ESTABLISHMENT OF 4-TUPLES CONCERNING INTEGERS WITH ELITE FEATURE

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Abstract

In this manuscript, a remarkable 4-tuples $(\alpha, \beta, \gamma, \delta)$ with elements are non-zero integers such that the arithmetic mean of any three elements among the four elements listed in this 4-tuples yields a square number is appraised by applying several techniques in Mathematics.

Keywords: Diophantine quadruples, Ternary quadratic, Diophantine equation

1. Introduction

"A set of m positive integers $\{a_1, a_2 \ldots . a_m\}$ is called a Diophantine $m-$tuple with the property $D(n)$, $n-\{0\} \in Z$ if $a_i . a_j + n$ is a perfect square for all $1 \leq i < j \leq m$". In $[3,4]$, the authors initiated how to find Diophantine triples with suitable properties. In $[5,6]$, the authors concentrated on evaluating Diophantine quadruples with an elegant property. For further review of quadruples one can refer $[1, 2,7 - 9]$. In this paper, an integer quadruples $(\alpha, \beta, \gamma, \delta)$where α, β, γ and δ are non-zero integers where the average of any three elements amid the four numbers deliver a square number is explained by following different methods.

2. Procedures of Examinations.

Let α, β, γ ,δ be distinct non-zero integers such that the average of any three among these four elements stay a perfect square. Originate with the exact postulation as given below:

$$\alpha + \beta + \gamma = 3p^2 \tag{1}$$

$$\alpha + \beta + \delta = 3q^2 \tag{2}$$

$$\alpha + \gamma + \delta = 3r^2 \tag{3}$$

$$\beta + \gamma + \delta = 3s^2 \tag{4}$$

concurrently with the supplementary proposition that

$$\alpha + \beta + \gamma + \delta = z^2 \tag{5}$$

By making simple arithmetical calculations in the simultaneous equations $(1), (2), (3)$ and (4), the values of α, β, γ and $\delta$are declared as follows

$$\alpha = p^2 + q^2 + r^2 - 2s^2 \tag{6}$$

$$\beta = p^2 + q^2 + s^2 - 2r^2 \tag{7}$$

$$\gamma = p^2 + r^2 + s^2 - 2q^2 \tag{8}$$

$$\delta = q^2 + r^2 + s^2 - 2p^2 \tag{9}$$

Addition of all the above four equations from (6) to (9) offers the successive equation

$$\alpha + \beta + \gamma + \delta = p^2 + q^2 + r^2 + s^2 \tag{10}$$

Comparing equation (5) and (10), it is noted that

$$z^2 = p^2 + q^2 + r^2 + s^2 \tag{11}$$

Procedure 1.

Undertaking the new linear modifications $p = m + n\,,\ q = m + 7n\,,\ r = m - n\,, s = m - 7n$

where $m \neq n \neq 0$ in $(6), (7), (8)$ and (9) , the corresponding values of $\alpha,\ \beta,\ \gamma$ and δ in two variables are offered subsequently by

$$\alpha = m^2 - 47\,n^2 + 42\,mn \tag{12}$$

$$\beta = m^2 + 97\,n^2 + 6\,mn \tag{13}$$

$$\gamma = m^2 - 47\,n^2 - 42\,mn \tag{14}$$

$$\delta = m^2 + 97\,n^2 - 6\,mn \tag{15}$$

Relieving the same conversions in (11) ensurethe following famous Pythagoreans equation

$$z^2 = (2m)^2 + (10n)^2 \tag{16}$$

The process of executing the quadruples$(\alpha, \beta, \gamma, \delta)$wherethe average of three among four elements stand for a square number by solving (16) in two different proposals are enlightened below.

Proposal 1.1.

