MEDIUM DOMINATION DECOMPOSITION NUMBER OF JOIN OF GRAPHS

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Abstract

The Medium domination Number is a notation which uses neighbourhood of each pair of vertices. If any two adjacent vertices that are dominating each other, then the domination number of that vertices is atleast one. For any connected, undirected, simple graph G of order p, the Medium Domination Number $MD(G) = \frac{TDV(G)}{pC_2}$, where $TDV(G)$ is the total number of vertices that dominate every pair of vertices. A decomposition $(G_1, G_2, \ldots, G_n)$ of a graph G is said to be Medium Domination Decomposition (MDD) if $\lfloor MD(G_i) \rfloor = i-1, i = 1,2,\ldots,n$.The number of subgraphs of a Medium Domination Decomposition $(G_1, G_2, \ldots, G_n)$ of a graph G is said to be Medium Domination Decomposition Number of G and is denoted by $\pi_{MD}(G)$. Here, we have investigated some new bounds on Medium Domination Decomposition Number of join of graphs.

Keywords: Join of Graphs , Central vertex join, Central edge join

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1. Introduction

The graph G is considered here simple, connected, undirected and finite graphs with p vertices and q edges and G_i be the subgraph of G with p_i vertices and q_i edges, where $1 \leq i \leq n$, n is the number of subgraphs of G. The length of a shortest $u-v$ path in a connected graph G is called the distance from a vertex u to a vertex v. $d(u,v)$ denotes the distance between u and v. Two $u-v$ paths are internally disjoint if they have no vertices in common, other than u and v. The degree of a vertex v in a graph G is the number of edges incident with v and denoted by $d(v)$. The minimum degree among the vertices of a graph G is denoted by $\delta(G)$. The maximum degree among vertices of a graph G is denoted by $\Delta(G)$. The concept of Medium Domination Number was introduced by Vargor and Dunder which finds the total number of vertices that dominate all pairs of vertices and evaluate the average of this value and call it Medium Domination Number. A graph join is a binary operation on graphs. Speacifically it is an operation that takes two graphs H_1 and H_2 and produces a graph G with the property that all the edges that connect the vertices of the graph $H_1$with the vertices of the graph H_2. It is a commutative operation. Let $C(G)$ be the central graph of G with $p_c$vertices and q_c edges. For basic terminologies in graph theory we refer [3], [4] and [5]. The following are the basic definitions and results needed for the main section.

Definition 1.1.[1] For $G = (V, E)$ and $\forall\, u, v \in V$, if u and v are adjacent they dominate each other, then atleast $dom\,(u, v) = 1$.

Definition 1.2.[1] For $G = (V, E)$ and $\forall\, u, v \in V$, the total number of vertices that dominate every pair of vertices is defined as $TDV(G) = \sum_{\forall u,v \in V(G)} dom\,(u, v)$.

Definition 1.3.[1] For any connected, undirected, loopless graph G of order p the Medium Domination Number of G is defined as $MD(G) = \frac{TDV(G)}{pC_2}$.

Theorem 1.4.[1] For G has p vertices, q edges and for $d(v_i) \geq 2$, $TDV(G) = q + \sum_{v_i \in V} \binom{d(v_i)}{2}$.

Definition 1.5.[5] Let H_1 and H_2 be any two graphs on s, t vertices and s', t' edges respectively. The join $H_1 \vee H_2$ of disjoint graphs H_1 and H_2 is the graph obtained from $H_1 + H_2$ by joining each vertex of H_1 to each vertex of H_2. Note that the join $H_1 \vee H_2$ has $s + t$ vertices and $s' + t' + st$ edges.

Definition 1.6. [7] The central graph $C(G)$ of a graph G of order p and size q is a graph of order $p_c (= p + q)$ and size $q_c \left(= \binom{p}{2} + q\right)$ which is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.

