On F-Index of Fuzzy Transformation Graphs

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Abstract
The F-index is sum of cubes of degrees of every vertex of G i.e. $F(G) = \sum_{i=1}^{n} d_G(v)^3$
In this paper, we have studied the F-index of generalized fuzzy transformation graphs and obtained some upper bounds for $F(G)$ in terms of elements of a graph G

Keywords: First Zagreb index; Second Zagreb index; F-index; .

Subject Classification: 05C90; 05C35; 05C12.

1 Introduction

Nowadays, due to various applications of fuzzy graph theory (FGT), a huge number of researchers are working on topological indices (TIs). In the field of molecular chemistry, TIs are molecular descriptors which are calculated on the molecular graph (MG) of a chemical compound. These TIs are numerical quantities of a graph which describes its topology. Zagreb index $M_1(G)$ is one such TIs which is degree-based TI and introduced by Gutman and Trinajstic in 1972 [5] and this TI is used to calculate π-electron energy of a conjugate system. One of the most useful topological indices are the Zagreb indices which are defined as:

$$M_1(G) = \sum_{i=1}^{n} d_G(v_i)^2 \quad (1)$$

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v) \quad (2)$$

where M_1 and M_2 are the first and second Zagreb indices respectively.

These indices are extensively studied in chemical and mathematical literature. Interested readers are referred to [4,6,7,8,13] for some recent results on the topic. In the same paper, where Zagreb indices were introduced, another topological index, was also shown to influence total π-electron energy. This index was defined as sum of cubes of degrees of the vertices of the graph. However, this index never got attention except recently Furtula et al. [2] studied this index and establish some basic properties of this index and showed that the predictive ability of this index is almost similar to that of first Zagreb index and for the entropy and acetic factor, both of them yield correlation coefficients greater than 0.95. They named this index as "forgotten topological index" or "F-index". Throughout this paper, we call this index as F-index and denote it by F(G). So,

$$F(G) = \sum_{i=1}^{n} d_G(v_i)^3 \tag{3}$$

But those indices are defined in a crisp graph. As fuzzy graphs is the generalization of crisp graphs and many real life problems cannot be handled by crisp graphs, those indices in fuzzy graphs have more applications.

In the modern day, fuzzy graph (FG) theory is one of the most applicable to regular life. So there are many researchers who are implementing FG theory, especially topological indices of fuzzy graphs. Rosenfeld [15], inspired by Zadeh's [19] classical set (fuzzy set) in 1975, introduced the fuzziness for a graph, then, it is called a fuzzy graph. Additionally, this time he introduced several connective parameters of an FG and some applications of these parameters by Yeh et al. [18]. In [16,17], Sunitha et al. studied fuzzy block, fuzzy bridge, FSG, CFG, PFSG, fuzzy tree, fuzzy forest, fuzzy cut vertex, etc. The degree of a vertex d(v)/deg(v) and degree of an edge (dG(e)) is also discussed in [11]. Further information on the FG hypothesis is provided in [1,3,12,]. The first Zagreb index is discussed in [14].

In this article, the $F-$index for fuzzy graphs is introduced which is a more generalization of the first Zagreb index for crisp graphs and it is extensively used in spectral fuzzy graph theory, fuzzy network theory, fuzzy graph theory and several fields of fuzzy mathematics and chemistry.

2 Preliminaries

Here, we provide some fundamental definitions and theorems which are crucial to developing the later sections. A fuzzy graph $\mathcal{G} = (\mathcal{V}, \sigma, \mu)$ corresponding to the crisp graph G^* is a non-empty set $\mathcal{V}$ together with a pair of functions $\sigma: \mathcal{V} \to [0,1]$ and $\mu: \mathcal{V} \times \mathcal{V} \to [0,1]$ such that for all $a, b \in \mathcal{V}$,