The Pythagorean equation (16) is fulfilled by

$$m = \frac{(r^2 - s^2)}{2} \tag{17}$$

$$n = \frac{rs}{10} \tag{18}$$

Meanwhile our concern is to treasure the parameters in integers, it is experiential that m and n are integers for the ensuing options of r and s

$$r = 2R \text{ and } s = 10S$$

These selections of r and s reduce (17) and (18)as

$$m = 2R^2 - 50S^2$$

$$n = 4RS$$

Replacement of the above values of m and n in (12), (13), (14) and (15) convert the preferable values of α , β, $\gamma$and δ as follows

$$\alpha = 4R^4 + 336R^3S - 952R^2S^2 - 8400RS^3 + 2500S^4$$

$$\beta = 4R^4 + 48R^3S + 1352R^2S^2 - 1200RS^3 + 2500S^4$$

$$\gamma = 4R^4 - 336R^3S - 952R^2S^2 + 8400RS^3 + 2500S^4$$

$$\delta = 4R^4 - 48R^3S + 1352R^2S^2 + 1200RS^3 + 2500S^4$$

The table below contains numerical values for a few parameters that support the proposal.

R	S	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	-277244	187588	613828	314884	$(418)^2$	$(274)^2$	$(466)^2$	$(610)^2$
3	2	-177404	62788	189508	115204	$(158)^2$	$(14)^2$	$(206)^2$	$(350)^2$
4	1	-23804	23428	388	26884	$(2)^2$	$(94)^2$	$(34)^2$	$(130)^2$
6	2	-349952	203008	166144	276736	$(80)^2$	$(208)^2$	$(176)^2$	$(464)^2$

Proposal 1. 2.

Let us designate an additional solution of (16) be

$$m = rs \tag{19}$$

$$n = \frac{(r^2 - s^2)}{10} \tag{20}$$

For receiving an integer solution of equation (16), let us prefer the fortuitous of r and s as given below.

$r = 10R$ and

$s = 10S$

Hence,(19) and (20) are developed into

$m = 100RS$

$n = 10R^2 - 10\,S^2$

Substituting the above-mentioned values of m and n in (12), (13), (14) and (15)), the

comparable values of α , β, γ and δ are turned into

$\alpha = -4700R^4 + 42000R^3S + 19400R^2S^2 - 42000RS^3 - 4700S^4$

$\beta = 9700R^4 + 6000R^3S - 9400R^2S^2 - 6000RS^3 + 9700S^4$

$\gamma = -4700R^4 - 42000R^3S + 19400R^2S^2 + 42000RS^3 - 4700S^4$

$\delta = 9700R^4 - 6000R^3S - 9400R^2S^2 + 6000RS^3 + 9700S^4$

Algebraic Computations of an essential triples are enumerated below for few chances of the newly introduced parameters R and S.

R	S	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	-1017500	422500	1502500	782500	$(550)^2$	$(250)^2$	$(650)^2$	$(950)^2$
3	2	1502500	782500	-1017500	422500	$(650)^2$	$(950)^2$	$(550)^2$	$(250)^2$
4	1	1622500	2702500	-3417500	1982500	$(550)^2$	$(1450)^2$	$(250)^2$	$(650)^2$
6	2	12755200	13676800	-19500800	9068800	$(1520)^2$	$(3440)^2$	$(880)^2$	$(1040)^2$

Procedure 2.

A tactic of an alternative translations $p = m + n$, $q = m - n$, $r = m + 3n$ and $s = m - 3n$ where $m \neq n \neq 0$ in (6), (7), (8) and (9)deliver the assessment of α, β , γ and δas follows

$$\alpha = m^2 - 7\,n^2 + 18\,mn \tag{21}$$

$$\beta = m^2 - 7n^2 - 18mn \quad (22)$$

$$\gamma = m^2 + 17n^2 + 6mn \quad (23)$$

$$\delta = m^2 + 17n^2 - 6mn \quad (24)$$

Reservation of the same alterations in (11)leads to an equation of the form as below

$$z^2 = (2m)^2 + 20n^2 \quad (25)$$

Applying few methods of solving (25), an estimation of a gorgeous integer quadruple fulfilling the condition that the average of three quantities stays a square number is established as follows.

Proposal 2.1.

