EXPLORING AMPLITUDE DEATH IN CONJUGATE COUPLED VAN DER POL OSCILLATORS: A COMPREHENSIVE REVIEW

Rohit Panda[1*], Amit Sharma[2]

Abstract.
Much of the recent studies concerned with coupled oscillator systems is focused on introducing the amplitude death phenomenon in oscillators whose equilibrium points are otherwise unstable in uncoupled state resulting in limit cycle oscillations. This article presents a review on a recent research work that induces amplitude death in coupled Van der Pol oscillators. The coupling is through diffusive flow of conjugate dynamical variables. Further exploration is done to understand the variation of amplitude death regime or range of values of coupling strength with that of the characteristic parameters of the oscillators, namely nonlinearity parameter and natural frequency. We also establish the condition between the characteristic parameters for which such type of conjugate coupling fails to instigate amplitude death regime. Oscillators showing amplitude death have been used extensively in many microscopic physical and biological applications. Finally, a rigorous discussion on the output characteristics of such coupled VdP oscillator pairs for different ranges of values of coupling strength is also furnished. Such knowledge of output characteristics will be of importance for a scientific personnel in the domain of applied physics who wish to use this type of conjugate coupled VdP oscillator in their specific applications.

[1*,2]Department of Physics, University Institute of Science, Chandigarh University, Mohali 140310 India
[2]Center of Excellence for Computational Physics, Department of Physics, University Institute of Science, Chandigarh University, Mohali 140310 India
Email: amit.e11102@cumail.in

***Corresponding Author:** Rohit Panda
*Department of Physics, University Institute of Science, Chandigarh University, Mohali 140310 India

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1. Introduction

The Van der Pol Oscillator named after the Dutch physicist Balthasar Van der Pol who conceived it incipiently, is a self sustained, non linear oscillating system [1]. The popularity of the oscillator can be attributed to its immense applications in the various branches of sciences. For example, it can be found in electrical systems that utilise vacuum tubes as one of its components. The oscillator has been implemented extensively in modelling several naturally occurring phenomenon. For instance, it has been used: to model two plates at the location of geological faults in seismology [2, 3], to model vocal fold oscillations in phonation studies [4], to model action potential of neurons in biological studies [5].

The dynamical equation that represents the Van der Pol oscillator (VdP) is given by the differential equation,

$$\ddot{x} + \epsilon(1 - x^2)\dot{x} + \omega^2 x = 0 \tag{1}$$

where x represents the dynamical variable and ω designates the natural frequency of the oscillator [6]. It can be seen that the oscillator is non-linear with, ε (>0) representing a measure of non-linearity and is hence known as the non-linearity parameter. It is a well known fact that the oscillator has a single equilibrium state which is unstable corresponding to the point $x = \dot{x} = 0$ in phase space [7]. Further close inspection reveals that the oscillator can be conceived as a perturbation of a simple harmonic oscillator. Thus for weak nonlinearities ($\varepsilon \ll 1$) the oscillator is seen to have sinusoidal output which is characteristic of simple harmonic oscillations whereas, there is a significant departure from this characteristic for stronger nonlinearities ($\varepsilon \gg 1$). Figure 1 is indicative of the preceding statements.

Majority of recent studies pertaining to the study of coupled oscillator systems (any in general) is focused on converting this unstable equilibrium point to a stable equilibrium point. This would mean the oscillation of the oscillators in coupled state would die down and such an effect is formally known as amplitude death phenomenon [8, 9, 10]. The most commonly used type of coupling is the conjugate coupling, which can successfully introduce amplitude death phenomenon in the majority of the oscillator systems. In other words, conjugate coupling is mainly used as a tool to convert the unstable fixed point to stable equilibrium creating an amplitude death regime. For instance, conjugate coupling has been employed to introduce amplitude death in coupled Landau-Stuart oscillators and Lorenz oscillators [11]. The same phenomenon can be perceived in a pair of conjugate coupled Chua oscillators [12]. An extensive study of such phenomena has been carried out in Rössler chaotic systems as well [13]. Mathematically, conjugate coupling implies coupling via diffusive row of 'dissimilar' variables or 'conjugate' variables, namely x and $\dot{x}$ [14, 15, 16].

