



Soft ρ -sets where $\rho \in \{p, q, Q\}$

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Abstract -In this paper soft p-set, soft q-set and soft Q-set are defined and their properties are characterized.

Keywords: Parametrized family, Soft sets, Soft Topology, Soft open sets.

I. Introduction

Molodtsov [4] initiated the concept of soft set theory, which is a set associated with parameters and completely new approach for modeling vagueness and uncertainty. He successfully applied the soft set theory into several directions such as smoothness of functions, game theory, Riemann Integration, theory of measurement, and so on. Soft set theory and its applications have shown great development because of the general nature of parametrization expressed by a soft set. Shabir and Naz [5] introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. Later, Zorlutuna et al. [8], Aygunoglu and Aygun [1] and Hussain [3] et al are continued to study the properties of soft topological space. Thangavelu and Rao [6,7] introduced the notion of p-set, q-set in topological spaces. In this paper soft p-set, soft q-set and soft Q-set are defined and their properties are characterized.

II. Preliminaries

Let (X, τ) be a topological space and $(X, \tilde{\tau}, E)$ be a soft topological space. Let A be a subset of X , and \tilde{F} be a soft set over X with parameter space E .

$IntA$ and ClA denote the interior and closure of A in (X, τ) respectively. $\tilde{S}cl\tilde{F}$ and $\tilde{S}int\tilde{F}$ denotes the soft closure of \tilde{F} and soft interior of \tilde{F} in the soft topological space.

Definition 2.1

A subset A of a space (X, τ) is called

- (i) a p-set if $ClIntA \subseteq IntClA$
- (ii) a q-set if $IntClA \subseteq ClIntA$

(iii) a Q -set if $IntCIA=CIIntA$.

Definition 2.2

The pair (F, E) is called a soft set over X , where $F:E \rightarrow 2^X$ is a mapping. Conveniently (F, E) is represented by F_E which is known as a soft set over (X, E) . If the parameter space E is fixed, we use F instead of F_E . $S(X, E)$ denotes the collection of all soft subsets of X with parameter set E . $S(X, E) = \{F:E \rightarrow 2^X\}$. If $F \in S(X, E)$ then we write $F = \{(e, F(e)):e \in E\}$. The next definition is due to Maji, Biswas [31].

Definition 2.3

Let F and G be any two soft sets over a common universe X with a parameter space E . Then

- (i) F is a soft subset of G denoted by $F \overset{s}{\subseteq} G$ if $F(e) \subseteq G(e)$ for all $e \in E$.
- (ii) $F=G$ if $F(e)=G(e)$ for all $e \in E$.
- (iii) The soft complement of F , denoted by F^c , is defined as

$$F^c = \{(e, F^c(e)):e \in E\} = \{(e, X \setminus F(e)):e \in E\}. \text{ That is } F^c(e) = X \setminus F(e).$$

Definition 2.4

Let F be a soft set over X and $x \in X$. Then $x \in F$ whenever $x \in F(e)$ for every $e \in E$.

The notions of a soft point and a quasi soft set are respectively studied by Zorlutuna[8] and Evanzalin [2]. These two notions are useful to study a link between topology and soft topology.

Definition 2.5

Let $x \in X, A \subseteq X$ and $e \in E$.

- (i) The soft set x_e is called a soft point if $x_e(e) = \{x\}$ and $x_e(\alpha) = \emptyset$ for every $\alpha \in E \setminus \{e\}$.
- (ii) The soft set A_e is called a quasi soft set if $A_e(e) = A$ and $A_e(\alpha) = \emptyset$ for every $\alpha \in E \setminus \{e\}$.

Remark 2.6

Every soft point is a quasi soft set. But the quasi soft set need not be a soft point.

Definition 2.7

A sub collection ${}^s\tau$ of $S(X, E)$ is said to be a soft topology over (X, E) if

- (i) \emptyset_E and X_E belong to ${}^s\tau$,
- (ii) ${}^s\tau$ is closed under finite intersection,

(iii) ${}^s\tau$ is closed under arbitrary union.