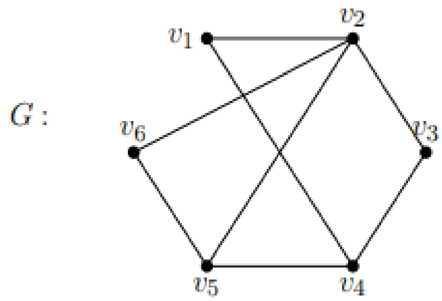
Definition 1.7. [6] Let H_1 and H_2 be any two graphs on s, t vertices and s', t' edges respectively. The central vertex join of H_1 and H_2 is the graph $H_1 \dot{\vee} H_2$, is obtained from $C(H_1)$and H_2 by joining each vertex of H_1 with every vertex of H_2. Note that the central vertex join $H_1 \dot{\vee} H_2$ has $s' + s + t$ vertices and $s' + t' + st + \frac{t(t-1)}{2}$ edges.

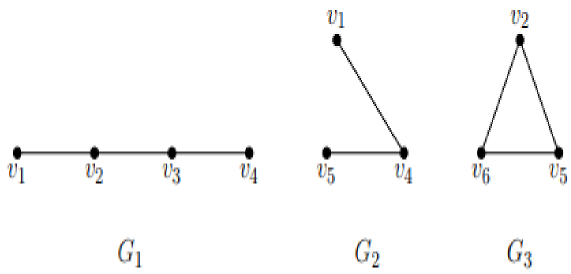
Definition 1.8. [6] Let H_1 and H_2 be any two graphs on s, t vertices and s', t' edges respectively. Then the central edge join of two graphs H_1 and H_2 is the graph $H_1 \underline{\vee} H_2$ is obtained from $C(H_1)$ and H_2 by joining each vertex corresponding to edges of $H_1$with every vertex of $V(H_2)$. Note that the central edge join $H_1 \underline{\vee} H_2$ has $s' + s + t$ vertices and $s' + t' + s't + \frac{t(t-1)}{2}$ edges.

2. Medium Domination Decomposition of Graphs:

Definition 2.1.[2] A decomposition $(G_1, G_2, \ldots, G_n)$ of a graph G is said to be a Medium Domination Decomposition (MDD) if $\lfloor MD(G_i) \rfloor = i - 1, i = 1,2,3, \ldots, n$.

Example: 2.2





Here, $\lfloor MD(G_1) \rfloor = 0, \lfloor MD(G_2) \rfloor = 1$ and $MD(G_3) = 2$

Definition 2.3. The number of subgraphs of a medium domination decomposition $(G_1, G_2, \ldots, G_n)$ of a graph G is said to be Medium Domination Decomposition Number of G and is denoted by $\pi_{MD}(G)$. From the above example, $\pi_{MD}(G) = 3$.

Theorem 2.4:[2] If a graph G admits MDD, then $p \geq 4$ and $q \geq 3$.

Theorem 2.5:[2] Star graph does not admit MDD.

Theorem 2.6:[2]The complete graph K_4 does not admit MDD.

Theorem 2.7: For any $p \geq 5$, $\pi_{MD}(K_p) \leq p-2$.

Theorem 2.8: For any $m_1 \geq 2$ and $m_2 \geq 2$, $\pi_{MD}(K_{m_1,m_2}) \leq \min\{m_1, m_2\}$.

3. Medium Domination Number and Medium Domination Decomposition Number of Join of Graphs:

Theorem 3.1: For any graphs H_1 and H_2, then MD $(H_1 \vee H_2) = \frac{1}{pC_2}\left[q + \sum_{k=1}^{s}(d(v_k)C_2) + \sum_{k=(s+1)}^{p}(d(v_k)C_2)\right]$, where $v_k \epsilon V(H_1 \vee H_2)$ and $k = 1,2,\dots,p$.

Theorem 3.2: Let H_1 be the r_1 regular graph and H_2 be the r_2 regular graph. Then $MD(H_1 \vee H_2) = \frac{1}{pC_2}[q + s((t + r_1)C_2) + t((s + r_2)C_2]$.

