The Upper Paired Monophonic Number of a Graph

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Abstract

A paired monophonic set M in a connected graph G is called a minimal a paired monophonic set if no proper subset of M is a paired monophonic set of G. The upper paired monophonic $m_p^+(G)$ of G is the maximum cardinality of a minimal paired monophonic set of G. Some general properties satisfied by this concept are studied. The paired monophonic number of some family of graphs is obtained. It is shown that for every pair of positive integers a and b with $4 \leq a \leq b$, there exists a connected graph G such that $m_p(G) = a$ and $m_p^+(G) = a + b$, where $m_p(G)$ is the paired monophonic number of G.

Keywords: perfect matching, monophonic path , monophonic number, paired monophonic number, upper paired monophonic number.

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1 Introduction

By a graph $G = (V, E)$, we mean a finite,undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [1]. For the *neighborhood* of the vertex v in $G, N(v) = \{u \in V(G) : uv \in E(G)\}$. The *degree* of a vertex v of a graph is $deg(v) = |N(V)|$. $\Delta(G)$ and $\delta(G)$ are the maximum and minimum degrees of the graph respectively. A vertex v is said to be a universal vertex if $deg(v) = n - 1$. Let uv be an edge of G such that $deg(G) = 1$. Then G is called an *end vertex* and u is the *support vertex* of G. For $S \subseteq V(G)$, the *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S. A

vertex v is called an *extreme vertex* of a graph G if $G[N(v)]$ is complete. A matching in a graph is a set of edges that do not have a set of common vertices A *perfect matching* M of a graph G is a set of disjoint edge that covers all vertices of G.

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. An $u-v$ path of length $d(u,v)$ is called an $u-v$ *geodesic*. An edge that connects two non-adjacent vertices of a path P is called the *chord* of P . A chordless $u-v$ path is referred to as a *monophonic path*. The closed interval $J[u,v]$ for two vertices u and v is consists of all the vertices along an $u-v$ monophonic path, including the vertices u and v. If $uv \in E$, then $J[u,v] = \{u,v\}$. For a set M of vertices, let $J[M] = \cup_{u,v\in M} J[u,v]$. Then certainly $M \subseteq J[M]$. If $J[M] = V$, a set $M \subseteq V(G)$ is referred to as a *monophonic set* of G. The *monophonic* number $m(G)$ of G is the minimum order of its monophonic sets.The monophonic number of a graph was studied in [1-15, 17].

A monophonic set M in a connected graph G is called a *paired monophonic set* if $M = V$ or the each component of $G[M]$ has perfect matching. The minimum cardinality of a paired monophonic set of G is the *paired monophonic number* of G and is denoted by $m_p(G)$. The paired monophonic number of G is studied in [16]. Throughout the following G denotes a connected graph with at least three vertices. The following theorems are used in the sequel.

Theorem 1.1. [16] Each extreme vertex of a graph G belongs to every paired monophonic set of G. In particular, each end-vertex of G belongs to every minimal paired monophonic set of G.

Theorem 1.2. [16] Each support vertex of a graph G belongs to every paired monophonic

set of G.

Theorem 1.3. [16]] For the complete graph $G = K_n$ $(n \geq 2), m_p(G) = n$

2 The Upper Paired Monophonic Number of a Graph

Definition 2.1. A paired monophonic set M in a connected graph G is called a *minimal a paired* monophonic *set* if no proper subset of M is a monophonic geodetic set of G. The *upper paired* monophonic $m_p^+(G)$ of G is the maximum cardinality of a minimal paired monophonic set of G.

Example2.2. For the graph G given in Figure 2.1, $M_1 = \{v_1, v_2, v_4, v_5\}$, $M_2 = \{v_1, v_4, v_5, v_7\}$, $M_3 = \{v_1, v_4, v_5, v_8\}$,
$M_4 = \{v_2, v_3, v_5, v_6\}$, $M_5 = \{v_2, v_3, v_6, v_7\}$ and $M_6 = \{v_2, v_3, v_6, v_8\}$, are minimum paired monophonic sets of G so that $m_p(G) = 4$. Also $W = \{ v_1, v_3, v_4, v_6, v_7, v_8\}$ is a minimal paired monophonic set of G and so $m_p^+(G) \geq 6$. It is easily verified that no 7-element subset of V is a minimal paired monophonic set of G, and thus $m_p^+(G) = 6$.