The case of VdP oscillators is no different and has been imposed with similar treatments. This article reviews a recent study, which attempts to introduce amplitude death phenomenon in VdP oscillator pairs via conjugate coupling [17]. The amplitude death is prevalent only for a certain range of values of the coupling strength, familiarly known as amplitude death regime [15, 18]. The review shall be made more rigorous by exploring the variation of amplitude death range (values of coupling strength) with the parameters that characterise the VdP oscillator pair, namely the nonlinearity parameter and the natural frequency. A more comprehensive knowledge is also gained by understanding the amplitude death characteristics (damped or under-damped) and the output characteristics (outside the amplitude death range) depending on the coupling strength values. Such understanding will be of immense importance in the domain applied science where a particular type of output for an oscillator is desirable.

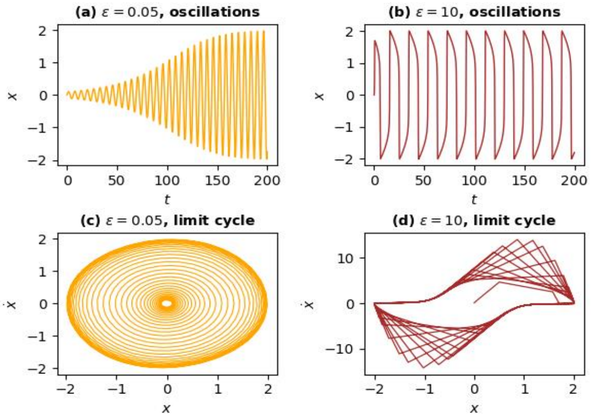

Figure 1: Limit cycle oscillations of VdP oscillator for weak ($\varepsilon \ll 1$) and strong nonlinearities ($\varepsilon \gg 1$).

2. Amplitude death phenomenon in conjugate coupled Van der Pol Oscillator

It has been made aware already that the Van der Pol oscillator has a single unstable equilibrium point at $x = \dot{x} = 0$ [19, 20]. In simpler terms this basically means that the oscillations will evolve into some kind of limit cycle depending on the values of its characteristic parameters, namely non-linearity parameter and natural frequency. We focus on a recent study of coupled VdP oscillators which aims to introduce the amplitude death via conjugate coupling as a result of which the unstable equilibrium point of the individual oscillators can be converted into stable fixed point, creating an amplitude death regime. The ensuing paragraphs of this section will present a very brief recap of the above mentioned study which has been published in Ref [21].

The type of conjugate coupling that has been reported to have amplitude death is defined by the following governing equation,

$$\begin{aligned} \dot{x} - \epsilon(1 - x^2)\dot{x} + \omega_0^2 x &= k(y - \dot{x}) \\ \dot{y} - \epsilon(1 - y^2)\dot{y} + \omega_0^2 y &= k(x - \dot{y}) \end{aligned} \quad (2)$$

where ω_0 and ε refers to the natural frequency and non-linearity parameters of the oscillators. The parameter, k indicates coupling strength between the two oscillators. It can be noted that, for ease of simple analysis we have assumed identical oscillators ($\varepsilon_1 = \varepsilon_2 = \varepsilon$) and ($\omega_1 = \omega_2 = \omega_0$). we begin by converting the governing equations to coupled first order equations to obtain,

$$\begin{aligned} \dot{x} &= v_x \\ \dot{v_x} &= (\epsilon - k)v_x - \omega_0^2 x + ky - \epsilon v_x x^2 \\ \dot{y} &= v_y \\ \dot{v_y} &= (\epsilon - k)v_y - \omega_0^2) - kx - \epsilon v_y y^2 \end{aligned} \quad (3)$$

Solving equation (3) for fixed point we get $x = \dot{x} = 0$ and $y = \dot{y} = 0$ and hence conclude such coupling does not alter the stable point. Now, one can employ Lyapunov analysis to ascertain its nature of stability. This method principally entails perturbation analysis of the linearized governing equations [22, 23]. The Jacobian matrix for linearized equations turn out to be