Corollary 3.3: For any r-regular graph H_1 and H_2, $MD(H_1 \vee H_2) = \frac{1}{pC_2}[q + s((t + r)C_2) + t((s + r)C_2]$.

Result 3.4: For any connected graphs H_1 and H_2 with $s \leq 2$ and $t \leq 2$, $H_1 \vee H_2$ does not admit MDD.

Proof: Let H_1 and H_2 be the connected graph with $s \leq 2$ and $t \leq 2$. Let $G = H_1 \vee H_2$. If $s = 2$ and $t = 2$ then $H_1 \vee H_2$ is a complete graph with four vertices. By Theorem 2.7, G does not admit MDD. By Theorem 2.5, the graph G does not admit MDD for all the remaining cases. Hence $H_1 \vee H_2$ does not admit MDD, for $s \leq 2$ and $t \leq 2$.

Theorem 3.5: Let H_1 and H_2 be null graphs.

(i) If $s = 1$ and $t \geq 1$, then $H_1 \vee H_2$ does not admit MDD.

(ii) If $s \geq 2$ and $t \geq 2$, then $\pi_{MD}(H_1 \vee H_2) \leq \min\{s, t\}$.

Proof: (i) Let H_1 be the null graph with one vertex and H_2 be the null graph with $t \geq 1$. Let $G = H_1 \vee H_2$. Then G is a star graph $K_{1,t}$. By Theorem 2.6, G does not admit MDD.

(ii)Consider the null graph with $s \geq 2$ and $t \geq 2$. Then the join of H_1 and H_2 is a complete bipartite graph with $s + t$ vertices. By Theorem 2.9, $\pi_{MD}(H_1 \vee H_2) \leq \min\{s, t\}$. Hence the proof.

Result 3.6: Let H_1 be the $(s-1)$-regular graph and H_2 be the $(t-1)$-regular graph with $s + t = 5$. Then $\pi_{MD}(H_1 \vee H_2) \leq s + t - 2$.

Proof: Since H_1 is the $(s-1)$-regular graph and H_2 is the $(t-1)$-regular graph with $s + t = 5$, H_1 and H_2 are the complete graph with s and t vertices respectively. Therefore, $H_1 \vee H_2$ is also a complete graph with $s + t$ vertices. Thus, by Theorem 2.8, $\pi_{MD}(H_1 \vee H_2) \leq s + t - 2$. Hence the proof.

Theorem 3.7: Let H_1 and H_2 be connected graphs and H_1 and H_2 admits MDD. Then $\pi_{MD}(H_1) + \pi_{MD}(H_2) \leq \pi_{MD}(H_1 \vee H_2)$.

Proof: For proving this theorem we have to consider the following two cases.

Case(i): H_1 and H_2 are complete graphs.

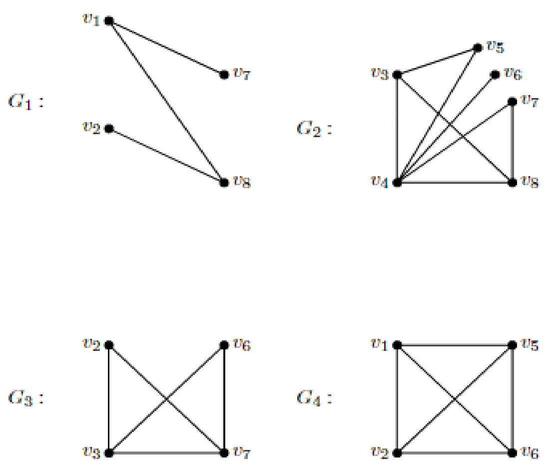
Since H_1 and H_2 admits MDD, H_1 and H_2 are complete graphs with s and $t \geq 5$ vertices. Then $H_1 \vee H_2$ is also a complete graph with $s + t$ vertices. By Result 3.6, $\pi_{MD}(H_1 \vee H_2) \leq s + t - 2$ .We have $\pi_{MD}(K_p) \leq p - 2$. Therefore $\pi_{MD}(H_1) + \pi_{MD}(H_2) \leq s - 2 + t - 2$. That is, $\pi_{MD}(H_1) + \pi_{MD}(H_2) \leq s + t - 4$. Thus, $\pi_{MD}(H_1 \vee H_2) > \pi_{MD}(H_1) + \pi_{MD}(H_2)$.