If ${}^s\tau$ is a soft topology on X then the triplet $(X, {}^s\tau, E)$ is called a softtopological space over (X, E) and ${}^s\tau$ is a soft topology over (X, E) . The members of ${}^s\tau$ are called the soft open sets in $(X, {}^s\tau, E)$. The soft complement of a soft open set is soft closed. The soft interior and soft closure of soft set can be defined in the usual way. sIntF and sClF denote respectively the soft interior and soft closure of F .

Lemma 2.8

Let $(X, {}^s\tau, E)$ be a soft topological space. Then for each $e \in E$, the collection of subsets $F(e)$ of X , $F \in {}^s\tau$ is a topology on X , denoted by $({}^s\tau)_e$.

The family $\{({}^s\tau)_e : e \in E\}$ is called the parametrized family of topologies induced by the soft topology ${}^s\tau$.

Lemma 2.9

Let $(X, {}^s\tau, E)$ be a soft topological space and $F \in S(X, E)$. Then for $e \in E$, $({}^sIntF)(e) \subseteq Int(F(e)) \subseteq Cl(F(e)) \subseteq ({}^sClF)(e)$,

Evanzalin [2] has proved some interesting properties of the soft interior and soft closure operators as given in the next two lemmas.

Lemma 2.10

Let $(X, {}^s\tau, E)$ be a soft topological space with $QS({}^s\tau)_e$ is contained in ${}^s\tau$ for every $e \in E$ and $F \in S(X, E)$. Then for every $e \in E$

- (i) $({}^sIntF)(e) = Int(F(e))$ and $({}^sClF)(e) = Cl(F(e))$
- (ii) $({}^sCl{}^sIntF)(e) = ClInt(F(e))$ and $({}^sInt{}^sClF)(e) = IntCl(F(e))$
- (iii) $({}^sInt{}^sCl{}^sIntF)(e) = IntClInt(F(e))$ and $({}^sCl{}^sInt{}^sClF)(e) = ClIntCl(F(e))$

III. Main Results

Definition 3.1

A soft subset F in a soft topological space $(X, {}^s\tau, E)$ is called

- (i) a soft p -set if ${}^sCl{}^sIntF \subseteq {}^sInt{}^sClF$.
- (ii) a soft q -set if ${}^sInt{}^sClF \subseteq {}^sCl{}^sIntF$.
- (iii) a soft Q -set if ${}^sCl{}^sIntF = {}^sInt{}^sClF$

Let $X = \{a, b, c, d\}$ and $E = \{\alpha, \beta\}$ in the following two illustrations.

Illustration 3.2

Let ${}^s\tau = \{\emptyset_E, X_E, F_1, F_2, F_3\}$ where $F_1 = \{(\alpha, \{a\}), (\beta, \emptyset)\}$, $F_2 = \{(\alpha, \{c\}), (\beta, \emptyset)\}$ and $F_3 = \{(\alpha, \{a, c\}), (\beta, \emptyset)\}$. Then $(X, {}^s\tau, E)$ is a soft topological space.

The corresponding soft closed sets $= \{\emptyset_E, X_E, H_1, H_2, H_3\}$ where, $H_1 = \{(\alpha, \{b, c, d\}), (\beta, X)\}$, $H_2 = \{(\alpha, \{a, b, d\}), (\beta, X)\}$ and $H_3 = \{(\alpha, \{b, d\}), (\beta, X)\}$.

Let $F = \{(\alpha, \{c\}), (\beta, \emptyset)\}$. Then ${}^sCl^sIntF = {}^sCl(\{(\alpha, \{a\}), (\beta, \emptyset)\}) = \{(\alpha, \{a, b, d\}), (\beta, X)\}$ and ${}^sInt^sClF = {}^sInt(\{(\alpha, \{a, b, d\}), (\beta, X)\}) = \{(\alpha, \{a\}), (\beta, \emptyset)\}$.