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_0^2 & \epsilon - k & k & 0 \\ 0 & 0 & 0 & 1 \\ -k & 0 & -\omega_0^2) & \epsilon - k \end{bmatrix} \quad (4)$$

whose characteristic equation is,

$$\lambda^4 + (2k - 2\epsilon)\lambda^3 + (\epsilon^2 - 2\epsilon k + 2\omega^2 + k^2)\lambda^2 + (2\omega^2 k - 2\epsilon\omega^2)\lambda + \omega^4 - k^2 = 0 \quad (5)$$

The real parts of the roots of the equation (5) indicate Lyapunov exponents at the equilibrium point. The exponent is defined as the rate at which infinitesimally close trajectories in phase space separate out [24, 25, 26, 27]. More negative the Lyapunov exponent, the more stable the fixed point will be and the more positive the exponent, the more unstable the fixed point shall be. A criterion known as Routh-Hurwitz (RH) criterion, is used extensively in the domain of control systems to determine the signs of the zeroes of the equation that designate output function [28, 29, 30, 31, 32]. The criterion states that a bi-quadratic polynomial, $a\lambda^4+b\lambda^3+c\lambda^2+d\lambda+e$ will have all roots comprising of negative real parts if the conditions $e > 1$, $d > 0$, $cd-be > 0$ and $bcd-ad^2 > 0$ are satisfied [33, 34, 35, 36]. This specific case of the criterion can be implemented to determine the sign of the roots of the equation (5). By using one of the conditions we immediately note that $k > \varepsilon$ should be one of the required conditions for amplitude death. However, this clearly is not the only inequality as the other conditions of the criterion also must be satisfied necessarily. Analytically manipulating the other inequality conditions in the context of this coupled oscillator is a rigorously tedious task. Thus we seek for graphical visualisation.

We plot the largest value of the real parts of the roots of equation (5) as a function of coupling strength. The result is indicated by figure 2. The values of coupling strength for which the largest Lyapunov exponent is negative testifies that amplitude death phenomenon is prevalent in this range of k values. As mentioned earlier this range is popularly termed as amplitude death regime. In other words, conjugate coupling has successfully converted the earlier unstable point into a stable attractor in this regime.

3. Variation of Amplitude death regime with nonlinearity parameter

We shall explore the work done in the recent study further by understanding the variation of amplitude death regime with the characteristic parameters of the oscillators, non-linearity parameter ε at fixed value of $\omega = 1.05$ in Figure. 3. We plot the largest Lyapunov exponent described by equation (3) against coupling strength k for nonlinearity parameter = 0.1 (keeping ω_0 fixed). In Figure. 2 (a), we observed the largest Lyapunov exponent (black line) is negative at $k = 0.11$ which evidence of amplitude death and It also have good agreement of the largest eigenvalue (obtained from Eq. (5)) of the coupled oscillators. While at higher coupling strength ($k>1.13$), the coupled system goes to unbound state (UBS).

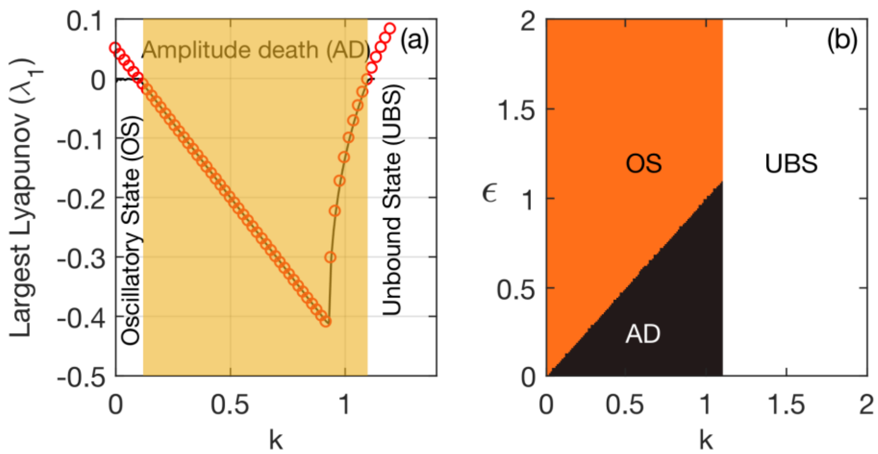

Figure. 2: (a) Variation of largest Lyapunov exponent (black line) for the range of coupling strength k signifying amplitude death at nonlinearity parameter $\varepsilon = 0.1$ and natural frequency, $\omega_0 = 1.05$. Red circle represents the largest eigenvalue of the coupled VdP oscillator. (b) Parameter space in (ε-k) for natural frequency $\omega_0 = 1.05$ for conjugate coupled VdP oscillators.