Case(ii): H_1 and H_2 are non-complete.

Subcase(i): H_1 and H_2 are acyclic graphs.

If $s = 4$ and $t = 4$, then the graph H_1 and H_2 is a path graph with four vertices. We have $\pi_{MD}(P_4) = 1$. Therefore, $\pi_{MD}(H_1) + \pi_{MD}(H_2) = 2$. Now to find the medium domination decomposition of G. Let $\{v_k: 1 \leq k \leq 8\}$ be the vertices of G .

Since, there exists a subgraph of G with medium domination number three, we can decompose the graph G into four subgraphs with medium domination number zero, one, two and three. Construct a subgraph G_1 from G with vertex set $V(G_1) = \{v_1, v_2, v_7, v_8\}$ and edge set $E(G) = \{v_7v_1, v_1v_8, v_8v_2\}$. Construct the subgraph G_2 with vertex set $V(G_2) = \{v_3, v_4, v_5, v_6, v_7, v_8\}$ and edge set $E(G_2) = \{v_3v_4, v_3v_5, v_3v_8\} \cup \{v_4v_i: 5 \leq i \leq 8\} \cup \{v_7v_8\}$. Construct the subgraph G_3 with vertex set $V(G_3) = \{v_2, v_3, v_6, v_7\}$ and edge set $E(G_3) = \{v_2v_3, v_2v_3, v_3v_6, v_3v_7, v_6v_7\}$. Construct the subgraph G_4 with vertex set $V(G_4) = \{v_1, v_2, v_5, v_6\}$ and $E(G_4) = \{v_1v_5, v_1v_2, v_1v_6, v_2v_5, v_2v_6, v_5v_6\}$. The graphical representation of G_1, G_2, G_3 and G_4 are given in figure.



Here, $\lfloor MD(G_1) \rfloor = 0$, $\lfloor MD(G_2) \rfloor = 1$, $\lfloor MD(G_3) \rfloor = 2$ and $\lfloor MD(G_4) \rfloor = 3$. Thus $\pi_{MD}(G) = 4$. Therefore $\pi_{MD}(H_1) + \pi_{MD}(H_2) < \pi_{MD}(H_1 \vee H_2)$.

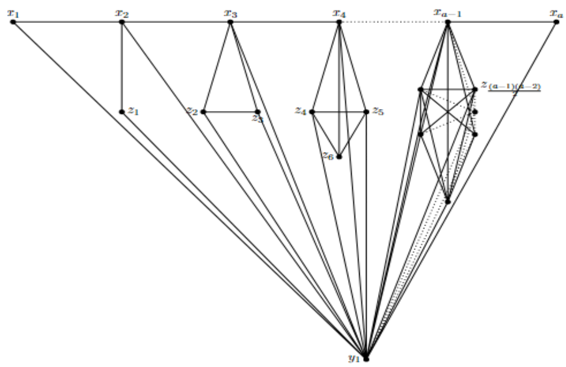
If $s > 4$ and $t > 4$, then $\pi_{MD}(H_1) = 2$ and $\pi_{MD}(H_2) = 2$. We have $\pi_{MD}(H_1 \vee H_2)$ is atleast four, since by case(i). Thus $\pi_{MD}(H_1) + \pi_{MD}(H_2) \leq \pi_{MD}(H_1 \vee H_2)$ whenever H_1 and H_2 are acyclic graphs.

subcase(ii): H_1 and H_2 contains atleast one cycle.