Also ${}^sInt^sClF = \{(\alpha, \{a\}), (\beta, \emptyset)\} \subseteq \{(\alpha, \{a, b, d\}), (\beta, X)\} = {}^sCl^sIntF$ that implies F is a soft q -set but neither a soft Q -set nor a soft p -set.

Suppose $H = \{(\alpha, \{a, c\}), (\beta, \emptyset)\}$. ${}^sCl^sIntH = {}^sCl(\{(\alpha, X), (\beta, X)\}) = X_E$ and ${}^sInt^sClH = {}^sInt(\{(\alpha, X), (\beta, X)\}) = X_E$. Since ${}^sCl^sIntH = {}^sInt^sClH$, it follows that H is a soft Q -set.

Illustration 3.3

Let ${}^s\tau = \{\emptyset_E, X_E, G_1, G_2, G_3\}$ where $G_1 = \{(\alpha, \{b, c, d\}), (\beta, X)\}$, $G_2 = \{(\alpha, \{a, b, d\}), (\beta, X)\}$ and $G_3 = \{(\alpha, \{b, d\}), (\beta, X)\}$. Then $(X, {}^s\tau, E)$ is a soft topological space. The corresponding soft closed sets $= \{\emptyset_E, X_E, H_1, H_2, H_3\}$ where $H_1 = \{(\alpha, \{a\}), (\beta, \emptyset)\}$, $H_2 = \{(\alpha, \{c\}), (\beta, \emptyset)\}$ and $H_3 = \{(\alpha, \{a, c\}), (\beta, \emptyset)\}$.

Let $F = \{(\alpha, \{a, b, c\}), (\beta, \{d\})\}$. Then ${}^sCl^sIntF = {}^sCl(\{(\alpha, \emptyset), (\beta, \emptyset)\}) = \emptyset_E$ and ${}^sInt^sClF = {}^sInt(\{(\alpha, X), (\beta, X)\}) = X_E$. Since ${}^sCl^sIntF \subseteq {}^sInt^sClF$, F is a soft p -set but it is neither a soft q -set nor a soft Q -set.

The soft complement of a p -set is again a p -set as established in the next proposition.

Proposition 3.4

Let F be a soft subset of a soft topological space $(X, {}^s\tau, E)$. Then

- (i) F is a soft p -set $\Leftrightarrow F^c$ is a soft p -set.
- (ii) F is a soft q -set $\Leftrightarrow F^c$ is a soft q -set.
- (iii) F is a soft Q -set $\Leftrightarrow F^c$ is a soft Q -set.
- (iv) F is a soft Q -set $\Leftrightarrow F$ is a soft p -set and a soft q -set.

Proof:

F is a soft p -set $\Leftrightarrow {}^sCl^sIntF \subseteq {}^sInt^sClF$

$$\Leftrightarrow ({}^s Cl^s Int F)^{cs} \supseteq ({}^s Int^s Cl F)^c$$

$$\Leftrightarrow {}^s Cl^s Int [(F)^c] \supseteq {}^s Int^s Cl [(F)^c]$$

$$\Leftrightarrow (F)^c \text{ is a soft } p\text{-set.}$$

The other assertions can be analogously proved and the proof is completed.

The following proposition shows that a soft set is soft ρ -set iff it is a ρ -set in a topology induced by the soft topology.

Proposition 3.5

Let $(X, {}^s \tau, E)$ be a soft topological space with the condition that $QS(({}^s \tau)_e)$ is contained in ${}^s \tau$ for every $e \in E$. Let $F \in S(X, E)$. Then F is a soft ρ -set if and only if $F(e)$ is a ρ -set in $(X, ({}^s \tau)_e)$ for every $e \in E$.