In figure 2(b), we draw parameter plane (ε-k) at fixed value of $\omega = 1.05$, and show the various dynamical regime: oscillatory state (OS), amplitude death (AD), and unbound state (UBS). It can be notice that the critical value of coupling strength k for amplitude death is increase with nonlinearity parameter ε up to the $\varepsilon = 1.1$. After this value amplitude death region is vanished and coupled oscillators have oscillation state. While at $k>1.1$, the coupled oscillators, wen to the unbound state (UBS) from oscillatory state or amplitude death state at any value of nonlinearity parameter ε. It can also be concluded that varying the nonlinearity results in variation of the lower limit of amplitude death state of coupling strength while the upper limit remains unchanged. Thus the nonlinearity parameter for this type of coupling acts as a controller for the lower limit of amplitude death.

4. Variation of Amplitude death regime with natural frequency

Figure 3(a) is indicative of the variation of largest Lyapunov exponent with the range of coupling strength k to signifying amplitude death for various value of natural frequency ω_0. Unlike the previous case, here adjusting the natural frequency causes variation in the upper limit of range of coupling strength k for which amplitude death can be attained. Thus natural frequency can act as a controller for the upper limit of amplitude death range. We have also draw the parameter plane between (ω_0-k) at constant nonlinearity parameter ε = 0.1 for conjugate coupled VdP oscillators in figure 3(b), and observed similar regime: oscillatory (OS), Amplitude death (AD), and unbound state (UBS).

Above mentioned facts form the basis for the most sought after property of this type of coupling. The beauty of this type of conjugate coupled VdP oscillator pair is that the lower and upper limits of amplitude death range can be controlled independently by varying the respective individual oscillator parameters (ε and ω_0) without one affecting the other. One may imagine that such coupling gas a wide ranges of uses that has such immense ease of control to obtain the desired output.

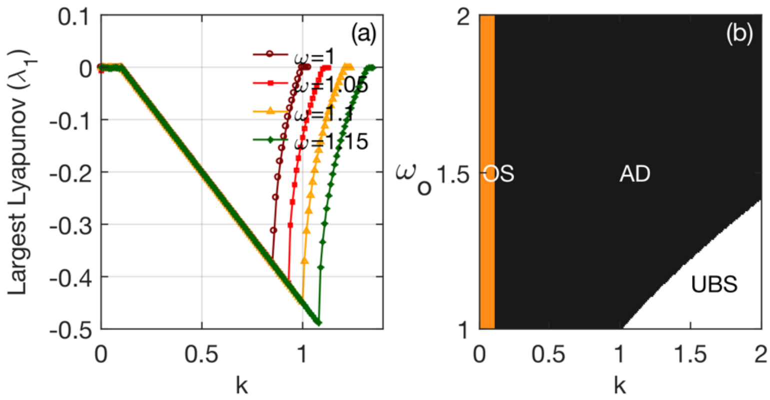

Figure 3: (a) Variation of largest Lyapunov exponent with coupling strength, k for various value of natural frequency ω at nonlinearity parameter ε = 0.1. (b) Different dynamical regime in parameter plane (ω-k) for conjugate coupled VdP oscillator.

5. Output characteristic of the coupling

For more exhaustive review in the light of the above discussions, we seek for graphical verification and analysis. In our context, understanding how the conjugate coupled oscillator behaves, particularly in the amplitude death region is of immense importance to experimentalists and engineers, who wish to get the desired output for their applications, by varying its parameters. Such desired output in precise applications can only be achieved with knowledge on output characteristics of an oscillator in correlation with its parameters. For this we solve the governing equations (2) and (3) numerically for fixed values of natural frequency ω_0 and nonlinearity parameter ε. The outcome is reflected in figure 4(a). A comprehensive understanding of the outputs can be achieved by more rigorous inspection of figure 4(a), which shows the variation of largest eigenvalue with coupling strength k.