Then we have $\pi_{MD}(H_1)$ and $\pi_{MD}(H_2)$ are atleast two. But $\pi_{MD}(H_1 \vee H_2)$ is atleast four. Thus, $\pi_{MD}(H_1) + \pi_{MD}(H_2) \leq \pi_{MD}(H_1 \vee H_2)$. Hence the proof.

Theorem 3.8: For any positive integers a and b with $a \geq 4$ and $= \sum_{i=1}^{a-2} i$, there exists a graph G such that $\pi_{MD}(G) = a$.

Proof: Let P_a be the path graph with $a \geq 4$ vertices, where $\{x_1, x_2, \ldots, x_a\}$ are the vertices of P_a. Let K_b , $b \geq 1$ be the complete graph with b vertices. Thus, the graph H_1 is formed by connecting the $\{x_2, x_3, \ldots, x_{a-1}\}$ vertices of P_a to $K_1, K_2, \ldots, K_{a-2}$ respectively by an edge. Let $G = H_1 \vee K_1$. The graphical representation of G is given below:



Now to find the medium domination decomposition number of G.

For $a = 4$, there exists a subgraph of G with medium domination number three, Therefore, we can decompose the graph G into four subgraphs with medium domination number zero, one, two and three. Construct the subgraph G_1 from G with vertex set $V(G_1) = \{x_i: 1 \leq i \leq 4\} \cup \{y_1\}$ and the edge set $E(G_1) = \{x_ix_{i+1}: 1 \leq i \leq 3\} \cup \{x_4y_1\}$. Construct the subgraph G_2 from G with vertex set $V(G_2) = \{x_1, y_1\}$ and edge set $E(G_2) = \{x_1y_1\}$.Construct the subgraph G_3 from G with vertex set $V(G_3) = \{x_2, z_1, y_1\}$ and edge set $E(G_3) = \{x_2z_1, x_2y_1, z_1y_1\}$. Construct the subgraph G_4 from G with vertex set $V(G_4) = \{x_3, z_2, z_3, y_1\}$ and edge set $E(G_4) = \{x_3z_2, x_3z_3, x_3y_1, z_2z_3, z_2y_1, z_3y_1\}$. Here $LMD(G_1) = 0, MD(G_2) = 1, MD(G_3) = 2$ and $MD(G_4) = 3$. Thus $\pi_{MD}(G) = 4$.

For $a = 5$, there exists a subgraph of G with medium domination number four, Therefore, we can decompose the graph G into five subgraphs with medium domination number zero, one, two, three and four. Construct the subgraph G_1 from G with vertex set $V(G_1) = \{x_i: 1 \le i \le 5\} \cup \{y_1\}$ and edge set $E(G_1) = \{x_ix_{i+1}: 1 \le i \le 4\} \cup \{x_5y_1\}$. The vertex set and edge set of G_2, G_3 and G_4 are same as in the above part. Construct the subgraph G_5 from G with vertex set $V(G_5) = \{x_4, z_4, z_5, z_6, y_4\}$ and edge set $E(G_5) = \{x_4z_4, x_4z_5, x_4z_6, x_4y_1, z_4z_5, z_4z_6, z_4y_1,$

$z_5z_6, z_5y_1, z_6y_1\}$. Here $MD(G_1) = 0$, $MD(G_2) = 1$, $MD(G_3) = 2$, $MD(G_4) = 3$, $MD(G_5) = 4$. Thus $\pi_{MD}(G) = 5$.

