FAIR INDEPENDENT DOMINATION IN JOIN, UNION AND LEXICOGRAPHIC PRODUCT OF GRAPHS

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Abstract

In a graph G, a fair independent dominating set (or FID-set) is an independent dominating set S_{ID} such that all vertices in $V(G)\setminus S_{ID}$ are independently dominated by the equal number of vertices from S_{ID}; that is, every two vertices in $V(G)\setminus S_{ID}$ have the equal number of neighbours in S_{ID}. The fair independent domination number, fid(G) is the minimum number of elements in a FID-set. In this paper, we characterize the fair independent dominating sets in the join, union and lexicographic product of graphs.

KEYWORDS: Independent domination, Fair independent domination, q-fair independent domination.

1 INTRODUCTION

Let $G = (V(G), E(G))$ be a graph and $b \in V(G)$. The neighbourhood of b is the set $N_G(b)=N(b) = \{a \in V(G): ab \in E(G)\}$. If $A \subseteq V(G)$, the *open neighbourhood* of A is theset $N_G(A)=N(A) = \cup_{b\in A} N_G(b)$. The *closed neighbourhood* of A is $N_G[A] = N[A] = A \cup N(A)$.

A *dominating set* of a graph G is a set S_D of vertices of G such that all vertices in $V(G)\setminus S_D$ is adjacent to a vertex in S_D. An *independent set* of a graph G is a set S_I of vertices of G if no two of its vertices are adjacent. An *independent dominating set* of graph G is a set S_{ID} that is both dominating and independent in G. An *independent domination number* of G, represented by $i(G)$, is the set $i(G) = min\{|S_{ID}|\}$. An independent dominating set of G of size $i(G)$ is called an $i - set$.

An independent domination theory was framed by Berge[4] and Ore [5] in 1962. An independent domination number and its notation $i(G)$ were introduced by Cockayne and Hedetniemi[6,7].

Consider the graph G in which the graph is not empty. For $q \geq 1$ an integer, a *q-fair independent dominating set*, abbreviated as qFID-set, in G is an independent dominating set ID such that $|Nd(b) \cap ID| = q$ for every vertex $b \in V \backslash ID$. As we note that the set $ID = V$ is a qFID-set since it is empty every vertex in V \ ID = ∅ satisfies the required property, where $Nd(b)$ is the neighbourhood of the vertex b. The *q-fair independent domination number* of graph G, represented by fid_q(G), is the minimum cardinality of a qFID-set. A qFID-set of G of cardinality fid_q(G) is called a $fid_q(G)$-set.

A *fair independent dominating set*, abbreviated as FID-set, in G is a qFID-set for some integer $q \geq 1$. Thus, an independent dominating set ID is a FID-set in G if $ID = V$ or if $ID \neq V$ and all vertices in $V(G) \backslash ID$ are dominated by the equal number of vertices from ID; that is, $|Nd(u) \cap ID| = |Nd(v) \cap ID| > 0$ for every two vertices $u, v \in V \backslash ID$. The *fair independent domination number*, represented by *fid*(G), of a non-empty graph G is the minimum cardinality of a FID-set in G. By convention, if $G = K_n$, we define *fid*(G)= *n*. Hence if G is the non-empty graph, then *fid*(G) = $min\{fid_q(G)\}$, where the minimum is taken over all integers q where $1 \leq q \leq |V| - 1$. A FID-set of G of cardinality *fid*(G) is called a *fid*(G)-set. Every FID-set in a graph G is an independent dominating set in G. Hence, we have some observations.

2 OBSERVATIONS

Observation 2.1. *For a graph G having order n, the following conditions holds.*

(a) $i(G) \leq fid(G)$.
(b) $fid((G) \leq n$, *with equality* $\Leftrightarrow$ $G = \overline{K_n}$.

Observation 2.1(b) can be improved as follows: If G be a graph of order n, then *fid*(G)$\leq n - 2$, but not if $G = \overline{K_n}$, in which case *fid*(G) = *n*, or G contains specifically one edge, in which case *fid*(G)= $n - 1$.

Observation 2.2. *([2]) Let G be a graph of order n. Then,* $\xi_{or}(G) \geq 0$, *with equality* $\Leftrightarrow$ $G = \overline{K_n}$, *where* $\xi_{or}(G)$ *is out regular number.*

A *perfect independent dominating set* S_{PID}, abbreviated as PID-set in G, then every vertex in $V(G) \backslash S_{PID}$ is dominated by a unique vertex in S_{PID}, and so S_{PID} is a 1FID-set, implying that *fid*(G)$\leq fid_1$(G)$\leq i(G)$. Consequently, by Observation 2.1, the following observation is made.