$$\mu(a,b) \leq min\{\sigma(a), \sigma(b)\}$$

We assume that S is finite and nonempty, μ is reflexive and symmetric. In all the examples σ is chosen suitably. Also, we denote the underlying graph by $G^*: (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in S: \sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in S \times S: \mu(u,v) > 0\}$. A fuzzy graph $H: (\tau, \upsilon)$ is called a partial fuzzy subgraph of $G: (\sigma, \mu)$ if $\tau(u) \leq \sigma(u)$ for every u and $\upsilon(u,v) \leq \mu(u,v)$ for every u, v. In particular we call a partial fuzzy subgraph $H: (\tau, \upsilon)$ a fuzzy subgraph of $G: (\sigma, \mu)$ if $\tau(u) = \sigma(u)$ for every $u \in \tau^*$ and $\upsilon(u,v) = \mu(u,v)$ for every arc $u, v \in \upsilon^*$. Now a fuzzy subgraph $H: (\tau, \upsilon)$ spans the fuzzy graph $G: (\sigma, \mu)$ if $\tau(u) = \sigma(u)$. A connected f-graph $G: (\sigma, \mu)$ is a fuzzy tree(f-tree) if it has a fuzzy spanning subgraph $F: (\tau, \upsilon)$ which is a tree, where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$) is a weakest arc of G. Here $P_1: x, w, v$ is an $\alpha-$ strongest $u - v$ path whereas $P_2: x, u, v$ is a β- strong $x - v$ path.

3 Fuzzy Transformation Graphs

Let $G = (V, \sigma, \mu)$ be a fuzzy graph with underlying crisp graph $G = (V, \sigma^*, \mu^*)$. Then fuzzy transformation graph $G^{ab} = (X, \sigma^{ab}, \mu^{ab})$, where $X = V \cup E$. Let α and β be any two elements of G^{ab} then we say that the associativity of $\alpha$and β is + if they are adjacent or incident in G otherwise. Let a and b be two permutation of the set $\{+.-\}$. For more details on transformation graphs refer [1,46,10,12,15,17]. Then we define the four kinds of fuzzy transformation graphs based on the permutations of a and b that is . These fuzzy transformation graphs are defined as follows:

Definition 1 *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph with underlying crisp graph* $G = (V, \sigma^*, \mu^*)$ *. Then the vertex set of fuzzy transformation graph* $V(G^{++}) = V \cup E$*. The adjacency and incidence relations and their values are defined as follows:*

- $\sigma^{++}(u)$ = $\{\sigma(u)$u $\mu(u)$ifu 0otherwise
- $\mu^{++}(v_i, v_j)$ = $\{\sigma(v_i) \wedge \sigma(v_j)$v_i,v_j $andareadjacentin$G0otherwise
- $\mu^{++}(e_i, e_j)$ = {0e_i,e_j
- $\mu^{++}(v_i, e_j)$ = $\{\mu(e_j)$v_$iliesontheedge$e_j 0otherwise

Definition 2 *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph with underlying crisp graph* $G = (V, \sigma^*, \mu^*)$ *. Then the vertex set of fuzzy transformation graph* $V(G^{+-}) = V \cup E$*. The adjacency and incidence relations and their values are defined as follows:*

- $\sigma^{+-}(u)$ = $\{\sigma(u)$u $\mu(u)$ifu 0otherwise
- $\mu^{+-}(v_i, v_j)$ = $\{\sigma(v_i) \wedge \sigma(v_j)$v_i,v_j $andareadjacentin$G0otherwise

- $\mu^{+-}(e_i, e_j)$ = {0e_i,e_j
- $\mu^{+-}(v_i, e_j)$ = $\begin{cases} 0 & \text{otherwise} \\ \mu(e_j) & \end{cases}$ v_$iliesontheedge$e_j

Definition 3 *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph with underlying crisp graph* $G = (V, \sigma^*, \mu^*)$ *. Then the vertex set of fuzzy transformation graph* $V(G^{-+}) = V \cup E$*. The adjacency and incidence relations and their values are defined as follows:*

- $\sigma^{-+}(u)$ = {$\sigma(u)$u $\mu(u)$ifu 0otherwise
- $\mu^{-+}(v_i, v_j)$ = {0(v_i,v_j)>0 $\sigma(v_i) \wedge \sigma(v_j)$otherwise
- $\mu^{-+}(e_i, e_j)$ = {0e_i,e_j
- $\mu^{-+}(v_i, e_j)$ = {$\mu(e_j)$v_$iliesontheedge$e_j 0otherwise

Definition 4 *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph with underlying crisp graph* $G = (V, \sigma^*, \mu^*)$ *. Then the vertex set of fuzzy transformation graph* $V(G^{--}) = V \cup E$*. The adjacency and incidence relations and their values are defined as follows:*

- $\sigma^{--}(u)$ = {$\sigma(u)$u $\mu(u)$ifu 0otherwise
- $\mu^{--}(v_i, v_j)$ = {0(v_i,v_j)>0 $\sigma(v_i) \wedge \sigma(v_j)$otherwise
- $\mu^{--}(e_i, e_j)$ = {0e_i,e_j
- $\mu^{--}(v_i, e_j)$ = {0v_$iliesontheedge$e_j $\mu(e_j)$otherwise

