ON A CONTRAST OF GENERALIZED DISSIMILARITIES (DIVERGENCES)

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Abstract
In this study, we developed a brand-new information inequality using generalised dissimilarity measures as a comparison. Obtain the bounds of underused dissimilarity in terms of other standard dissimilarity measures and help obtain new dissimilarities compared to that contrast. Lastly, understand the relationship between unused dissimilarity and Renyi's entropy.

Keywords: Contrast of generalized dissimilarity measures, unused information inequalities, unused dissimilarity and its bounds, convex and normalized function.

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1. Introduction

Let $\Gamma_k = \{M = (m_1, m_2, m_3, \dots, m_k): m_i > 0, \sum_{i=1}^{k} m_i = 1\}$, $k \geq 2$ be the collection of all discrete finite probabilities.. The restriction here to discrete distributions is exclusively for convenience, proportionate results hold for continuous distributions. Let's consider $m_i \geq 0$ for some $i = 1,2,3\dots,k$ at that point we have to assume that

$0\Theta(0) = 0\Theta\left(\frac{0}{0}\right) = 0.$

Csiszar's Θ- dissimilarity **[**2**]** defined as

$$C_\Theta(M,N) = \sum_{i=1}^{k} n_i \Theta\left(\frac{m_i}{n_i}\right) \tag{1.1}$$

Comparably **[7]** initiated a generalized measure of information determined by

$$S_\Theta(M,N) = \sum_{i=1}^{k} n_i \Theta\left(\frac{m_i+n_i}{2n_i}\right) \tag{1.2}$$

where $\Theta:(0,\infty) \to R$ (set of real no.) is real, continuous as well as convex function and $M = (m_1, m_2, \dots, m_k), N = (n_1, n_2, \dots, n_k) \in \Gamma_k$, where m_i and n_i are probability mass functions. By properly describing the convex function, many studied dissimilarities may be obtained from such generalised measures. the following standards.

The measures (1.3), (1.4), and (1.5) below are all generalised dissimilarity measures with one specification [12], where $r \in R$ is a criteria.

Relative Information of type "*r*"

$$\psi_r(M,N) = [r(r-1)]^{-1}\left[\sum_{i=1}^{k} m_i^r n_i^{1-r} - 1\right], r \neq 0,1. \tag{1.3}$$

Unified Relative *JS* and *AG* Dissimilarity of type "*r*"

$$\Omega_r(M,N) = [r(r-1)]^{-1}\left[\sum_{i=1}^{k} m_i \left(\frac{m_i+n_i}{2m_i}\right)^k - 1\right], r \neq 0,1. \tag{1.4}$$

Relative Entropy [8**]**

$$K(M,N) = \sum_{i=1}^{k} m_i \log\frac{m_i}{n_i}\ . \tag{1.5}$$

Chi-square dissimilarity [9**]**

$$\chi^2(M,N) = \sum_{i=1}^{k} \frac{(m_i - n_i)^2}{n_i}\ . \tag{1.6}$$

Relative Jensen-Shannon Dissimilarity [11**]**

$$F(M,N) = \sum_{i=1}^{k} m_i \log\frac{2m_i}{m_i+n_i}. \tag{1.7}$$

Renyi's "*p*" order entropy [10**]**

$$R_p(M,N) = \sum_{i=1}^{k} \frac{m_i^p}{n_i^{p-1}}, p > 1 \tag{1.8}$$

2. Difference of Generalized Dissimilarities

In literature, the corresponding theorem is widely recognised [1].

Theorem 2.1. *If the function Θ is convex as well as normalized, i.e., $\Theta''(u) \geq 0\ \forall\ u > 0$ and $\Theta(1) = 0$ individually, at that point then $C_\Theta(M,N)$ and its adjoint $C_\Theta(N,M)$ are both non-negative and convex within the combine of probability distribution $(M,N) \in \Gamma_k \times \Gamma_k$.*

The inequalities we now have with reference to the variance of generalised dissimilarity measurements and evaluation follow the same lines as the conclusion put out by [12].