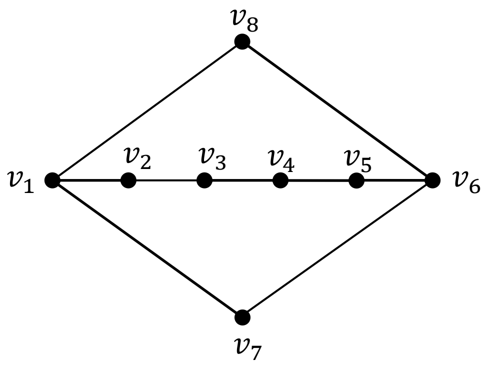


Figure 2.1

Remark 2.3. Every minimum monophonic set of G is a minimal paired monophonic set of G, but the converse is not true. For the graph G given in Figure 2.1, $W = \{ v_1, v_3, v_4, v_6, v_7, v_8\}$ is a minimal paired monophonic set but not a minimum paired monophonic set of G.

Theorem 2.4. For a connected graph $G, 2 \leq m_p(G) \leq m_p^+(G) \leq n$.

Proof. Since any paired monophonic set needs at least two vertices, $m_p(G) \geq 2$. Let S be a minimum paired monophonic set of G so that $m_p(G) = |W|$. Since S is also minimal paired monophonic set of G, it is clear that $m_p^+(G) \geq m_p(G) = |W|$. Hence $m_p(G) \leq m_p^+(G)$. Since the vertex set $V(G)$ is a paired monophonic set, $m_p^+(G) \leq n$. Thus $2 \leq m_p(G) \leq m_p^+(G) \leq n$. ■

Remark 2.5. By Theorem 1.3, for $G = K_2$, the set of all end-vertices of G is the unique minimum paired monophonic set of G and so $m_p(G) = m_p^+(G)$. Also, for the graph given in Figure 2.1, $m_p(G) = 4$ and $m_p^+(G) = 6$ so that strict inequality can hold in Theorem 2.4.

Theorem 2.6. Each extreme vertex of a graph G belongs to every minimal paired monophonic set of G. In particular, each end-vertex of G belongs to every minimal paired monophonic set of G.

Proof. Since every minimal paired monophonic set is a paired geodetic set of G, the theorem follows from Theorem 1.1. ■

Theorem 2.7. Each supportive vertex of a graph G belongs to every minimal paired monophonic set of G.

Proof. Since every minimal paired monophonic set is a paired monophonic set of G, the theorem follows from Theorem 1.2. ■

Theorem 2.8. Let G be a connected graph with v a cut-vertex of G and let M be a minimal paired monophonic set of G. Then every component of $G - v$ contains an element of M.

Proof. Suppose that there is a component G_1 of $G - v$ such that G_1 contains no vertex of S. By Theorem 2.3, G_1 does not contain any end-vertex of G. Thus G_1 contains at least one vertex, say u. Since M is a monophonic set, there exists vertices $x, y \in M$ such that u lies on the $x - y$ monophonic $P : x = u_0, u_1, u_2, \ldots, u, .., u_t = y$ in G. Let P_1 be a $x - u$ sub path of P and P_2 be a $u - y$ subpath of P. Since v is a cut-vertex of G, both P_1 and P_2 contain v so that P is not a path, which is a contradiction. Thus every component of $G-v$ contains an element of M. ■

Corollary 2.9. For any star $G = K_{1,n-1}$ $(p \geq 3), m_p(G) = m_p^+(G) = n.$

Proof : This follows from Theorems 2.6 and 2.7 ■

Corollary 2.10. For any path $P_n, m_p(G) = m_p^+(G) = 4.$

Proof : This follows from Theorems 2,6 and 2.7 ■

Theorem 2.11. For any cycle C_n $(n \geq 4), g_p^+(G) = 4.$

Proof. Let n be even. Let $\{u, v\}$ be a set of antipodal vertices in G and $ux, vy \in E(G)$ such that $x \neq y$. Then $M = \{u, v, x, y\}$ is a minimal paired monophonic set of G and so $m_p^+(G) \geq 4$. Now, we show that there is no minimal paired monophonic set X of G with $|X| \geq 5$. Suppose that there exists a minimal monophonic set X of G such that $|X| \geq 5$. Then, it follows that X contains two antipodal vertices, say u and v. Then $ux, vy \in E(G)$ such that $x \neq y$. Hence $M = \{u, v, x, y\}$ with $S \subset X$, which is a contradiction to X a minimal minimal paired monophonic set of G. Therefore $m_p^+(G) = 4$. Let n be odd. Let u and v be any two vertices of G. Then it is clear that $\{u, v\}$ is not a monophonic set of C_n. For any vertex u, let v, w be the antipodal vertices of u. Then clearly $M = \{u, v, w, x\}$ is a minimal paired monophonic set of G so that $m_p^+(G) \geq 4$, where $ux \in E(G)$. By similar argument in first part of this theorem, we prove that $m_p^+(G) = 4$. ■

Theorem 2.12. For the complete bipartite graph $G = K_{r,s}$,

$$m_p^+(G) = \begin{cases} r + s; r = 1, s \geq 1 \\ 4; \quad r \geq 2, s \geq 2 \end{cases}$$