Example 1. Let $G = (V, \sigma, \mu)$ be a fuzzy graph shown in Fig.1 with vertex set $V = \{v_1, v_2, v_3, v_4\}$ such that $\sigma(v_1) = 0.8$, $\sigma(v_2) = 0.7$, $\sigma(v_3) = 0.6$, $\sigma(v_4) = 0.4$,where as $\mu(v_1v_2) = 0.6$, $\mu(v_2v_3) = 0.5$ and $\mu(v_3v_4) = 0.4$. Then $d_G(v_1) = 0.6$, $d_G(v_2) = 1.1$,$d_G(v_3) = 0.9$ and $d_G(v_4) = 0.4$ Therefore the first Zagreb index of G is given by

$$F(G) = \sum_{u \in V(G)} [\sigma(u) d_G(u)]^3$$

$$= [(0.8)(0.6)]^3 + [(0.7)(1.1)]^3 + [(0.6)(0.9)]^3 + [(0.4)(0.4)]^3$$

$$= 0.728$$

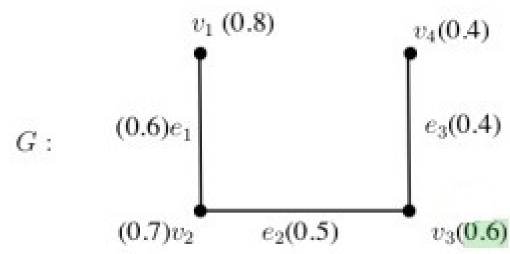

Figure 1. A fuzzy graph G on 4-vertices

Example for G^{++}: Consider a graph G in Figure 1. Then according Definition 1, the fuzzy transformation graph G^{++} shown in Figure 2 has vertex set $V(G^{++}) = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3\}$ such that $\sigma(v_1) = 0.8$, $\sigma(v_2) = 0.7$, $\sigma(v_3) = 0.6$, $\sigma(v_4) = 0.4$,$\sigma(e_1) = 0.6$,$\sigma(e_2) = 0.5$ and $\sigma(e_3) = 0.4$.Further, $\mu(v_1, v_2) = 0.6$, $\mu(v_2, v_3) = 0.5$, $\mu(v_3, v_4) = 0.4$, $\mu(v_1, e_1) = 0.6$, $\mu(v_2, e_1) = 0.6$, $\mu(v_2, e_2) = 0.5$, $\mu(v_3, e_2) = 0.5$, $\mu(v_3, e_3) = 0.4$ and $\mu(v_4, e_3) = 0.4$. By Lemma 1,

$$d_G^{++}(u) = 2\sum_{u \neq v} \mu(u, v) \ \ and \ \ d_G^{++}(e) = 2\mu(u, v).$$

Therefore, we have

$$d_G(v_1) = \sum_{v_1 \neq v_2} \mu(v_1, v_2) = 2(0.6) = 1.2$$

$$d_G(v_2) = 2(1.1) = 2.2$$

$$d_G(v_3) = 2(0.9) = 1.8$$

$$d_G(v_4) = 2(0.4) = 0.8$$

$$d_G(e_1) = 2\mu(v_1 v_2) = 2(0.6) = 1.2$$

$d_G(e_2) = 2\mu(v_2v_3) = 2(0.5) = 1$

$d_G(e_3) = 2\mu(v_3v_4) = 2(0.4) = 0.8$

Therefore, the F-index of G^{++} is given by

$F(G^{++}) = \sum_{u\in V(G^{++})} [\sigma(u)d_G^{++}(u)]^3$

$= \sum_{u\in V(G^{++})\cap V(G)} [\sigma(u)d_G(u)]^3 + \sum_{u\in V(G^{++})\cap E(G)} [\sigma(u)d_G(u)]^3$