Theorem 2.2. *Let* $\Theta_1, \Theta_2 : I \subset R_+ \to R$ *be two convex as well as normalized functions, i.e.,* $\Theta''_1(u), \Theta''_2(u) \geq 0 \;\forall\; u > 0$ *and* $\Theta_1(1) = \Theta_2(1) = 0$ *correspondingly and let the subsequent supposition.*
(i) Θ_1 and Θ_2 are twice differentiable on (a, b), $0 < a \leq 1 \leq b < \infty$ with $a \neq b$.
(ii) There exist the real constants x, X such that $x < X$ and

$$x \leq \frac{\Theta''_1(t)}{\Theta''_2(t)} \leq X, \Theta''_2(u) \neq 0 \;\forall\; u \in (a, b) \tag{2.1}$$

If $M, N \in \Gamma_n$, then we have the subsequent inequalities

$$x\left[C_{\Theta_2}(M,N) - S_{\Theta_2}(M,N)\right] \leq C_{\Theta_1}(M,N) - S_{\Theta_1}(M,N) \leq X\left[C_{\Theta_2}(M,N) - S_{\Theta_2}(M,N)\right] \tag{2.2}$$

and $C_\Theta(M,N), S_\Theta(M,N)$ are provided respectively, by (1.1) and (1.2).

Proof. Let's take a look at two functions.

$$\Theta_x(u) = \Theta_1(u) - x\Theta_2(u) \tag{2.3}$$

and

$$\Theta_X(u) = X\Theta_2(u) - \Theta_1(u). \tag{2.4}$$

where x and X are the max. and min. values of the function $\frac{\Theta''_1(u)}{\Theta''_2(u)} \;\forall\; u \in (a, b)$.

Since

$$\Theta_1(1) = \Theta_2(1) = 0 \Rightarrow \Theta_x(1) = \Theta_X(1) = 0 \tag{2.5}$$

and the functions $\Theta_1(t)$ and $\Theta_2(t)$ are twice differentiable. Then displayed of (2.1), we hold

$$\Theta''_x(t) = \Theta''_1(t) - x\Theta''_2(t) = \Theta''_2(t)\left[\frac{\Theta''_1(u)}{\Theta''_2(u)} - x\right] \geq 0 \tag{2.6}$$

and

$$\Theta''_X(t) = X\Theta''_2(t) - \Theta''_1(t) = \Theta''_2(t)\left[X - \frac{\Theta''_1(u)}{\Theta''_2(u)}\right] \geq 0. \tag{2.7}$$

We can assert that the functions $\Theta_x(u)$ and $\Theta_X(u)$ are convex and normalised on (a,b) in view (2.5), (2.6), and (2.7).

Now, by using linearity property, we hold

$$C_{\Theta_x}(M,N) - S_{\Theta_x}(M,N) = C_{\Theta_1 - x\Theta_2}(M,N) - S_{\Theta_1 - x\Theta_2}(M,N)$$

$$= C_{\Theta_1}(M,N) - xC_{\Theta_2}(M,N) - S_{\Theta_1}(M,N) + xS_{\Theta_2}(M,N)\,. \tag{2.8}$$

And

$$C_{\Theta_X}(M,N) - S_{\Theta_X}(M,N) = C_{X\Theta_2 - \Theta_1}(M,N) - S_{X\Theta_2 - \Theta_1}(M,N)$$

$$= XC_{\Theta_2}(M,N) - C_{\Theta}(M,N) - XS_{\Theta_2}(M,N) + S_{\Theta_1}(M,N)\,. \tag{2.9}$$

Since, we know that $C_\Theta(M,N) \geq S_\Theta(M,N)$ in [7] Therefore (2.8) and (2.9) can be put together as the followings correspondingly

$$\left[C_{\Theta_1}(M,N) - S_{\Theta_1}(M,N)\right] - x\left[C_{\Theta_2}(M,N) - S_{\Theta_2}(M,N)\right] \geq 0$$

and

$$X\left[C_{\Theta_2}(M,N) - S_{\Theta_2}(M,N)\right] - \left[C_{\Theta_1}(M,N) - S_{\Theta_1}(M,N)\right] \geq 0.$$

Or

$$\left[C_{\Theta_1}(M,N) - S_{\Theta_1}(M,N)\right] \geq x\left[C_{\Theta_2}(M,N) - S_{\Theta_2}(M,N)\right] \tag{2.10}$$

and

$$X\left[C_{\Theta_2}(M,N) - S_{\Theta_2}(M,N)\right] \geq \left[C_{\Theta_1}(M,N) - S_{\Theta_1}(M,N)\right] \tag{2.11}$$

Execute the result (2.2) simultaneously in (2.10) and (2.11).