Proof. If $r = 1, s \geq 2$, then G is a star. Hence the result follows from Corollary 2.9. So let $r \geq 2, s \geq 2$. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_{ns}\}$ be a bipartition of G. Let $W = \{x_i, x_j, y_l, y_m\}$, where $i \neq j$ and $l \neq m$. Then W is a minimal paired monophonic set of G. Hence $m_p^+(G) \geq 4$. Now, we show that there is no minimal paired geodetic set M of G with $|M| \geq 5$. Suppose that there exists a minimal monophonic set M of G such that $|M| \geq 5$. Then, it follows that M contains at least two vertices of X and at least two vertices of Y. Then $W \subset M$, which is a contradiction to M a minimal minimal paired monophonic set of G. Therefore $m_p^+(G) = 4$.

■

Theorem 2.13. For a connected graph $G, m_p(G) = n$ if and only if $m_p^+(G) = n$.

Proof. Let $m_p^+(G) = n$. Then $M = V(G)$ is the unique minimal paired monophonic set of of G. Since no proper subset of M is a paired monophonic set, it is clear that M is the unique minimum paired monophonic set of G and so $m_p(G) = n$. The converse follows from Theorem 2.4. ■

Theorem 2.14. If G is a connected graph of order p with $m_p(G) = n-1$, then $m_p^+(G) =$

$n-1$.

Proof. Since $m_p(G) = n - 1$, it follows from Theorem 2.4 that $m_p^+(G) = n$ or $n - 1$. If $m_p^+(G) = n$, then by Theorem 2.13, $m_p(G) = n$, which is a contradiction. Hence $m_p^+(G) = n - 1$. ■

Remark 2.15. The converse of Theorem 2.15 need not be true. For the graph G given in Figure 2.2, $M = \{v_2, v_3, v_4, v_5, v_6, v_7\}$ is a minimal paired monophonic set of G and so $m_p^+(G) = 6 = n-1$. Also $M_1 = \{v_1, v_5, v_6, v_7\}$ is a g_p–set of G so that $m_p(G) = 4 < n - 1$.

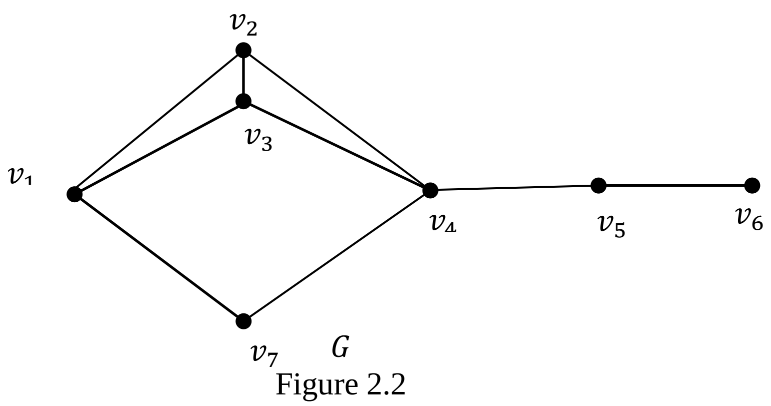


Figure 2.2

Theorem 2.16. For any positive integers a and b with $4 \leq a < b$, there exists a connected graph G such that $m_p(G) = a$ and $m_p^+(G) = a + b$.

Proof: For $a = b$, let $P : x, y, z$ be a path on three vertices and R_i: y_i, z_i $(1 \leq i \leq a - 1)$ be copy of path on two vertices. Let G be te graph obtained from P and Ri $(1 \leq i \leq a - 1)$ by joining each y_i $(1 \leq i \leq a - 1)$. The graph G is shown in Figure 2.3.