Observation 2.3. *If a graph G has a PID-set, then id(G)=* $fid_1(G) = fid(G)$.

Observation 2.4. *If* $G \in \{P_n, K_n, C_n, \overline{K_n}, K_{m,n}\}$, *for* $m,n \geq 1$, *then* $fid(G) = i(G)$.

3 MAIN RESULTS

Following that, we construct a relationship between the fair independent domination number (*fid*) and an out-regular number (ξ_{or}) of a graph G.

Theorem 3.1. *For every graph G of order* $n \geq 2$, $fid(G) + \xi_{or}(G) = n$.

Proof. If a graph $G=\overline{K_n}$, then fid(G)= n and we know that, $\xi_{or}(G) = 0$. Thus, we can assume that $G \neq K_n$, for otherwise the required result holds. Let ID be a fid(G)$-set$. By Observation 2.1(b), fid(G)< n. Let M=V\ID. Then, M is an OR-set ([2]) in G, and so ξ_{or}(G) $\geq$ $|M|$ = $n-$ fid(G), or likewise,

$$fid(G) + \xi_{or}(G) \geq n. \qquad (1)$$

Conversely, let M be an $\xi_{or}(G)$-set. By Observation 2.2, $\xi_{or}(G) > 0$.
By definition, $\xi_{or}(G) < n$. Let ID = V\M. Then, ID is a FID-set, and so
$fid(G) \leq |ID| = n - \xi_{or}(G)$, or, likewise

$$fid(G) + \xi_{or}(G) \leq n. \qquad (2)$$

From equation (1) and (2), we get

$$fid(G)+\xi_{or}(G) = n.$$

□

Remark 1: *Let G be a connected graph of order n* $\geq$ 2 *and any +ve integer q*, $1 \leq i(G) \leq fid(G) \leq q fid(G)$.

Theorem 3.2. *For any connected graph G of order n. Then fid(G)*= 1 $\Leftrightarrow$ $i(G) = 1$.

Proof. If $G = K_1$, then i(G) = fid(G) = 1. Let $|V(G)| \geq 2$. Suppose fid(G)= 1. By Remark 1, $i(G) = 1$.

Conversely, if $i(G) = 1$. Let ID = $\{i\}$ be an independent dominating set of G. Then, for all $j \in V(G)\backslash ID$, $Nd(j) \cap$ ID = $\{i\}$. Thus, for all $k, j \in V(G)\backslash ID$ with $k \neq j$, we have $|Nd(k) \cap ID| = 1 = |Nd(j) \cap ID|$. This shows that ID is a 1FID-set of G. By Remark 1, $1 \leq fid$(G) $\leq 1fid$(G) = 1 and therefore, fid(G) = 1. □

Corollary 3.2.1. *Let G be* W_n, $K_{1,n-1}$ *or* K_n. *Then fid(G)* = 1.

Proof. The proof of this corollary is immediately follows from Theorem 3.2. □

Theorem 3.3. *For any connected graph G of order* $n \geq 2$. *Then it holds following conditions.*
(i) If the graph $\overline{G}$ *is connected, then fid(G)= fid (* $\overline{G}$ *).*
(ii) If the graph $\overline{G}$ *has* $c \geq 2$ *components, then fid(G)* $\leq n/c \leq n/2$.

Proof. (i) Assume that the graph $\overline{G}$ is connected. Let ID be a fid(G)$-set$, then every vertex $v \in V \backslash ID$ is adjacent to precisely q vertices in ID for some integer q, $1 \leq q \leq |ID|$. If $q = |ID|$, then in $\overline{G}$ there are no edges between ID and $V \backslash ID$, this contradicting our assumption that $\overline{G}$ is connected. Therefore, $q < |ID|$. But then in $\overline{G}$ every vertex in $V \backslash ID$ is adjacent to precisely $|ID| - q > 0$ vertices in ID, and so ID is a FID-set in G. Hence,

$$fid(\overline{G}) \leq |ID| = fid(G). \qquad (3)$$

Now reversing the roles of G and $\overline{G}$, we have that

$$fid(G) \leq fid(\overline{G}). \tag{4}$$

From equation (3) and (4), we get

$$fid(G) = fid(\overline{G}).$$

(ii) Suppose that $\overline{G}$ is disconnected and has 'c' components. Clearly, there exists smallest component in $\overline{G}$ has cardinality at most n/c. Let S be the smallest component in $\overline{G}$ and let $ID = V(S)$. Then in G every vertex in $V \setminus ID$ is adjacent to all vertices in ID, and so ID is a FID-set in G. Thus,