3. New Dissimilarity Measure and Properties

For this subdivision, we'll get a measure of recent dissimilarity, let $\Theta : (0, \infty) \to R$ (set of real numbers) be a function determined as

$$\Theta(u) = \Theta_1(u) = \frac{1}{u^2} - 1, u > 0, \Theta_1(1) = 0\ \ . \tag{3.1}$$

and

$$\Theta'_1(u) = -\frac{2}{u^3}, \Theta''_1(u) = \frac{6}{u^4}\ . \tag{3.2}$$

Since $\Theta''_1(u) > 0 \ \forall \ u > 0$ and $\Theta_1(1) = 0$, therefore $\Theta_1(t)$ is convex as well as normalized function.
Now substitute $\Theta_1(u)$ in (1.1) we record the new dissimilarity measure below for $M, N \in \Gamma_k$ and $0 < a \leq 1 \leq b < \infty, a \neq b$.

$$C_{\Theta_1}(M,N) - S_{\Theta_1}(M,N) = CS(M,N) = \sum_{i=1}^{k} \frac{n_i^3(3m_i+n_i)(n_i-m_i)}{m_i^2(m_i+n_i)^2} \geq 0. \tag{3.3}$$

Here, measure $CS(M,N)$ is a non- negative and convex in the couple of probability distribution $M, N \in \Gamma_k$ and $CS(M,N) = 0$ iff $m_i = n_i \ \forall \ i = 1,2,3\ldots,k$.

Therefore $CS(M,N) \neq CS(N,M)$ which shows that $CS(M,N)$ is a non- symmetric dissimilarity measure.
Now, this can be verified by following example:
Let M be the binomial probability distribution with parameters (l= no. of trials=10,m=Probability of success of each trial=0.7 and n= proability of failure of each trial=1-m=0.3) and N is the parameterized Poisson probability distribution

(λ= average parameter=lm=7) for the random variable Z, then we have

Table 1 Discrete Probability Distributions for (l=10, m=0.7, n=0.3) are evaluated.

z_i	0	1	2	3	4	5	6	7	8	9	10
m_i	0.0000059	0.000137	0.00144	0.009	0.036	0.102	0.2	0.266	0.233	0.121	0.0282
n_i	0.000911	0.00638	0.022	0.052	0.091	0.177	0.199	0.149	0.13	0.101	0.0709

By using above Table, we get the followings:

$$CS(M,N) = \sum_{i=1}^{11} \frac{n_i^3(3m_i+n_i)(n_i-m_i)}{m_i^2(m_i+n_i)^2}$$

$$= \frac{n_1^3(3m_1+n_1)(n_1-m_1)}{m_1^2(m_1+n_1)^2} + \frac{n_2^3(3m_2+n_2)(n_2-m_2)}{m_2^2(m_2+n_2)^2} \cdots \frac{n_{11}^3(3m_{11}+n_{11})(n_{11}-m_{11})}{m_{11}^2(m_{11}+n_{11})^2}$$

$$= 43.04518 \quad . \tag{3.4}$$

And

$$CS(N,M) = \sum_{i=1}^{11} \frac{m_i^3(3n_i+m_i)(m_i-n_i)}{n_i^2(n_i+m_i)^2}$$

$$= \frac{m_1^3(3n_1+m_1)(m_i-n_i)}{n_1^2(n_1+m_1)^2} + \frac{m_2^3(3n_2+m_2)(m_2-n_2)}{n_2^2(n_2+m_2)^2} \cdots \frac{m_{11}^3(3n_{11}+m_{11})(m_{11}-n_{11})}{n_{11}^2(n_{11}+m_{11})^2}$$

$$= 0.77422 \quad . \tag{3.5}$$

Using results (3.4) and (3.5), it is verified that $CS(M,N) \neq CS(N,M)$ i.e. that $CS(M,N)$ is a non-symmetrical dissimilarity measure.