Proof:

By using Lemma 1.5.5, for every $e \in E$, $({}^s Cl^s Int F)(e) = Cl Int(F(e))$ and $({}^s Int^s Cl F)(e) = Int Cl(F(e))$.

$$F \text{ is a soft } p\text{-set} \Leftrightarrow {}^s Cl^s Int F \supseteq {}^s Int^s Cl F$$

$$\Leftrightarrow ({}^s Cl^s Int F)(e) \supseteq ({}^s Int^s Cl F)(e) \text{ for every } e \in E$$

$$\Leftrightarrow Cl Int(F(e)) \supseteq Int Cl(F(e))$$

$$\Leftrightarrow F(e) \text{ is a } p\text{-set in } (X, ({}^s \tau)_e)$$

$$F \text{ is a soft } q\text{-set} \Leftrightarrow {}^s Int^s Cl F \supseteq {}^s Cl^s Int F$$

$$\Leftrightarrow ({}^s Int^s Cl F)(e) \supseteq ({}^s Cl^s Int F)(e) \text{ for every } e \in E$$

$$\Leftrightarrow Int Cl(F(e)) \supseteq Cl Int(F(e))$$

$$\Leftrightarrow F(e) \text{ is a } q\text{-set in } (X, ({}^s \tau)_e)$$

$$F \text{ is a soft } Q\text{-set} \Leftrightarrow {}^s Cl^s Int F = {}^s Int^s Cl F$$

$$\Leftrightarrow ({}^s Cl^s Int F)(e) = ({}^s Int^s Cl F)(e) \text{ for every } e \in E$$

$$\Leftrightarrow Cl Int(F(e)) = Int Cl(F(e))$$

$$\Leftrightarrow F(e) \text{ is a } Q\text{-set in } (X, ({}^s \tau)_e). \text{ Hence the proof is completed.}$$

A soft topology and its α -soft topology have the same collection of soft ρ -sets as shown in the next proposition.

Proposition 3.6

Let $(X, {}^s \tau, E)$ be a soft topological space with $QS(({}^s \tau)_e)$ is contained in ${}^s \tau$ for every $e \in E$ and ${}^s \alpha O(X, {}^s \tau, E)$ be the collection of soft α -open sets in $(X, {}^s \tau, E)$.

Let $F \in S(X, E)$. Then F is a soft p -set in $(X, {}^s\tau, E)$ if and only if F is a soft p -set in $(X, {}^s\alpha O(X, {}^s\tau, E), E)$.

Proof:

F is a soft p -set $\Leftrightarrow F(e)$ is a soft p -set in $(X, ({}^s\tau)_e)$ for every $e \in E$.

$$\Leftrightarrow (ClInt(F(e)) \subseteq IntCl(F(e)))$$

$$\Leftrightarrow ({}^s\alpha Cl {}^s\alpha Int F)(e) \subseteq ({}^s\alpha Int {}^s\alpha Cl F)(e)$$

$$\Leftrightarrow {}^s\alpha Cl {}^s\alpha Int F \subseteq {}^s\alpha Int {}^s\alpha Cl F$$

$\Leftrightarrow F$ is a soft p -set in $(X, {}^s\alpha O(X, {}^s\tau, E), E)$.

F is a soft q -set $\Leftrightarrow F(e)$ is a soft q -set in $(X, ({}^s\tau)_e)$ for every $e \in E$.

$$\Leftrightarrow (IntCl(F(e)) \subseteq ClInt(F(e)))$$

$$\Leftrightarrow (\alpha Int \alpha Cl(F(e)) \subseteq (\alpha Cl \alpha Int(F(e))))$$

$$\Leftrightarrow \alpha Int \alpha Cl F \subseteq \alpha Cl \alpha Int F$$

$\Leftrightarrow F$ is a soft q -set in $(X, {}^s\alpha O(X, {}^s\tau, E), E)$.

F is a soft Q -set $\Leftrightarrow F(e)$ is a soft Q -set in $(X, ({}^s\tau)_e)$ for every $e \in E$.

$$\Leftrightarrow (ClInt(F(e)) = IntCl(F(e)))$$

$$\Leftrightarrow ({}^s\alpha Cl {}^s\alpha Int F)(e) = ({}^s\alpha Int {}^s\alpha Cl F)(e)$$

$$\Leftrightarrow {}^s\alpha Cl {}^s\alpha Int F = {}^s\alpha Int {}^s\alpha Cl F$$

$\Leftrightarrow F$ is a soft Q -set in $(X, {}^s\alpha O(X, {}^s\tau, E), E)$. This completes the proof of the proposition.