$$fid(G) \leq |ID| \leq n/c \leq n/2.$$

Hence the proof. □

4 FAIR INDEPENDENT DOMINATION IN THE JOIN OF GRAPHS

Definition:([2]) Let A and B be sets which are not necessarily disjoint. The *disjoint union* of A and B, represented by $A \uplus B$, is the set obtained by taking the union of A and B treating each element in A as distinct from each element in B. The *join* of two graphs A and B is the graph $A + B$ with vertex-set $V(A + B) = V(A) \uplus V(B)$ and edge-set

$$E(A + B) = E(A) \uplus E(B) \cup \{xy : x \in V(A), y \in V(B)\}.$$

Theorem 4.1. *Let A and B be connected graphs. Then $J \subseteq V(A+B)$ is an FID-set of $A+B$ $\Leftrightarrow$ one of the following statements holds:*

(i) $J \subseteq V(A)$ and J is a $|J|$FID-set of A.

(ii) $J \subseteq V(B)$ and J is a $|J|$FID-set of B.

(iii) $J = V(A) \cup J_B$, where J_B is a pFID-set of B for some +ve integer p.

(iv) $J = J_A \cup V(B)$, where J_A is a qFID-set of B for some +ve integer q.

(v) $J = J_A \cup J_B$, where J_A is a qFID-set of A and J_B is a pFID-set of B for some +ve integers q and p such that $q + |J_B| = p + |J_A|$.

Proof. Assume that J is a FID-set of $A + B$. Let $J_A = V(A) \cap J$ and $J_B = V(B) \cap J$. Then $J = J_A \cup J_B$. Consider the following cases:

Case 1. $J_B = \emptyset$ or $J_A = \emptyset$

Assume that $J_B = \emptyset$. Then $J = J_A \subseteq V(A)$. Let $a \in V(B)$. Then $|N_{A+B}(i) \cap J| = |J|$. Thus J is a $|J|$FID-set of $A + B$. Since $J \in V(A)$, J is a $|J|$FID-set of G. Likewise, J is a $|J|$FID-set of B if $J_A = \emptyset$.

Case 2. $J_B \neq \emptyset$ and $J_A \neq \emptyset$

Assume that $J_A = V(A)$. If $J_B \neq V(B)$, then J_B is a pFID-set for any +ve integer p. So, suppose $J_B \neq V(B)$ and let $b \in V(B) \backslash J_B$. Then $N_{A+B}(b) \cap J = V(A) \cup (N_B(b) \cap J_B)$. Since J is a FID-set, it follows that $|N_B(b) \cap J_B| = p$ for some integer p for each $b \in V(B) \backslash J_B$. This implies that $J = V(A) \cup J_B$ and J_B is a pFID-set of B for some +ve integer p. Likewise, (iv) holds if $J_B = V(B)$.

Next, suppose that $J_A \neq V(A)$ and $J_B \neq V(B)$. Assume that further that J_A is not a FID-set. Then there exist $x, y \in V(A)\backslash J_A$ such that

$$|N_A(x) \cap J_A| = |N_A(y) \cap J_A|.$$

Hence,

$$|N_{A+B}(x) \cap J| = |N_A(x) \cap J_A| + |J_B| = |N_A(y) \cap J_A| + |J_B| = |N_{A+B}(y) \cap J|,$$

this contradicts to our assumption that J is a FID-set. Thus, J_A is a FID-set of A. Likewise, S_B is a FID-set of B. Let q and p be +ve integers such that J_A is a qFID-set and J_B is a pFID-set. Let $x \in V(A)\backslash J_A$ and $a \in V(B)\backslash J_B$. Since J is a FID-set of $A + B$, it follows that

$$|N_A(x) \cap J_A| + |J_B| = |N_{A+B}(x) \cap J| = |N_{A+B}(a) \cap J| = |N_B(a) \cap J_B| + |J_A|.$$

Thus, $q + |J_B| = p + |J_A|$, showing that (v) holds.

For the converse, assume that the statement (v) holds. Let $u, v \in V(A + B)\backslash J$. If $u, v \in V(A)$ or $u, v \in V(B)$, then $|N_{A+B}(u) \cap J| = q + |J_B| = |N_{A+B}(v) \cap J|$ or $|N_{A+B}(u) \cap J| = p + |J_A| = |N_{A+B}(v) \cap J|$. So suppose $u \in V(A)$ and $v \in V(B)$. Then by assumption,

$$|N_{A+B}(u) \cap J| = q + |J_B| = p + |J_A| = |N_{A+B}(v) \cap J|.$$

Therefore, J is a FID-set of $A + B$. Clearly, J is a FID-set of $A + B$ if (i), (ii), (iii) or (iv) holds. □

Theorem 4.2. *Let A and B be connected graphs. Then fid(A+B)*= 1 $\Leftrightarrow$ $i(A) = 1$ *or* $i(B) = 1$.