4. Bounds of New Dissimilarity Measure

Using Theorem 2.2, we will be able to calculate the bounds of CS(M,N) with regard to other dissimilarity in this subdivision.

Proposition 4.1. *Suppose $M, N \in \Gamma_k, 0 < a \leq 1 \leq b < \infty, a \neq b$ and $r \geq -2$, later we have*

$$\frac{6}{b^{r+2}}[\psi_r(M,N) - \Omega_r(N,M)] \leq CS(M,N) \leq \frac{6}{a^{r+2}}[\psi_r(M,N) - \Omega_r(N,M)], \tag{4.1}$$

where $\psi_r(M,N), \Omega_r(N,M), CS(M,N)$ are given by (1.3),(1.4) and (3.3) correspondingly.
Proof. Let us observe

$$\Theta_2(u) = [r(r-1)]^{-1}(u^r - 1), r \neq 0,1, u > 0, \Theta_2(1) = 0 \tag{4.2}$$

and

$$\Theta'_2(u) = (r-1)^{-1}u^{r-1}, r \neq 1, \Theta''_2(u) = u^{r-2} \quad . \tag{4.3}$$

Since $\Theta''_2(u) > 0 \ \forall \ u > 0$ and $\Theta_2(1) = 0$, consequently $\Theta_2(u)$ is convex and normalized function correspondingly.
Now by substituting $\Theta_2(u)$ in (1.1) we get:

$$C_{\Theta_2}(M,N) = [r(r-1)]^{-1}\left[\sum_{i=1}^{k} m_i^r n_i^{1-r} - 1\right] = \psi_r(M,N), r \neq 0,1 \quad . \tag{4.4}$$

Now, let

$$\Theta_r(u) = \frac{\Theta''_1(u)}{\Theta''_2(u)} = \frac{6}{u^{r+2}}, u > 0, r \in R \quad . \tag{4.5}$$

where $\Theta''_1(u)$ and $\Theta''_2(u)$ accordingly, as indicated by 3.2 and 4.3.

Also $\Theta'_r(u) = -\frac{6(r+2)}{u^{r+3}}$. (4.6)

Now, we can check that

$$\Theta'_r(u) = \begin{cases} \leq 0 & \text{if } r \geq -2 \\ \geq 0 & \text{if } r \leq -2 \,. \end{cases}$$

It means $\Theta_r(u)$ is monotonically decreasing in $r \geq -2$ and monotonically increasing in $r \leq -2$. Therefore, for $r \geq -2$, we have

$$X = \sup_{u\in(a,b)} \Theta_r(u) = \Theta_r(a) = \frac{6}{a^{r+2}} \tag{4.7}$$

and

$$x = \inf_{u\in(a,b)} \Theta_r(u) = \Theta_r(b) = \frac{6}{b^{r+2}} \quad . \tag{4.8}$$

The inequality (4.1) is obtained by utilizing (3.3),(4.4), (4.7) and (4.8) in (2.2).
We should now have a look at some exceptional cases at $r = 0, r = 1$ and $r = 2$.

Result 4.1. *Let $M, N \in \Gamma_k, 0 < a \leq 1 \leq b < \infty, a \neq b$ and $r = 0$, then we have*

$$\frac{6}{b^2}[K(N,M) - F(N,M)] \leq CS(M,N) \leq \frac{6}{a^2}[K(N,M) - F(N,M)] \quad . \tag{4.9}$$

Proof. Substitute $r = 0$ in (4.6) we acquire the following.

$$\psi_0(M,N) = \lim_{r\to 0}\psi_s(M,N) = \sum_{i=1}^{k} n_i \log\frac{n_i}{m_i} = K(N,M) \quad . \tag{4.10}$$

Substitute (4.10) in (4.1) at $r = 0$, and acquire the outcome (4.9).