Continuing in this way, for any a, there exists a subgraph of G with medium domination number $a - 1$, Therefore, we can decompose the graph G into a subgraphs with medium domination number 0,1,2,…, $a - 1$. Construct the subgraph G_1 from G with vertex set $V(G_1) = \{x_i: 1 \le i \le a\} \cup \{y_1\}$ and edge set $E(G_1) = \{x_ix_{i+1}: 1 \le i \le a_1\} \cup \{x_ay_1\}$. The remaining subgraphs are $G_2 = K_2, G_3 = K_3, \dots, G_a = K_a$. Thus the medium domination number of $G_1, G_2, \dots, G_a$ is $0,1,2, \dots, a - 1$ respectively. Thus $\pi_{MD}(G) = a$. Hence the proof.

4. Medium Domination Number and Medium Domination Decomposition Number of Central Vertex Join and Central edge Join of Graphs:

Theorem 4.1: For any graphs H_1 and H_2, then $MD(H_1 \dot{\vee} H_2) = \frac{1}{pC_2}[q + s' + \sum_{k=1}^{s}(d(v_k)C_2) + \sum_{k=(s+1)}^{p}(d(v_k)C_2)]$.

Proof: Let H_1 be the (s,t) graph and H_2 be the (s',t') graph. Let $V(H_1) = \{x_i: 1 \le i \le s\}$ and $V(H_2) = \{y_j: 1 \le j \le t\}$. Let $C(H_1)$ be the central graph of H_1 with $s + s'$ vertices and $\binom{s}{2} + s'$ edges. Let $V(C(H_1)) = \{x_i: 1 \le i \le s\} \cup \{c_k: 1 \le k \le s'\}$. Let $H_1 \dot{\vee} H_2$ be the central vertex join of the graphs H_1 and H_2 with $p(= s + s' + t)$ vertices and $q\left(= s' + t' + st + \frac{t(t-1)}{2}\right)$ edges, where $\{v_k: 1 \le k \le p\}$ are the vertices in $H_1 \dot{\vee} H_2$. Since the graph $H_1 \dot{\vee} H_2$ is obtained from $C(H_1)$ and H_2 by joining each vertex of H_1 with every vertex of H_2, each vertex of H_1 have degree $t + deg\, x_i$, for each $i = 1,2, \dots, s$,each vertex of H_2 have degree $s + deg\, y_j$, for each $j = 1,2, \dots, t$ and s' vertices have degree two. Therefore, $TDV(H_1 \dot{\vee} H_2) = q + s'(2C_2) + \sum_{i=1}^{s}((t + degx_i)C_2) + \sum_{i=1}^{t}((s + degy_j)C_2)$. That is, $TDV(H_1 \dot{\vee} H_2) = q + s' + \sum_{k=1}^{s}(d(v_k)C_2) + \sum_{k=(s+1)}^{p}(d(v_k)C_2)$, where $\sum_{i=1}^{s}(t + degx_i) = \sum_{k=1}^{s}(d(v_k)C_2)$, $\sum_{i=1}^{t}(s + degy_j) = \sum_{k=(s+1)}^{p}(d(v_k))$. Hence $MD(H_1 \dot{\vee} H_2) = \frac{1}{pC_2}[q + s' + \sum_{k=1}^{s}(d(v_k)C_2) + \sum_{k=(s+1)}^{p}(d(v_k)C_2)]$.

Hence the proof.

Theorem 4.2: For any graphs H_1 and H_2, then $MD(H_1 \underline{\vee} H_2) = \frac{1}{pC_2}[q + \sum_{k=1}^{s}(d(v_k)C_2) + \sum_{k=(s+1)}^{p}(d(v_k)C_2)) + s'((t + 2)C_2)]$.

Proof: The proof is similar to Theorem 4.1.

Result 4.3:(i) Let H_1 be the null graph with $s = 2$ and H_2 be the connected graph with $t = 2$, then $H_1 \dot{\vee} H_2$ does not admit MDD. (ii)Let H_1 be the connected graph with $s = 2$ and H_2 be the null graph with $t = 2$, then$H_1 \underline{\vee} H_2$ does not admit MDD.