It can be seen that for the values of coupling strength such that $k < \varepsilon$ referred to as region R1 in Figure 4(a) the largest eigenvalue is positive and there will be no amplitude death. This is verified in figure 4(b). We also infer that the frequency of stable oscillations decreases as the coupling strength increases and the reason for this can be attributed to the fact that the largest eigenvalue decreases and becomes less positive in this interval. For the range of coupling strength given by $0.09 < k < 1$, where k is the value of coupling strength at which the largest eigenvalue is least as depicted by R2 in figure 4(a), the largest eigenvalue is negative. Thus it can be concluded that amplitude death phenomena can be encountered in this range. This is indicated in figure 4(c). Furthermore, the oscillatory damping is found to be of under-damping type. The under-damping characteristics are also found to decrease with the increase in coupling strength, meaning as the coupling strength increases the amplitude dies down quicker. This is because the largest eigenvalue decreases and becomes more negative in this region.

For the region R3 shown in figure 4(a) the largest eigenvalue is still negative and thus it can be inferred that amplitude death shall transpire in this region. This fact is supported by figure 4(d). Moreover, it can be noticed that the amplitude death is an over-damped type. Further inspection shows that the characteristics of over-damping decreases with increase in coupling strength, essentially meaning the rate of amplitude death decreases. This can be accredited to the fact that the largest eigenvalue increases and becomes less negative in this region.

On the other hand, region R4 depicted that the largest eigenvalue exponent is positive which basically translates that amplitude death is not possible. In fact, interestingly in this region there is growth in amplitude with no oscillations refer as unbound state as depicted in figure 4(e). We can term this phenomenon as explosive amplitude growth. Furthermore, it can be discerned that the rate of amplitude growth increases as coupling strength increases and this owes to the fact that the coupling strength increases and becomes more positive in this interval.

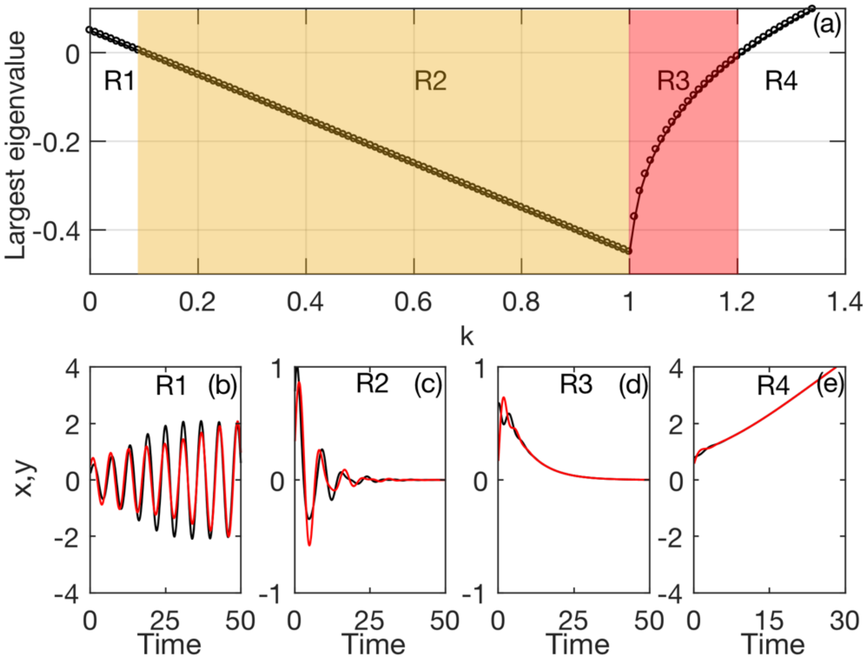

Figure 4: (a) Largest eigenvalue plotted as a function of coupling strength k. The time series of x and y variable outputs of the coupled oscillator for values of coupling strength (b) k = 0.1, (c) k = 0.5 (d) k = 1.1, and (e) k = 1.3 at nonlinearity parameters $\varepsilon = 0.1$ and natural frequency $\omega_0 = 1.1$.