$= ([(0.8)(1.2)]^3 + [(0.7)(2.2)]^3 + [(0.6)(1.8)]^3 + [(0.4)(0.8)]^3)$

$+([(0.6)(1.2)]^3 + [(0.5)(1)]^3 + [(0.4)(0.8)]^3)$

$= 6.3602.$

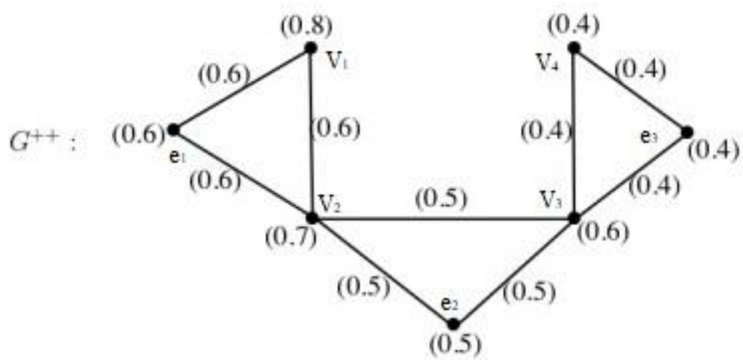

Figure 2: The transformation graph G^{++}

Example for G^{+-}: Consider a graph G in Figure 1. Then according Definition 2, the fuzzy transformation graph G^{+-} shown in Figure 3 has vertex set $V(G^{+-}) = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3\}$ such that $\sigma(v_1) = 0.8$, $\sigma(v_2) = 0.7$, $\sigma(v_3) = 0.6$, $\sigma(v_4) = 0.4$,$\sigma(e_1) = 0.6$,$\sigma(e_2) = 0.5$ and $\sigma(e_3) = 0.4$.Further, $\mu(v_1, v_2) = 0.6$, $\mu(v_2, v_3) = 0.5$, $\mu(v_3, v_4) = 0.4$, $\mu(v_1, e_3) = 0.4$, $\mu(v_2, e_3) = 0.4$, $\mu(v_3, e_1) = 0.6$, $\mu(v_4, e_1) = 0.6$, $\mu(v_1, e_2) = 0.5$ and $\mu(v_4, e_2) = 0.5$. By Lemma 1,

$$d_G^{+-}(u) = 2\sum_{u,v\in VXV} \mu(u, v) \ \ and \ \ d_G^{+-}(e) = (|V| - 2)\mu(u, v).$$

Therefore, we have

$d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_4) = 1.5$

$d_G(e_1) = (|V| - 2)\mu(u, v) = (4 - 2)(0.6) = 1.2$

$d_G(e_2) = (4 - 2)(0.5) = 1$

$d_G(e_3) = (4 - 2)(0.4) = 0.8$

Therefore, the F-index of G^{+-} is given by

$F(G^{+-}) = \sum_{u \in V(G^{+-})} [\sigma(u) d_G^{+-}(u)]^3$

$= \sum_{u \in V(G^{+-}) \cap V(G)} [\sigma(u) d_G(u)]^3 + \sum_{u \in V(G^{+-}) \cap E(G)} [\sigma(u) d_G(u)]^3$

$= ([(0.8)(1.5)]^3 + [(0.7)(1.5)]^3 + [(0.6)(1.5)]^3 + [(0.4)(1.5)]^3)$

$+([(0.6)(1.2)]^3 + [(0.5)(1)]^3 + [(0.4)(0.8)]^3)$

$= 4.7799.$

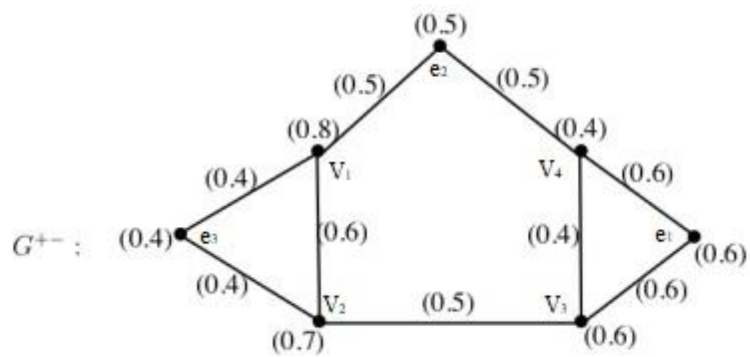

Figure 3: The transformation graph G^{+-}

Example for G^{-+}: Consider a graph G in Figure 1. Then according Definition 3, the fuzzy transformation graph G^{-+} shown in Figure 4 has vertex set $V(G^{-+}) = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3\}$ such that $\sigma(v_1) = 0.8, \sigma(v_2) = 0.7, \sigma(v_3) = 0.6$, $\sigma(v_4) = 0.4, \sigma(e_1) = 0.6, \sigma(e_2) = 0.5$ and $\sigma(e_3) = 0.4$.Further, $\mu(v_1, v_3) = 0.6$, $\mu(v_1, v_4) = 0.4$, $\mu(v_2, v_4) = 0.4$, $\mu(v_1, e_1) = 0.6$, $\mu(v_2, e_1) = 0.6$, $\mu(v_2, e_2) = 0.5$, $\mu(v_3, e_2) = 0.5$, $\mu(v_3, e_3) = 0.4$ and $\mu(v_4, e_2) = 0.4$. By Lemma 1,