Theorem 4.4: For any connected graphs H_1 and H_2 with $s \geq 2$ and $t \geq 2$, $H_1 \dot{\vee} H_2$ and $H_1 \underline{\vee} H_2$ admits MDD.

Proof: Since H_1 and H_2 are connected graphs with $s \geq 2$ and $t \geq 2$, $H_1 \dot{\vee} H_2$ and $H_1 \underline{\vee} H_2$ are also a connected graph with $p \geq 4$ and $q \geq 4$. Thus, by Theorem 2.5, $H_1 \dot{\vee} H_2$ and $H_1 \underline{\vee} H_2$ and admits MDD.

Theorem 4.5: Let H_1 and H_2 be null graphs with $s \geq 2$ and $t \geq 2$. Then $\pi_{MD}(H_1 \dot{\vee} H_2) \leq s - 1 + min\{s, t-1\}$.

Proof: Let H_1 and H_2 be null graphs with $s \geq 2$ and $t \geq 2$. Let $\{x_1, x_2, \dots, x_s\}$ be the vertices in H_1 and $\{y_1, y_2, \dots, y_t\}$ be the vertices in H_2. Let $C(H_1)$ be the central graph of H_1. By the definition of central graph, we join all the non-adjacent vertices in H_1. Therefore $C(H_1)$ is a complete graph with s vertices. Consider $H_1 \dot{\vee} H_2$. Since we join all the vertices in H_1 to all the vertices in H_2, $\{x_1, x_2, \dots, x_s\} \cup \{y_1\}$ form a complete graph K_{s+1} and the remaining graph form a complete bipartite graph $K_{s,t-1}$. We have $\pi_{MD}(K_{s,t-1}) \leq min\{p_1, p_2 - 1\}$. Thus $\pi_{MD}(H_1 \dot{\vee} H_2) \leq s - 1 + min\{s, t-1\}$. Hence the proof.

Remark 4.6: The bound in Theorem 4.5 is sharp. For the graphs H_1 and H_2 with $p_1 = 2$ and $p_2 = 2$, $\pi_{MD}(H_1 \dot{\vee} H_2) = 2$.

Corollary 4.7: Let H_1 and H_2 be null graphs with $s \geq 4$ and $t = 1$. Then $\pi_{MD}(H_1 \dot{\vee} H_2) \leq s - 1$.

Proof: By Theorem 4.5, the vertices in H_1 and H_2 form a complete graph with $s + 1$ vertices. We have $\pi_{MD}(K_{s+1}) \leq s + 1 - 2$. Thus $\pi_{MD}(H_1 \dot{\vee} H_2) \leq s - 1$. Hence the proof.

Theorem 4.8: Let H_1 be the null graph and H_2 be the complete graph with $s + t \geq 5$. Then $\pi_{MD}(H_1 \dot{\vee} H_2) \leq s + t - 2$.

Proof: Let $\{x_i : 1 \leq i \leq s\}$ be the vertices of H_1 and let $\{y_j : 1 \leq j \leq t\}$ be the vertices of H_2. Since H_1 is a null graph with s vertices, $C(H_1)$ is a complete graph with s vertices. Consider the graph $H_1 \dot{\vee} H_2$. Then $H_1 \dot{\vee} H_2$ form a complete graph K_{s+t} with $s + t \geq 5$. Therefore, $\pi_{MD}(H_1 \dot{\vee} H_2) \leq s + t - 2$. Hence the proof.

