< \mu_P, \mathrm{P}_P >$ is E's prime IF ideal. Since P is non constant, it follows that $0+P =< 0+\mu_P, 0+\mathrm{P}_P >$ is a E/P's proper ideal. Let $0 \neq n \in N, < x+\mu_P, x+\mathrm{P}_P > \in E/P$ be such that $n(x+\mu_P) = 0+\mu_P$ and $n(x+\mathrm{P}_P) = 0+\mathrm{P}_P$

Now $(x+\mu_P) = 0+\mu_P \Rightarrow nx+\mu_P = 0+\mu_P$

$$\Longrightarrow (nx+\mu(0) = (0+\mu_P)(0)$$

$$\Longrightarrow \mu(nx-0) = \mu_P(0)$$

$$\Longrightarrow \mu(x) = \mu_P(0) \qquad (\because P \text{ is prime IF ideal})$$

$$\Longrightarrow x+\mu_P = 0+\mu_P \text{ (by theorem4.2)}$$

and $(x+\mathrm{P}_P) = 0+\mathrm{P}_P \Rightarrow nx+\mathrm{P}_P = 0+\mathrm{P}_P$

$$\Longrightarrow (nx+\mathrm{P}_P)(0) = (0+\mathrm{P}_P)(0)$$

$$\Longrightarrow \mathrm{P}(nx-0) = \mathrm{P}_P(0)$$

$$\Longrightarrow \mathrm{P}(x) = \mathrm{P}_P(0) \qquad (\because P \text{ is prime IF ideal})$$

$$\Longrightarrow x+\mathrm{P}_P = 0+\mathrm{P}_P \text{ (by theorem4.2)}$$

Therefore $0+P =< 0+\mu_P, 0+\mathrm{P}_P >$ is E/P's prime IF ideal.

Theorem4.8. Consider $P =< \mu_P, \mathrm{P}_P >$ be an E's prime IF ideal. Then IFS $< \theta_\mu, \theta_\mathrm{P} >$ defined as

$\theta(x+< \mu_A, \mathrm{P}_A >) = \mu_A(x)$; $\theta_\mathrm{P}(x+< \mu_A, \mathrm{P}_A) = \mathrm{P}_A(x)$ is a E/P's prime IF ideal.

Proof: Consider $P =< \mu_P, \mathrm{P}_P >$ as an E's prime IF ideal. Then $< \theta_\mu, \theta_\mathrm{P} >$ is an E/P is IF ideal by Theorem 4.5. Since P is prime ideal so it is non constant IF ideal of E and thus $< \theta_\mu, \theta_\mathrm{P} >$ is a non constant IF ideal of E/P. Let $0 \neq n \in N, x+< \mu_P, \mathrm{P}_P > \in E/P$.

Now $\theta(n(x+< \mu_P, \mathrm{P}_P >) = \theta_\mu(nx+< \mu_P, \mathrm{P}_P >)$

$= \mu_P(nx)$

$= \mu(x)$ (∵ P is prime IF ideal)

$= \theta_\mu(nx + < \mu_P, \mathrm{P}_P >)$

and $\theta_\mathrm{P}(n(x + < \mu_P, \mathrm{P}_P >) = \theta_\mathrm{P}(nx + < \mu_P, \mathrm{P}_P >)$

$= \mathrm{P}_P(nx)$

$= \mathrm{P}(x)$ (∵ P is prime IF ideal)

$= \theta_\mathrm{P}(nx + < \mu_P, \mathrm{P}_P >)$

Therefore $< \theta_\mu, \theta_\mathrm{P} >$ is prime IF ideal of E/P.

Theorem4.9. Let $A =< \mu_A, \mathrm{P}_A >$ and $B =< \mu_B, \mathrm{P}_B >$ be two IF ideal of E such that $A \subseteq B$ with $\mu(0) = \mu_A(0)$; $\mathrm{P}_B(0) = \mathrm{P}_A(0)$ and B is prime IF ideal of E. Then IFS $B' =< \mu_{B'}, \mathrm{P}_{B'} >$ defined as $\mu_B(x + A) = \mu_B(x)$ and $\mathrm{P}_{B^F}(x + A) = \mathrm{P}_B(x)$,$\forall\ x + A \in E/A$, is E/A's prime IFideal.

Proof: Consider E's prime IF ideal B. Then B' is clearly E/A's IF ideal by theorem 4.5. Since B is a non constant IF ideal of E so as B' of E/A. Let $x + A \in E/A$, $0 \neq n \in$.

Then $\mu_{B'}(n(x + A))= \mu_{B'}(nx + A)$

$= \mu_B(nx)$

$= \mu(x)$ (∵ B is prime IF ideal)

$= \mu_{B'}(x + A)$

and $\mathrm{P}_{B'}(n(x + A))= \mathrm{P}_{B'}(nx + A)$

$= \mathrm{P}_B(nx)$

$= \mathrm{P}(x)$ (∵ B is prime IF ideal)

$= \mathrm{P}_{B^F}(x + A)$

Hence $B' =< \mu_{B'}, \mathrm{P}_{B'} >$ is prime IF ideal of E/A.

References.

1. G. Pilz, Near-rings,The Theory and its applications,2nd ed., North Holand,Amsterdam,1983
2. H. Saikia and L. K. Barthakur,On fuzzy N-subgroups and fuzzy ideals of near-rings and near-ring groups, J. Fuzzy Math. 11(30)(2003)567-580
3. J. Clay, Near-rings:Geneses and applications,Oxford,1992
4. K.T. Atanassov ,Intuitionistic fuzzy sets,Fuzzy sets and systems,vol.20(1986)139-143
5. L.A. Zadeh, Fuzzy sets, Information and controls. Vol.83(1965)338-353
6. O Ratnabala and M. Ranjit, On fuzzy ideals of N-groups, Bull. Pure and Appl. Sciences, vol. 26E(No.1)(2007),11-23
7. O Ratnabala, Intuitionistic fuzzy near-ring and its properties, Journal of the Indian Math. Soc.,Vol.74(No.s 3-4)(2007),203-219.
8. P. Saikia and L. K. Barthakur, Some properties of an intuitionistic fuzzy N-subgroup of an N-group, Advances in Fuzzy sets and systems,vol.25,(No.1)(2020)37-57