$$d_G^{-+}(u) = 2\sum_{uv\notin E}[\sigma(u)\wedge\sigma(v)] + \sum_{uv\notin E}\mu(u,v) \;\; and \;\; d_G^{-+}(e) = 2\mu(u,v).$$

Therefore, we have

$$d_G(v_1) = 0.4 + 0.6 + 0.6 = 1.6$$

$$d_G(v_2) = 0.5 + 0.6 + 0.4 = 1.5$$

$$d_G(v_3) = 0.4 + 0.4 + 0.5 = 1.3$$

$$d_G(v_4) = 0.4 + 0.4 + 0.4 = 1.2$$

$$d_G(e_1) = 2\mu(v_1v_2) = 2(0.6) = 1.2$$

$$d_G(e_2) = 2\mu(v_2v_3) = 2(0.5) = 1$$

$$d_G(e_3) = 2\mu(v_3v_4) = 2(0.4) = 0.8$$

Therefore, the F-index of $G^{-+}$is given by

$$F(G^{-+}) = \sum_{u\in V(G^{-+})}[\sigma(u)d_G^{-+}(u)]^3$$

$$= \sum_{u\in V(G^{-+})\cap V(G)}[\sigma(u)d_G(u)]^3 + \sum_{u\in V(G^{-+})\cap E(G)}[\sigma(u)d_G(u)]^3$$

$$= ([(0.8)(1.6)]^3 + [(0.7)(1.5)]^3 + [(0.6)(1.3)]^3 + [(0.4)(1.2)]^3)$$

$$+([(0.6)(1.2)]^3 + [(0.5)(1)]^3 + [(0.4)(0.8)]^3)$$

$$= 4.3706.$$

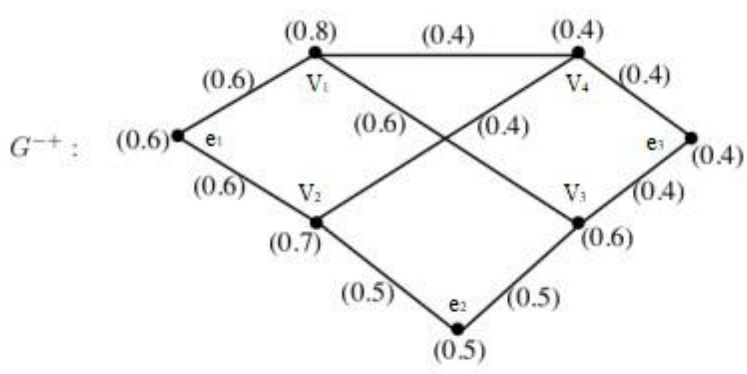

Figure 4: The transformation graph G^{-+}

Example for G^{--}: Consider a graph G in Figure 1. Then according Definition 4, the fuzzy transformation graph G^{--} shown in Figure 5 has vertex set $V(G^{--}) = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3\}$ such that $\sigma(v_1) = 0.8$, $\sigma(v_2) = 0.7$, $\sigma(v_3) = 0.6$, $\sigma(v_4) = 0.4$,$\sigma(e_1) = 0.6$,$\sigma(e_2) = 0.5$ and $\sigma(e_3) = 0.4$.Further, $\mu(v_1, v_3) = 0.6$, $\mu(v_1, v_4) = 0.4$, $\mu(v_2, v_4) = 0.4$, $\mu(v_1, e_3) = 0.4$, $\mu(v_2, e_3) = 0.4$, $\mu(v_3, e_1) = 0.6$, $\mu(v_4, e_1) = 0.6$, $\mu(v_1, e_2) = 0.5$ and $\mu(v_4, e_2) = 0.5$. By Lemma 1,

$$d_G^{--}(u) = 2\sum_{uv\notin E} [\sigma(u) \wedge \sigma(v)] + \sum_{uv\notin E} \mu(u, v) \ and \ d_G^{--}(e) = (|V| - 2)\mu(u, v).$$