Theorem 4.9: Let H_1 be the r- regular graph with $s \geq 3$ and H_2 be the null graph with $t \geq 1$. Then

(i) $\pi_{MD}(H_1 \dot{\vee} H_2) \leq \left\lfloor \frac{s}{2} \right\rfloor$+min$\{s, t\}$

(ii) $\pi_{MD}(H_1 \underline{\vee} H_2) \leq \left\lfloor \frac{s}{2} \right\rfloor$+min$\{s', t\}$

Proof:(i) Since $H_1$is the r- regular graph, $\pi_{MD}(C(H_1)) \leq \left\lfloor \frac{s}{2} \right\rfloor$. Also since by the definition of $H_1 \dot{\vee} H_2$, we join all the s vertices in H_1 to all the vertices in H_2. Let $G = H_1 \dot{\vee} H_2$ and let $H = G \backslash E(C(H_1)$. Consider the graph H. Then H is a complete bipartite graph with $s + t$ vertices. We have $\pi_{MD}(H) \leq$ min$\{s, t\}$. Therefore, $\pi_{MD}(H_1 \dot{\vee} H_2) \leq \left\lfloor \frac{s}{2} \right\rfloor$+min$\{s, t\}$.

(ii)The proof is similar to (i).

Corollary 4.10: Let H_1 be the $(s-1)$- regular graph with $s \geq 2$ and H_2 be the null graph with $t = 1$. Then $\pi_{MD}(H_1 \dot{\vee} H_2) = \pi_{MD}(H_1 \underline{\vee} H_2) = 2$.

Proof: Since H_1 is the $(s-1)$- regular graph, $\lfloor MD(C(H_1)) = 0 \rfloor$. Consider $G = H_1 \dot{\vee} H_2$ and let $H = G \backslash E(C(H_1)$. Then it is a star graph with s end vertices. We have $MD(H) = 1$. Thus, we can decompose the graph G in to two subgraphs G_1 and G_2 with medium domination number 0 and 1 respectively. Hence $\pi_{MD}(H_1 \dot{\vee} H_2) = 2$. Similarly, we can prove that $\pi_{MD}(H_1 \underline{\vee} H_2) = 2$. Hence the proof.

Theorem 4.11: Let H_1 be the star graph $K_{1,m}$ with $m \geq 2$ end vertices. Let H_2 be the null graph with $t \geq 1$. Then

(i) $\pi_{MD}(H_1 \dot{\vee} H_2) \leq m + 1$

(ii) $\pi_{MD}(H_1 \underline{\vee} H_2) \leq m$

Proof: Let H_1 be the star graph $K_{1,m}$ with $m \geq 2$ end vertices, where $\{x_i: 1 \leq i \leq s\}$ are the vertices in H_1 and $\{x_i: 2 \leq i \leq s\}$ are the m end vertices in H_1.Let H_2 be the null graph with $t \geq 1$, where $\{y_j: 1 \leq j \leq t\}$ are the vertices in H_2.

(i) Consider $H_1 \dot{\vee} H_2$. Here m end vertices from H_1 and one vertex from H_2 form a complete graph with $m+1$ vertices. Therefore, $\pi_{MD}(K_{m+1}) \leq m+1-2 = m+1$. Also the medium domination decomposition number of remaining graph is two. Therefore $\pi_{MD}(H_1 \dot{\vee} H_2) \leq m-1+2 = m+1$. Hence $\pi_{MD}(H_1 \dot{\vee} H_2) \leq m+1$.

(ii) Consider $H_1 \underline{\vee} H_2$. Here m end vertices form a complete graph with m vertices. Therefore $\pi_{MD}(K_m) \leq m-2$. Also the medium domination decomposition number of remaining graph is two. Therefore $\pi_{MD}\left(H_1 \underline{\vee} H_2\right) \leq m-2+2 = m$. Hence $\pi_{MD}\left(H_1 \underline{\vee} H_2\right) \leq m$. Hence the proof.

5. Conclusion

In this paper, we calculated the number of vertices that are capable of dominating both of u and v. The total number of vertices that dominate every pair of vertices is examined and the average of this value is calculated which is called "the medium domination number" of graph. Some theorems and bounds on the Medium Domination Decomposition Number of join of graphs, central vertex join of graphs and central edge join of graphs are given. Also a realization theorem for such decomposition is obtained. Further this concept can be extended to product of graphs.

6. References

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