By considering the general solutions to the equation of the form $z^2 = Dx^2 + y^2$ where D is a square free integer, the choice of m, n and z attained from (25) are as given below

$$\left.\begin{aligned} m &= \frac{1}{2}[a^2 - 20b^2] \\ n &= 2ab \\ z &= a^2 + 20b^2 \end{aligned}\right\} \quad (26)$$

In order to find out an integer solution, striking the replacement as $a = 2A$ and b is an arbitrary value in (26) , the equivalent values m, n and z are revealed by

$$m = 2A^2 - 10b^2$$

$$n = 4Ab$$

$$z = 4A^2 + 20b^2$$

Exchanging the values of m and n in $(21), (22), (23)$ and (24), the exact chances of the non-zero parameters α, β, γ and δ are turned out to be

$$\alpha = 4A^4 + 144A^3b - 152A^2b^2 - 720Ab^3 + 100b^4$$

$$\beta = 4A^4 - 144A^3b - 152A^2b^2 + 720Ab^3 + 100b^4$$

$$\gamma = 4A^4 + 48A^3b + 232A^2b^2 - 240Ab^3 + 100b^4$$

$$\delta = 4A^4 - 48A^3b + 232A^2b^2 + 240Ab^3 + 100b^4$$

Numerical illustrations for some positive values of A and b filling the supposition are presented in the below table.

A	b	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	-32732	38116	4708	28324	$(58)^2$	$(106)^2$	$(10)^2$	$(154)^2$
3	2	-13052	5956	7108	13444	$(2)^2$	$(46)^2$	$(50)^2$	$(94)^2$
4	1	5028	-7644	6948	2724	$(38)^2$	$(6)^2$	$(70)^2$	$(26)^2$
6	2	12544	-42752	49408	30976	$(80)^2$	$(16)^2$	$(176)^2$	$(112)^2$

Proposal 2.2.

Subcase 2.2.1.

Factorization of (25) provides the following equation

$$(z+2m)(z-2m)=20n^2 \tag{27}$$

The overhead equation can be inscribed in the fraction form as

$$\frac{(z-2m)}{(20n)}=\frac{(n)}{(z+2m)}=\frac{a}{b}, b\neq 0 \tag{28}$$

Modify (28) into the subsequent form of double equations

$$zb-2bm-20an=0 \tag{29}$$

$$az+2am-bn=0 \tag{30}$$

Resolving (29) and (30) by cross multiplication rule, the values of $m, n$and z needed for accomplishing an integer solution of (25) are furnished as

$$m=-20a^2+b^2$$

$$n=4ab$$

$$z=40a^2+2b^2$$

Transmitting the values of m and n in $(21), (22), (23)$ and (24), the precise options of α, β, γ and $\delta$are converted into

$$\alpha=400a^4-144a^3b-152a^2b^2+72ab^3+b^4$$

$$\beta=400a^4+144a^3b-152a^2b^2-72ab^3+b^4$$

$$\gamma=400a^4-480a^3b+232a^2b^2+24ab^3+b^4$$

$$\delta = 400a^4 + 480a^3b + 232a^2b^2 - 24ab^3 + b^4$$

Examples for few positive values of a and b gratifying the proposition are given in the succeeding table.

a	b	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	−29663	31681	4609	25057	$(47)^2$	$(95)^2$	$(1)^2$	$(143)^2$
3	5	−168575	166225	29425	141025	$(95)^2$	$(215)^2$	$(25)^2$	$(335)^2$
4	1	8097	191841	75489	136737	$(303)^2$	$(335)^2$	$(271)^2$	$(367)^2$
6	2	−122096	1115152	3456616	758032	$(668)^2$	$(764)^2$	$(572)^2$	$(860)^2$

Subcase 2.2.2.

Rewrite (27) in the ratio as

$$\frac{(z-2m)}{n} = \frac{20n}{(z+2m)} = \frac{a}{b}, b \neq 0$$

Applying the same technique as explained in subcase (i), the particular selections of the elements in the quadruple are expressed by

$$\alpha = a^4 - 72a^3b - 152a^2b^2 + 1440ab^3 + 400b^4$$

$$\beta = a^4 + 72a^3b - 152a^2b^2 - 1440ab^3 + 400b^4$$

$$\gamma = a^4 - 24a^3b + 232a^2b^2 + 480ab^3 + 400b^4$$

$$\delta = a^4 + 24a^3b + 232a^2b^2 - 480ab^3 + 400b^4$$

Some models are acknowledged in the subsequent table.

a	b	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	102976	−49088	66112	15424	$(200)^2$	$(152)^2$	$(240)^2$	$(104)^2$
3	5	746161	−314399	479041	125521	$(551)^2$	$(431)^2$	$(671)^2$	$(311)^2$
4	1	−624	−2928	4752	3984	$(20)^2$	$(12)^2$	$(52)^2$	$(44)^2$
6	2	23824	−52208	53776	28432	$(92)^2$	$(4)^2$	$(188)^2$	$(100)^2$

Subcase 2.2.3.