Therefore, we have

$$d_G(v_1) = 0.4 + 0.6 + 0.5 + 0.4 = 1.9$$

$$d_G(v_2) = 0.4 + 0.4 = 0.8$$

$$d_G(v_3) = 0.6 + 0.6 = 1.2$$

$$d_G(v_4) = 0.4 + 0.4 + 0.6 + 0.5 = 1.9$$

$$d_G(e_1) = 2\mu(v_1v_2) = 2(0.6) = 1.2$$

$$d_G(e_2) = 2\mu(v_2v_3) = 2(0.5) = 1$$

$$d_G(e_3) = 2\mu(v_3v_4) = 2(0.4) = 0.8$$

Therefore, the F-index of $G^{--}$is given by

$$F(G^{--}) = \sum_{u\in V(G^{--})} [\sigma(u)d_G^{--}(u)]^3$$

$$= \sum_{u\in V(G^{--})\cap V(G)} [\sigma(u)d_G(u)]^3 + \sum_{u\in V(G^{--})\cap E(G)} [\sigma(u)d_G(u)]^3$$

$$= ([(0.8)(1.9)]^3 + [(0.7)(0.8)]^3 + [(0.6)(1.2)]^3 + [(0.4)(1.9)]^3)$$

$$+([(0.6)(1.2)]^3 + [(0.5)(1)]^3 + [(0.4)(0.8)]^3)$$

$$= 5.0304.$$

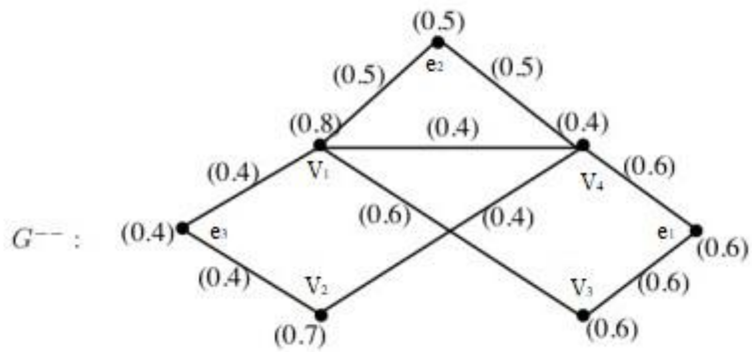

Figure 5: The transformation graph G^{--}

4 Results

Lemma 1. *Let* $G = (V, \sigma, \mu)$*be a fuzzy graph and* $H = G^{ab} = (V \cup E, \sigma^*, \mu^*)$*be fuzzy transformation graph. Then the degree of point vertex and line vertex are given by*

1. $d_G^{++}(u) = 2\sum_{u \neq v} \mu(u, v)$ *and* $d_G^{++}(e) = 2\mu(u, v)$.
2. $d_G^{+-}(u) = 2\sum_{u,v \in VXV} \mu(u, v)$ *and* $d_G^{+-}(e) = (|V| - 2)\mu(u, v)$.
3. $d_G^{-+}(u) = 2\sum_{uv \notin E} [\sigma(u) \wedge \sigma(v)] + \sum_{uv \notin E} \mu(u, v)$ *and* $d_G^{-+}(e) = 2\mu(u, v)$.
4. $d_G^{--}(u) = 2\sum_{uv \notin E} [\sigma(u) \wedge \sigma(v)] + \sum_{uv \notin E} \mu(u, v)$ *and* $d_G^{--}(e) = (|V| - 2)\mu(u, v)$.

Lemma 2. $[Handshaking\ Lemma\ For\ Fuzzy\ Graphs]$ The sum of degrees of all vertices in a fuzzy graph $G = (V, \sigma, \mu)$ is equal to twice the sum of membership values of all edges in G. *i.e* $\sum_{u \in V(G)} d_G(u) = 2\sum_{e \in E(G)} \mu(e)$.