Proof. Assume that $i(A) = 1$, say ID= $\{x\}$ is an independent dominating set in A. By Theorem 3.2, $fid(A) = 1$. Hence, ID is a FID-set in A. Furthermore, $|N_{A+B}(x) \cap ID| = |ID|$ for all $x \in V(A + B)\backslash$ ID. By Theorem 4.1, ID is a FID-set of $A + B$. It follows that fid(A+B) $\leq |ID| = 1$. By Remark 1, fid(A+B)= 1. Likewise, fid(A+B)= 1 if $i(A) = 1$.

Assume that fid(A+B)= 1. Then $i(A + B) = 1$ by Theorem 3.2.
It follows that $i(A) = 1$ or $i(B) = 1$. □

Corollary 4.2.1. *Let A be a connected graph and B be* W_n, F_n, $K_{1,n-1}$,*or* K_n. *Then fid(A+B)*= 1.

5 FAIR INDEPENDENT DOMINATION IN LEXICOGRAPHIC PRODUCT OF GRAPHS

Definition:([2])The *lexicographic product* of two graphs X and Y is the graph X[Y] with vertex-set $V(X[Y]) = V(X) \times V(Y)$ and edge-set $E(X[Y])$ satisfying the following conditions: $(x, u)(y, v) \in E(X[Y]) \Leftrightarrow$ either $xy \in E(X)$ or $x = y$ and $uv \in E(Y)$.

Theorem 5.1. *Let X and Y be connected graphs. Then fid(X[Y])*= 1 $\Leftrightarrow$ $i(X) = i(Y) = 1$.

Proof. The result is clearly holds if either X or Y is the trivial graph. So we assume that X and Y are non-trivial. Assume that *fid*(X[Y])= 1. Then $i(X[Y]) = 1$ by Theorem 3.2. Let $L = \{(x, a)\}$ be a $i-set$ of $X[Y]$. Let $y \in V(X)\setminus\{x\}$. Then $(x, a)(y, a) \in E(X[Y])$. This implies that $xy \in E(X)$. Thus, $\{x\}$ is an independent dominating set of X. Likewise, $\{a\}$ is an independent dominating set of Y. Thus, $i(X) = i(Y) = 1$.

For the converse, suppose $i(X) = i(Y) = 1$. Let $\{x\}$ be an $i-set$ of X and let $\{a\}$ be an $i-set$ of Y for some $x \in V(X)$, $a \in V(Y)$. Let $L = \{(x, a)\} \subseteq V(X[Y])$ and let $(y, b) \in V(X[Y])\setminus L$.If $y = x$, then $b \neq a$ and $(x, a)(y, b) \in E(X[Y])$ since $ab \in E(Y)$. Assume that $y \neq x$. Then $xy \in E(X)$. Hence, $(x, a)(y, b) \in E(X[Y])$. Thus, L is an $i - set$ of $X[Y]$ and $i(X[Y]) = 1$. By Theorem 3.2, *fid*(X[Y])= 1. □

6 FAIR INDEPENDENT DOMINATION IN THE UNION OF GRAPHS

Definition: The *union* of two graphs A and B is the graph $A \cup B$ with vertex-set $V(A \cup B) = V(A) \cup V(B)$ and edge-set $E(A \cup B) = E(A) \cup E(B)$.

Theorem 6.1. *Let A and B are two connected graphs. Then fid(A ∪ B)= 1* $\Leftrightarrow$ $i(A) = 1$ *and* $i(B) = 1$.

Proof. Assume that $i(A) = 1$, say ID= $\{x\}$ is an independent dominating set in A. By Theorem 3.2, *fid*(A)= 1. Hence, ID is an FID-set in A. Furthermore, $|N_{A\cup B}(x) \cap ID| = |ID|$ for all $x \in V(A \cup B)\setminus$ ID. Then, ID is a FID-set of $A \cup B$. It follows that *fid*(A∪B) $\leq |ID| = 1$. By Remark 1, *fid*(A ∪ B)= 1. Similarly, *fid*(A ∪ B)= 1 if $i(A) = 1$ and $i(B) = 1$.

Assume that *fid*(A ∪ B)=1. Then $i(A \cup B) = 1$ by Theorem 3.2. It follows that $i(A) = 1$ and $i(B) = 1$. □

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