Result 4.2. *Let $M, N \in \Gamma_k, 0 < a \leq 1 \leq b < \infty, a \neq b$ and $r = 1$, then we have*

$$\frac{6}{b^3}[K(M,N) - G(N,M)] \leq CS(M,N) \leq \frac{6}{a^3}[K(M,N) - G(N,M)] \,. \tag{4.11}$$

Proof. By substituting $r = 1$ in (4.4) we acquire:

$$\psi_1(M,N) = \lim_{r\to 1}\psi_r(M,N) = \sum_{i=1}^{k} m_i \log\frac{m_i}{n_i} = K(M,N) \,. \tag{4.12}$$

Substitute (4.12) in (4.1) at $r = 1$, and acquire the outcome (4.11).

Result 4.3. *SupposeM, $N \in \Gamma_k, 0 < a \leq 1 \leq b < \infty, a \neq b$ and $r = 2$, then we have*

$$\frac{9}{4b^4}\chi^2(M,N) \leq CS(M,N) \leq \frac{9}{4a^4}\chi^2(M,N) \quad . \tag{4.13}$$

Proof. By substituting $r = 2$ in (4.4) we acquire:

$$\psi_2(M,N) = \frac{1}{2}\left[\sum_{i=1}^{k} \frac{m_i^2}{n_i} - 1\right] = \frac{1}{2}\left[\sum_{i=1}^{k} \frac{m_i^2}{n_i} - 2m_i + n_i\right] = \frac{1}{2}\left[\sum_{i=1}^{k} \frac{(m_i - n_i)^2}{n_i}\right]$$
$$= \frac{1}{2}\chi^2(M,N) \quad . \tag{4.14}$$

By subsituting (4.14) in (4.1) at $r = 2$, we acquire the outcome (4.13).

5. Relation Among $CS(M,N)$ and Renyi's Entropy

As

$$\frac{\alpha+\beta}{2} \geq \sqrt{\alpha\beta} \;\; \forall \;\; \alpha,\beta \geq 0 \quad (\text{A.M.} \geq \text{G.M.}). \tag{5.1}$$

Therefore

$$(\alpha + \beta)^2 \geq 4\alpha\beta \,. \tag{5.2}$$

By substituting $\alpha = m_i$ and $\beta = n_i$ in (5.2), we obtain

$$(\mathrm{m_i} + \mathrm{n_i})^2 \geq 4\mathrm{m_i n_i} \quad . \tag{5.3}$$

Proposition 5.1. *Let $M, N \in \Gamma_k$, then possesses the successive new relation.*

$$CS(M,N) \leq \frac{1}{2}R_3(N,M) - \frac{3}{4}R_2(N,M) + \frac{1}{4}R_4(N,M) \,. \tag{5.4}$$

where $CS(M,N)$ is given by (3.3).

Proof. Multiply (5.3) by $\frac{m_i^2}{n_i^3(3m_i+n_i)(n_i-m_i)}$ and sum over all $i=1,2,3...,k$, we obtain

$$\sum_{i=1}^{k}\frac{m_i^2(m_i+n_i)^2}{n_i^3(3m_i+n_i)(n_i-m_i)} \geq \sum_{i=1}^{k}\frac{4n_im_i^3}{n_i^3(3m_i+n_i)(n_i-m_i)}$$

$$\Rightarrow \sum_{i=1}^{k}\frac{n_i^3(3m_i+n_i)(n_i-m_i)}{m_i^2(m_i+n_i)^2} \leq \sum_{i=1}^{k}\frac{n_i^3(3m_i+n_i)(n_i-m_i)}{4n_im_i^3}$$

$$\Rightarrow CS(M,N) \leq \frac{1}{2}\sum_{i=1}^{k}\frac{n_i^3}{m_i^2}-\frac{3}{4}\sum_{i=1}^{k}\frac{n_i^2}{m_i}+\frac{1}{4}\sum_{i=1}^{k}\frac{n_i^4}{m_i^3}$$

$$\Rightarrow CS(M,N) \leq \frac{1}{2}R_3(N,M)-\frac{3}{4}R_2(N,M)+\frac{1}{4}R_4(N,M).$$

Hence the relation (5.4).
Or

$$\frac{3}{4}R_2(N,M) \leq \frac{1}{2}R_3(N.M)+\frac{1}{4}R_4(N,M)-CS(M,N)\,. \tag{5.5}$$