Observation 1. *For every fuzzy graph* $G = (V, \sigma, \mu)$ *the following are true:*

1. $\sigma(u) \leq 1$
2. $\sum_{u \in V(G)} \sigma(u) \leq n$

3. $\mu(u,v) = \wedge(\sigma(u).\sigma(v)) \leq 1$

4. $\sum_{u,v\in E(G)} \mu(u,v) = \sum_{e\in E(G)} \mu(e)$

Theorem 1 *Let* G *be n-vertex fuzzy graph with m-edges. Then*

$$F(G^{++}) \leq 8F(G) + m\sum_{u\neq v} \mu(u,v)^3.$$

Proof. Let $G = (V,\sigma,\mu)$ be any fuzzy graph. Then we have

$$F(G^{++}) = \sum_{u\in V(G^{++})} [\sigma(u)d_G^{++}(u)]^3$$

$$= \sum_{u\in V(G^{++})\cap V(G)} [\sigma(u)d_G(u)]^3 + \sum_{u\in V(G^{++})\cap E(G)} [\sigma(u)d_G(u)]^3$$

By Lemma 1, we have

$$F(G^{++}) = \sum_{v\in V(G)} [\sigma(v)2\sum_{u\notin V} \mu(u,v)]^3 + \sum_{v\in V(G^{++})\cap E(G)} [\sigma(v)2\mu(e)]^3$$

$$= 8\sum_{v\in V(G)} [\sigma(v)d_G(v)]^3 + \sum_{v\in V(G^{++})\cap E(G)} [\sigma(v)2\mu(e)]^3$$

$$F(G^{++}) \leq 8F(G) + 8[\sum_{v\in E(G)} [\sigma(v)]^3][\sum_{e\in E(G)} [\mu(e)]^3]$$

$$\leq 8F(G) + 8m[\sum_{e\in E(G)} [\mu(e)]^3]$$

By Lemma 2, we have

$$F(G) \leq 8F(G) + 8m[\frac{\sum_{v\in V(G)} d_G(v)}{2}]^3$$

$$\leq 8F(G) + \frac{8m}{8}[\sum_{v\in V(G)} d_G(v)]^3$$

$$\leq 8F(G) + m[\sum_{u\neq v} \mu(u,v)]^3.$$

Theorem 2 *Let* G *be n-vertex fuzzy graph with m-edges. Then*

$$F(G^{+-}) \leq nm^3 + \frac{n(n-2)^3}{8}\sum_{u\neq v} \mu(u,v)^3.$$

Proof. Let $G = (V,\sigma,\mu)$ be any fuzzy graph. Then we have

$$F(G^{+-}) = \sum_{u\in V(G^{+-})} [\sigma(u)d_G^{+-}(u)]^3$$

$$= \sum_{u\in V(G^{+-})\cap V(G)} [\sigma(u)d_G(u)]^3 + \sum_{u\in V(G^{+-})\cap E(G)} [\sigma(u)d_G(u)]^3$$

By Lemma 1, we have

$$F(G^{+-}) = \sum_{v\in V(G)} [\sigma(v)\sum_{(u,v)\in VXV} \mu(u,v)]^3 + \sum_{v\in V(G^{+-})\cap E(G)} [\sigma(v)(|V|-2)\mu(u,v)]^3$$

$$= \sum_{v\in V(G)} [\sigma(v)m]^3 + \sum_{v\in V(G^{++})\cap E(G)} [\sigma(v)(n-2)\mu(u,v)]^3$$

$$F(G^{+-}) \le m^3 \sum_{v\in V(G)} [\sigma(v)]^3 + (n-2)^3[\sum_{v\in E(G)} [\sigma(v)\mu(u,v)]]^3$$

By Observation 1, we have

$$F(G^{+-}) \le nm^3 + (n-2)^3(\sum_{v\in V(G)} \sigma(v))^3(\sum_{e\in E(G)} \mu(e))^3$$

By Lemma 2, we get

$$F(G^{+-}) \le nm^3 + n(n-2)^3 \sum_{v\in V(G)} d_G(v)^3$$

$$\le nm^3 + n(n-2)^3 \sum_{u\neq v} (\frac{\mu(u,v)}{2})^3$$

$$\le nm^3 + \frac{n(n-2)^3}{8}\sum_{u\neq v} \mu(u,v)^3.$$

Theorem 3 *Let* G *be n-vertex fuzzy graph with m-edges and let* $\overline{m}$ *be be edge set of* $\overline{G}$*. Then*

$$F(G^{-+}) \le\le n\overline{m}^3(\overline{m}+1)^3 + m\sum_{u\neq v} \mu(u,v)^3.$$

Proof. Let $G = (V,\sigma,\mu)$ be any fuzzy graph. Then we have

$$F(G^{-+}) = \sum_{v\in V(G^{-+})} [\sigma(v)d_G^{-+}(v)]^3$$

$$= \sum_{u\in V(G^{-+})\cap V(G)} [\sigma(v)d_G(v)]^3 + \sum_{v\in V(G^{-+})\cap E(G)} [\sigma(v)d_G(v)]^3$$