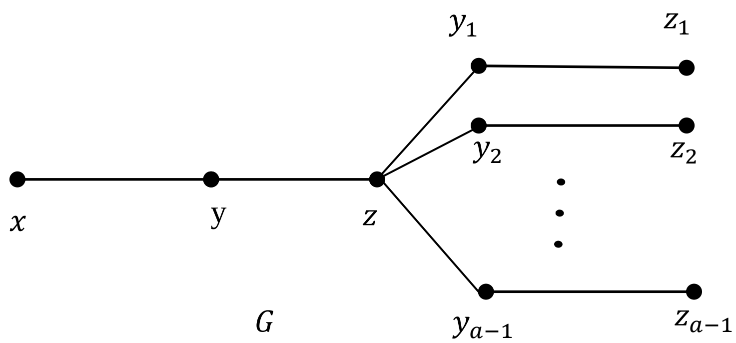
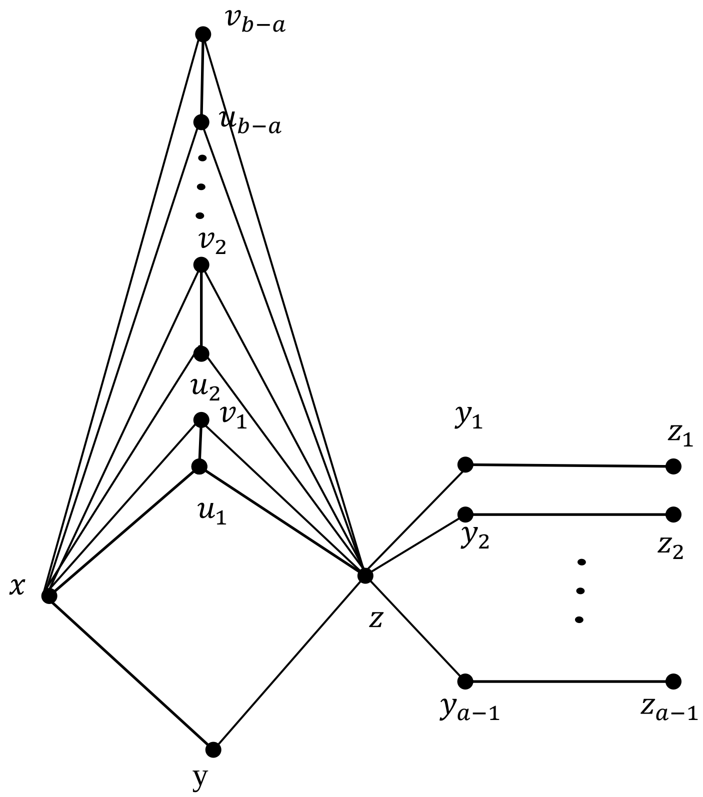


Figure 2.3

Let $Z = \{x, y, y_1, y_2, \ldots, y_a, z_1, z_2, \ldots, z_{a-1}\}$ be the set end vertices and support vertices of G. By Theorem Z is a subset of paired monophonic set of G and so $g_p(G) \geq 2a$. Now $I[Z] = V$ and each components of $G[Z]$ has perfect matching and so Z is a paired monophonic set of G. Therefore $m_p(G) = 2a = n - 1$. By Theorem 2.14, $g_p^+(G) = 2a$. So,let $4 \leq a \leq b$. Let $P : x, y, z$ be a path on three vertices. Let $P_i : y_i, z_i$ $(1 \leq i \leq a - 1)$ be a copy of path on two vertices and Q_i: u_i, v_i $(1 \leq i \leq b - a)$ be a copy of path on two vertices. Let G be the graph obtained from P_i $(1 \leq i \leq a - 1)$, $Q_i(1 \leq i \leq b - a)$ and P by joining each y_i $(1 \leq i \leq a - 1)$ with z and joining each ui and vi $(1 \leq i \leq b - a)$ with x and y respectively. The graph G is given in Figure 2.4. Let $Z = \{y_1, y_2, \ldots, y_{a-1}, z_1, z_2, \ldots, z_{a-1}\}$ be the set of end and supportive vertices of G. Then by Theorems 1.1 and 1.2, Z is a subset of every paired monophonic set of G and so $m_p(G) \geq 2a - 2$. It is easily observed that $Z \cup \{v\}$, where $v \notin Z$ is not a paired monophonic set of G and so $m_p(G) \geq 2a$. Let $M = Z \cup \{x, y\}$. Then M is a paired monophonic set of G so that $m_p(G) = 2a$.

Next we prove that $m_p^+(G) = a + b$. Let $M' = Z \cup \{u_1, u_2, \ldots, u_{b-a}, v_1, v_2, \ldots, v_{b-a}\} \cup \{y, z\}$. Then M' is a paired monophonic set of G. We prove that M' is a minimal paired monophonic set of G. On the contrary suppose that $M'$is not a minimal paired set of G. Then there exists a paired monophonic set M'' of G such that $M'' \subset M'$ such that $u \in S'$. Then by Theorems 1.1 and 1.2, $u \neq y_i$ and $u \neq z_i$ $(1 \leq i \leq a - 1)$. If u is either u_i or v_i for some i $(1 \leq i \leq b - a)$ or u is either y or z, then $G[M'']$ has no perfect matching, which is a contradiction. Therefore M'' is not a paired monophonic set of G, which is a contradiction. Therefore M' is a minimal paired monophonic set of G so that $m_p^+(G) \geq b - a + 2a - 2 + 2 = a = b$. We prove that $m_p^+(G) = a + b$. Since $|V(G)| = a + b + 1$, by Theorem2.13, $m_p^+(G) = a + b$.



G

Figure 2.4

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