***Note* 5.1.** $K(M,N), \chi^2(M,N), F(M,N)$ and $R_p(M,N)$ have been taken from (1.5), (1.6), (1.7), (1.11) and (1.8) correspondingly in result 4.1 and proposition 5.1.
***Note* 5.2**. Limits have been determined in outcomes 4.1 and 4.2 using the L' Hospital Rule.
Note 5.3. since the function $\Theta_r(u)$ for $r \leq -2$ is rising (as demonstrated in the verification of claim 4.1), the ensuing inequality results.

$$\frac{6}{a^{r+2}}[\psi_r(M,N)-\Omega_r(N,M)] \leq CS(M,N) \leq \frac{6}{b^{r+2}}[\psi_r(M,N)-\Omega_r(N,M)]\,.$$

We remove the special cases and proof for these inequalities for various r values.

6. Conclusion

Since dissimiarity measures have numerous applications in a few areas, it is always fundamentally curious to discover modern abberations that also appear in numerical shapes so that they can be connected as applications in numerical shapes. The goal of this study is to identify a new information inequality on the comparison of generalised Θ -Divergences CS(M,N), a new divergence measure that is appropriate for this Θ – Divergences and some constraints for the new divergence that are determined in relation to existing common divergence measures. Finally, a previously unknown link between the new divergence measure and Renyi's Entropy is found.

We hope that our research will encourage readers to think about the applications of information theory's divergence metrics. Such types of divergences are very helpful in determining the value of an occasion, or how useful it is in comparison to other events.

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8. References

1. S.M. Ali and S.D. Silvey, A general class of coefficients of divergence of one distribution from another, *Journal of the Royal Statistical Society*, B, **28**(1966),131-142.
2. I. Csiszar, Information type measures of differences of probability distribution and indirect observations, *Studia Scientiarum Mathematicarum Hungarica*, **2**(1967), 299- 318.
3. D. Dacunha-Castelle, H. Heyer, and B. Roynette, *Ecoled'Ete de Probabilites de, Saint- Flour* VII-1977, Springer, Berlin, Heidelberg, New York,1978.
4. S.S. Dragomir, V. Gluscevic and C.E.M Pearce, *Approximation for the Csiszar f- divergence via midpoint inequalities, in inequality theory and applications* ,Y.J. Cho, J.K. Kim and S.S. Dragomir (Eds.), Nova Science Publishers, Inc., Huntington, New York, **1**, 2001), 139-154.
5. K.C. Jain and P. Chhabra, Establishing relations among various measures by using well known inequalities, *International Journal of Modern Engineering Research (IJMER),* **4** (1)(2014), 238- 246.
6. K.C. Jain and P. Chhabra, New information inequalities and its special cases, *Journal of Rajasthan Academy of Physical Sciences*, **13**(1) (2014), 39-50.
7. K.C. Jain and R.N. Saraswat, Some new information inequalities and its applications in information

theory, *International Journal of Mathematics Research*, **4**(3) (2012), 295-307

8. S. Kullback and R.A. Leibler, On information and sufficiency, *The Annals of Mathematical Statistics*, **22**(1) (1951), 79-86, doi: 10.1214/aoms/1177729694.
9. K. Pearson, On the Criterion that a given system of deviations from the probable in the case of correlated system of variables is such that it can be reasonable supposed to have arisen from random sampling, The London, *Edinburgh and Dublin Philosophical Magazine and Journal of Science Series 5*, **50**(302) (1900),157-172.
10. A. Rényi, On measures of entropy and information, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*,**1***: Contributions to the Theory of Statistics*, **4**, University of California Press, (1961), 547-562.
11. R. Sibson, Information radius , *Z. Wahrs. Undverw. Geb.*, **14**(1969),149-160. [12] I.J. Taneja and P. Kumar, Generalized non-symmetric divergence measures and inequalities, *Journal of Interdisciplinary Mathematics*, **9**(3) (2006), 581-599.
12. I.J. Taneja, New developments in generalized information measures, Chapter in: *Advances in Imaging and Electron Physics*, Ed. P.W. Hawkes, **91**(1995), 37-135.