Proposition 3.7

Suppose F is soft $b^\#$ -open. Then if F is soft open then it is soft regular closed. The converse is not true.

Proof:

Suppose F is soft $b^\#$ -open.

If F is soft open then $F = {}^sCl {}^sInt F \cup {}^sInt {}^sCl F$

$$= {}^sCl F \cup {}^sInt {}^sCl F$$

$$= {}^sCl F$$

$$= {}^sCl {}^sInt F \text{ that implies } F \text{ is soft regular closed. Hence the}$$

proof is completed.

Examples can be constructed to show that a soft regular closed set that is also soft $b^\#$ -open is not soft open.

Let $(X, {}^s\tau, E)$ be a soft space given in Example 3.1.7.

Define the soft set F_5 as $F_5(\alpha) = \begin{cases} \{b, c, d\} & \text{for } \alpha = e \\ X & \text{otherwise} \end{cases}$. Then ${}^sCl^sIntF_5 = F_5$

and ${}^sCl^sIntF_5 \cup {}^sInt^sClF_5 = F_5$ but ${}^sIntF_5 = F_2$. Hence F_5 is soft regular closed and soft $b^\#$ -open in $(X, {}^s\tau, E)$ but not soft open.

The following proposition shows that a soft $b^\#$ -open set is soft α -open iff it is a soft Q-set.

Proposition 3.8

Suppose F is soft $b^\#$ -open. Then F is soft α -open if and only if it is a soft Q-set.

Proof:

If F is soft α -open then

$$F \subseteq {}^sInt^sCl^sIntF \subseteq {}^sCl^sIntF \subseteq {}^sCl^sIntF \cup {}^sInt^sClF = F \text{ and}$$

$F \subseteq {}^sInt^sCl^sIntF \subseteq {}^sInt^sClF \subseteq {}^sCl^sIntF \cup {}^sInt^sClF = F \Rightarrow {}^sCl^sIntF = {}^sInt^sClF = F$ which proves that F is a soft Q-set.

Conversely let F be a soft Q-set. Since F is soft $b^\#$ -open it follows that ${}^sCl^sIntF = {}^sInt^sClF = F$ that implies ${}^sInt^sCl^sIntF = {}^sInt^sClF = F$ so that F is soft α -open. This proves the proposition.

Proposition 3.9

Suppose F is soft $b^\#$ -open. The following are equivalent

- (i) F is soft semi-open.
- (ii) F is a soft q-set.
- (iii) F is soft regular closed.

Proof:

If F is soft semi-open, then ${}^sCl^sIntF \subseteq {}^sCl^sIntF \cup {}^sInt^sClF = F \subseteq {}^sCl^sIntF$ that implies ${}^sInt^sClF \subseteq {}^sCl^sIntF = F$ so that F is soft regular closed and a soft q-set. This proves (i) \Rightarrow (ii) and (i) \Rightarrow (iii).

Since every soft regular closed set is soft semi-open (iii) \Rightarrow (i) follows. Now suppose F is a soft q-set. Then ${}^sInt^sClF \subseteq {}^sCl^sIntF$. Since F is soft $b^\#$ -open, it follows that $F = {}^sCl^sIntF \cup {}^sInt^sClF = {}^sCl^sIntF$ so that F is soft semi-open. This proves (ii) \Rightarrow (i). This completes the proof.

Proposition 3.10

Suppose F is soft $b^\#$ -open. The following are equivalent

- (i) F is soft pre-open.
- (ii) F is a soft p -set.
- (iii) F is soft regular open.

Proof:

If F is soft pre-open then ${}^sInt^s C I F^s \subseteq {}^s C I {}^s Int F^s \cup {}^s Int^s C I F^s = F^s \subseteq {}^s Int^s C I F^s$ that implies ${}^s C I {}^s Int F^s \subseteq {}^s Int^s C I F^s = F^s$ so that F is soft regular open and a soft p -set. This proves (i) \Rightarrow (ii) and (i) \Rightarrow (iii).