6. No amplitude death condition

It is to be made aware that, in any case natural frequency becomes less than or equal to the non-linearity parameter ($\omega_0 \leq \varepsilon$), amplitude death cannot be achieved for any value of coupling strength. This has been verified in figure 5. The Lyapunov exponent is positive for all values of coupling strength, signifying that the amplitude death regime is non-existent. This testifies that conjugate coupling will be unsuccessful in creating a stable attractor for the oscillators whose characteristic parameters are such that $\omega_0 \leq \varepsilon$. Further verification can be done by solving the governing equations (2) and (3) numerically by implying the condition, $\omega_0 \leq \varepsilon$. This is portrayed in figure 5 by plotting the parameter plane between (ε-k) for different value of frequency.

7. Conclusion

The Van der Pol oscillator has a single equilibrium state which is unstable in nature. The incentive is to review a recent research study that induces amplitude death phenomenon by coupling two Van der Pol oscillators via diffusive flow of conjugate variables which is commonly referred to as conjugate coupling, consequently converting the initially unstable equilibrium state into a stable attractor. Amplitude death under conjugate coupling is prevalent for a range of values of coupling strength which is termed as amplitude death regime.

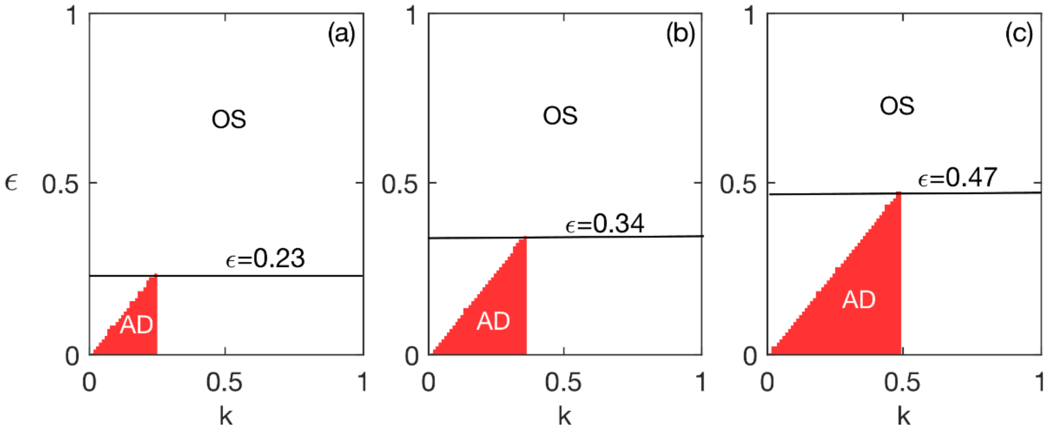

Figure 5: Figure depicting region of amplitude death in parameter plane (ε-k) for various value of natural frequency (a) $\omega_0 = 0.5$, (b) $\omega_0 = 0.6$, and (c) $\omega_0 = 0.7$ in the conjugate coupled VdP oscillators.

We have also explored further by appreciating the variation of this amplitude death regime with the characteristic parameters of the oscillators (ω_0 and ε). We find that the nonlinearity parameter (ε) acts as the controller for the lower limit of the amplitude death regime. The lower limit value of coupling strength, k is also exactly equal to the value of ε. On the other hand, the natural frequency acts as the controller for the upper limit of the amplitude death regime. Thus, it is concluded that the limits of the amplitude death regime can be controlled independently by varying the respective controlling characteristic parameters of the oscillators without one affecting the other. This gives extensive ease of control to obtain desired output and can be thought to have far-reaching applications. We have also presented a comprehensive understanding of the output characteristics of such conjugate coupled VdP oscillator pairs.

We see that based on the output characteristics the entire range of coupling strength can be broadly classified into four distinct types. Amplitude death is seen only in region R2 and R3. Region R1 and R4 fail to induce amplitude death phenomenon. As mentioned, region 1 represents limit cycle oscillations with decreasing frequency, while R4 indicates growth of amplitude with no oscillations. R2 signifies amplitude death with decreasing under-damped characteristics whereas, the oscillator pairs show Amplitude death with decreasing over-damped characteristics in R3. The coupled VdP oscillators possessing amplitude death can find applications in many field.

Furthermore, we realise that this type of conjugate coupling is unsuccessful in creating an amplitude death regime for oscillators, that have natural frequency lesser than the non- linearity parameter ($\omega_0 < \varepsilon$), with the output being oscillations for lower values of k to being non oscillatory growth of amplitude for higher values of k.

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7.1. A subsection

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