Redraft (27) as

$$\frac{(z+2m)}{20n} = \frac{n}{(z-2m)} = \frac{a}{b}, b \neq 0$$

By manipulating the identical procedure as clarified in subcase (i), the specific collections of the elements in an essential quadruples are articulated by

$$\alpha = 400a^4 + 1440a^3b - 152a^2b^2 - 72ab^3 + b^4$$

$$\beta = 400a^4 - 1440a^3b - 152a^2b^2 + 72ab^3 + b^4$$

$$\gamma = 400a^4 + 480a^3b + 232a^2b^2 - 24ab^3 + b^4$$

$$\delta = 400a^4 - 480a^3b + 232a^2b^2 + 24ab^3 + b^4$$

The following table offered numerical values for limited choice of the parameters rewarding the hypothesis.

a	b	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	17169	−5583	11121	3537	$(87)^2$	$(71)^2$	$(103)^2$	$(55)^2$
3	2	102976	−49088	66112	15424	$(200)^2$	$(152)^2$	$(248)^2$	$(104)^2$
3	4	152464	−130928	113296	18832	$(212)^2$	$(116)^2$	$(308)^2$	$(20)^2$
6	3	1390689	−452223	900801	286497	$(783)^2$	$(639)^2$	$(927)^2$	$(495)^2$

Subcase 2.2.4.

Alternate (27) as

$$\frac{(z+2m)}{n} = \frac{20n}{(z-2m)} = \frac{a}{b}, b \neq 0$$

As in subcase (i), the gathering of the elements in the crucial quadruple are expressed by

$$\alpha = a^4 + 72a^3b - 152a^2b^2 - 1440ab^3 + 400b^4$$

$$\beta = a^4 - 72a^3b - 152a^2b^2 + 1440ab^3 + 400b^4$$

$$\gamma = a^4 + 24a^3b + 232a^2b^2 - 480ab^3 + 400b^4$$

$$\delta = a^4 - 24a^3b + 232a^2b^2 + 480ab^3 + 400b^4$$

Numerical characters for selected choices of a and b gratifying the proposal are presented in the below table.

a	b	α	β	γ	δ	$\frac{\alpha+\beta+\gamma}{3}$	$\frac{\alpha+\beta+\delta}{3}$	$\frac{\alpha+\gamma+\delta}{3}$	$\frac{\beta+\gamma+\delta}{3}$
2	3	-2496	2112	576	2112	$(8)^2$	$(24)^2$	$(8)^2$	$(40)^2$
3	2	-29663	31681	4609	25057	$(47)^2$	$(95)^2$	$(1)^2$	$(143)^2$
3	4	-188111	349297	46321	225457	$(263)^2$	$(359)^2$	$(167)^2$	$(455)^2$
6	3	-202176	171072	46656	171072	$(72)^2$	$(216)^2$	$(72)^2$	$(360)^2$