By Lemma 1, we have

$$F(G^{-+}) = \sum_{v\in V(G)} [\sigma(v)[\sum_{uv\notin E} \sigma(u)\wedge\sigma(v) + \sum_{uv\notin E} \mu(u,v)]]^3 + \sum_{v\in V(G^{-+})\cap E(G)} [\sigma(v)2\mu(e)]^3$$

By Observation 1, we have $\sigma(u)\wedge\sigma(v) \le 1$. Therefore

$$F(G^{-+}) \leq \sum_{v\in V(G)} [\sigma(v)[\sum_{uv\in E(\overline{G})} 1 + \sum_{uv\in E} \overline{m}]]^3 + \sum_{v\in V(G^{-+})\cap E(G)} [\sigma(v)2\mu(e)]^3$$

$$\leq (\overline{m} + \overline{m}^2)^3 \sum_{v\in V(G)} \sigma(v)^3 + 8\sum_{v\in V(G)} \sigma(v)^3[\sum_{e\in E(G)} \mu(e)]^3$$

$$\leq \overline{m}^3(\overline{m} + 1)^3 \sum_{v\in V(G)} \sigma(v)^3 + 8\sum_{v\in V(G)} \sigma(v)^3[\sum_{e\in E(G)} \mu(e)]^3$$

By Lemma 2, we have

$$F(G^{-+}) \leq n\overline{m}^3(\overline{m} + 1)^3 + 8m[\frac{\sum_{v\in V(G)} d_G(v)}{2}]^3$$

$$\leq n\overline{m}^3(\overline{m} + 1)^3 + m\sum_{u\neq v} \mu(u, v)^3.$$

Theorem 4 *Let* G *be n-vertex fuzzy graph with m-edges and let* $\overline{m}$ *be edge set of* $\overline{G}$*.Then*

$$F(G^{--}) \leq n\overline{m}^3(\overline{m} + 1)^3 + n(n-2)^3 \sum_{u\neq v} \mu(u, v)^3.$$

Proof. Let $G = (V, \sigma, \mu)$ be any fuzzy graph. Then we have

$$F(G^{--}) = \sum_{v\in V(G^{--})} [\sigma(v)d_G^{--}(v)]^3$$

$$= \sum_{u\in V(G^{--})\cap V(G)} [\sigma(v)d_G(v)]^3 + \sum_{v\in V(G^{--})\cap E(G)} [\sigma(v)d_G(v)]^3$$

By Lemma 1, we have

$$F(G^{--}) = \sum_{v\in V(G)} [\sigma(v)[\sum_{uv\notin E} \sigma(u) \wedge \sigma(v) + \sum_{uv\notin E} \mu(u, v)]]^3 + \sum_{v\in V(G^{--})\cap E(G)} [\sigma(v)(|V| - 2)2\mu(u, v)]^3$$

By Observation 1, we have $\sigma(u) \wedge \sigma(v) \leq 1$. Therefore

$$F(G^{--}) \leq \sum_{v\in V(G)} [\sigma(v)[\sum_{uv\in E(\overline{G})} 1 + \sum_{uv\in E} \overline{m}]]^3 + \sum_{v\in V(G^{-+})\cap E(G)} (n-2)^2[\sigma(v)\mu(e)]^3$$

$$\leq (\overline{m} + \overline{m}^2)^3 \sum_{v\in V(G)} \sigma(v)^3 + 8(n-2)^3 \sum_{v\in V(G)} \sigma(v)^3[\sum_{e\in E(G)} \mu(e)]^3$$

$$\leq n\overline{m}^3(\overline{m} + 1)^3 + 8(n-2)^3 \sum_{v\in V(G)} \sigma(v)^3[\sum_{e\in E(G)} \mu(e)]^3$$

By Lemma 2, we have

$$F(G^{--}) \leq n\overline{m}^3(\overline{m}+1)^3 + 8n(n-2)^3(\frac{\sum_{v\in V(G)} d_G(v)}{2})^3$$

$$\leq n\overline{m}^3(\overline{m}+1)^3 + n(n-2)^3(\sum_{u\neq v} \mu(u,v))^3$$

5 Conclusion:

In this paper we have studied the F-index of generalized transformation graphs and obtained some upper bounds for $F(G)$ in terms of elements of a graph G.

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