Since every soft regular open set is soft pre-open, (iii) \Rightarrow (i) follows. Now suppose F is a soft p -set. Then ${}^s C I {}^s Int F^s \subseteq {}^s Int^s C I F^s$. Since F is soft $b^\#$ -open, it follows that $F = {}^s C I {}^s Int F^s \cup {}^s Int^s C I F^s = {}^s Int^s C I F^s$ so that F is soft pre-open. This proves (ii) \Rightarrow (i). Thus the proof is completed.

A soft $*b$ -open set in the soft topology is soft α -closed iff it is soft closed and soft open in the soft topology as shown in the next proposition.

Proposition 3.11

Suppose F is soft $*b$ -open. F is soft α -closed iff it is soft closed and soft open.

Proof:

Suppose F is soft $*b$ -open. Then $F^s \subseteq {}^s C I {}^s Int F^s \cap {}^s Int^s C I F^s$. If F is soft α -closed then $F^s \subseteq {}^s C I {}^s Int F^s \cap {}^s Int^s C I F^s \subseteq {}^s C I {}^s Int^s C I F^s \subseteq F^s$ so that $F = {}^s C I {}^s Int^s C I F^s$ that implies F is soft closed so that $F^s \subseteq {}^s C I {}^s Int F^s \cap {}^s Int^s C I F^s = {}^s Int F^s$ which shows that F is soft open. The converse part is trivial.

Proposition 3.12

Suppose F is soft $*b$ -open. Consider the following statements.

- (i) F is soft semi-closed
- (ii) F is a soft q -set
- (iii) F is soft regular open
- (iv) F is soft pre-open

Then (i) \Rightarrow (ii), (i) \Leftrightarrow (iii), (ii) \Rightarrow (iv) and (iii) \Rightarrow (iv) always hold.

Proof:

If F is soft semi-closed then $F^s \subseteq {}^s C I {}^s Int F^s \cap {}^s Int^s C I F^s \subseteq {}^s Int^s C I F^s \subseteq F^s$ so that $F = {}^s Int^s C I F^s \subseteq {}^s C I {}^s Int F^s$ that implies F is soft regular open and a soft q -set. This proves (i) \Rightarrow (iii) and (i) \Rightarrow (ii).

Since every soft regular open set is soft semi-closed (iii) \Rightarrow (i) is established. Now suppose F is a soft q -set. Then $Int^s Cl^s F \subseteq^s Cl^s Int^s F$. Since F is soft $*b$ -open it follows that $F \subseteq^s Cl^s Int^s F \cap^s Int^s Cl^s F = Int^s Cl^s F$ so that F is soft pre-open.

Proposition 3.13

Suppose F is soft $*b$ -open. Consider the following statements.

- (i) F is soft pre-closed.
- (ii) F is a soft p -set.
- (iii) F is soft regular closed.
- (iv) F is soft semi-open.

Then (i) \Rightarrow (ii), (i) \Leftrightarrow (iii), (ii) \Rightarrow (iv) and (iii) \Rightarrow (iv) always hold.

Proof:

If F is soft pre-closed then $F \subseteq^s Cl^s Int^s F \cap^s Int^s Cl^s F \subseteq^s Cl^s Int^s F \subseteq F$ so that $F = Int^s Cl^s Int^s F \subseteq^s Int^s Cl^s F$ that implies F is soft regular closed and a soft p -set. This proves (i) \Rightarrow (iii) and (i) \Rightarrow (ii).

Since every soft regular closed set is soft pre-closed (iii) \Rightarrow (i) is proved. Now suppose F_E is a soft p -set. Then $Cl^s Int^s F \subseteq^s Int^s Cl^s F$. Since F is soft $*b$ -open it follows that $F \subseteq^s Cl^s Int^s F \cap^s Int^s Cl^s F = Int^s Cl^s F$ so that F is soft semi-open.

IV. CONCLUSION

In this paper, soft p -set, soft q -set, soft Q -set and their operators are characterized by the parametrized family of topologies induced by the soft topology.

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