The Python Program for the authentication of the needed quadruples fulfilling our statement is epitomized below.

```
import mat□
w□ile True:
    T = input("Enter your c□oice part A of B  : ")
    if T in ('A'):
        c = input("Enter c□oice(1/2): ")
        if c in ('1', '2'):
            r = int(input('Enter a Number r : '))
            s = int(input('Enter t□e second number s : '))

        if (c == '1'):
            m = 2 * r ** 2 - 50 * s ** 2
            n = 4 * r * s

        elif(c == '2'):

            m = 100 * r * s
            n = 10 * r ** 2 - 10 * s ** 2
        else:
            print('Invalid Input')
            break

        p = m ** 2 - 47 * n ** 2 + 42 * m * n
        q = m ** 2 + 97 * n ** 2 + 6 * m * n
        r = m ** 2 - 47 * n ** 2 - 42 * m * n
        s = m ** 2 + 97 * n ** 2 - 6 * m * n
        a1 = (p + q + r)/3
        a2 = (p + q + s)/3
```

```
a3 = (p + r + s)/3
a4 = (q + r + s)/3
if (a1 < 0):
    a11 = −1 ∗ a1
    x = pow(a11,1/2)
else:
    x = pow(a1,1/2)

if (a1 < 0):
    x = −x

if (a2 < 0):
    a21 = −1 ∗ a2
    y = pow(a21,1/2)
else:
    y = pow(a2,1/2)

if (a2 < 0):
    y = −y

if (a3 < 0):
    a31 = −1 ∗ a3
    z = pow(a31,1/2)
else:
    z = pow(a3,1/2)
if (a3 < 0):
    z = −z


if (a4 < 0):
    a41 = −1 ∗ a4
    xx = pow(a41,1/2)
else:
    xx = pow(a4,1/2)

if (a4 < 0):
    xx = −xx

print('p : ',p)
print('q : ',q)
print('r : ',r)
print('s : ',s)
print('(p + q + r)/3 :  ',int(x),'^2')
print('(p + q + s)/3 :  ',int(y),'^2')
print('(p + r + s)/3 :  ',int(z),'^2')
print('(q + r + s)/3 :  ',int(xx),'^2')
```

```
elif T in ('B'):
    c = input('Enter your c□oice 3/4/5/6/7  : ')

    if c in ('3','4','5','6','7'):
        a = int(input('Enter a Number a : '))
        b = int(input('Enter t□e second number b : '))

    if(c == '3'):

        m = -10 * b ** 2 + 2 * a ** 2
        n = 4 * a * b

    elif(c == '4'):

        m = b ** 2 - 20 * a ** 2
        n = 4 * a * b

    elif(c == '5'):

        m = -a ** 2 + 20 * b ** 2
        n = 4 * a * b
    elif(c == '6'):

        m = -20 * a ** 2 + b ** 2
        n = -4 * a * b

    elif(c == '7'):

        m = -a ** 2 + 20 * b ** 2
        n = -4 * a * b
    else:
        print('Invalid Input')
        break

    p = m ** 2 - 7 * n ** 2 + 18 * m * n
    q = m ** 2 - 7 * n ** 2 - 18 * m * n
    r = m ** 2 + 17 * n ** 2 + 6 * m * n
    s = m ** 2 + 17 * n ** 2 - 6 * m * n
    a1 = (p + q + r)/3
    a2 = (p + q + s)/3
    a3 = (p + r + s)/3
    a4 = (q + r + s)/3
    if (a1 < 0):
        a11 = -1 * a1
        x = pow(a11,1/2)
```

```
else:
    x = pow(a1,1/2)

if (a1 < 0):
    x = -x

if (a2 < 0):
    a21 = -1 * a2
    y = pow(a21,1/2)
else:
    y = pow(a2,1/2)
if (a2 < 0):
    y = -y

if (a3 < 0):
    a31 = -1 * a3
    z = pow(a31,1/2)
else:
    z = pow(a3,1/2)
if (a3 < 0):
    z = -z

if (a4 < 0):
    a41 = -1 * a4
    xx = pow(a41,1/2)
else:
    xx = pow(a4,1/2)
if (a4 < 0):
    xx = -xx

print('p : ',p)
print('q : ',q)
print('r : ',r)
print('s : ',s)
print('(p + q + r)/3 :  ',int(x),'^2')
print('(p + q + s)/3 :  ',int(y),'^2')
print('(p + r + s)/3 :  ',int(z),'^2')
print('(q + r + s)/3 :  ',int(xx),'^2')
```

Conclusion.

In this communication, the quadruple $(\alpha,\ \beta,\ \gamma,\ \delta)$in which the average of any three numbers remains a square of an integer is examined. To conclude, one can explore stimulating quadruples and quintuples such that geometric mean of two or three numbers is a square of an integer.

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