ISSN 2063-5346 *Section A-Research paper*

A Semi-Analytic Solution For Time-Fractional Heat Like And Wave Like Equations Via Novel Iterative Method

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Article History: **Received:** 10.02.2023 **Revised:**07.06.2023 **Accepted:** 20.07.2023

ABSTRACT

Fractional differential equations have gained significant attention in recent years due to their ability to model various physical phenomena involving memory and non-locality accurately. Time-fractional heat-like and wave-like equations are significant as they describe critical processes in diverse fields such as physics, engineering, and biology. This research article introduces a novel Sumudu transform iterative method for calculating semi-analytic soluton of time-fractional heat-like and wave-like equations. By harnessing the power of the Sumudu transform and iterative techniques, this method offers a promising approach to solving such equations effectively, enabling better understanding and analysis of complex dynamical systems.

Keywords: Sumudu Transform,Heat and Wave like Equations,Fractional Differential equations,Mittag-Leffler function

Subject Classification: 35K05, 35L05, 26A33, 34A08.

DOI: 10.48047/ecb/2023.12.si8.524

1.0 Introduction

The study of time-fractional differential equations has emerged as a compelling research area, driven by the need to accurately describe and analyze dynamic systems exhibiting non-local and memory-dependent behaviours. Fractional derivatives provide a convenient mathematical framework for capturing these phenomena, enabling the modelling of processes that exhibit long-range interactions and exhibit memory effects. In this context, the time-fractional heat-like and wave-like equations have been of considerable interest due to their broad applicability and fundamental nature.

There are plenty of analytical techniques for evaluating the differential, partial differential, fractional partial differential equations and systems of functional equations such as VIM, HPTM, HAM -That is, The variational iteration method homotopy, perturbation transform method [17, 18, 24, 16], homotopy perturbation method respectively[1, 10, 25]. Iterative Laplace transforms, Rahmat Darzi et al. Applied the Sumudu transform to solve fractional diffusion-wave

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and fractional differential equations. Kielbasa, A.A., and Srivastava, H.M.,[6, 12, 27, 28, 29, 23] derived the formulae for Sumudu transform of R-L, Caputo, and Miller-ross sequential fractional derivatives by using Laplace -Sumudu duality. Modified variational iterative method (MVIP)[7, 8] for solving Klein-Gordon equations. There are other popular tools of fractional calculus and Integral transforms for solving the problems of applied science, mathematical physics, mathematical biology, dynamics and etc.[5, 6, 11, 14, 15].

In 2001, a novel approach introduced by Khuri evaluated the problems of solving nonlinear differential equations by Laplace Adomian Decomposition Method(LADM) [20, 26, 21]. LADM process has been used to find Volterra differential equations[19, 13], newton - homotopy method for solving nonlinear euations"[16]. Many problems in fractional derivatives [7], hydrodynamics [33], chemical diffusion [31], option pricing [30], computational fluid dynamics [32], and control theory [34] can be modelled using partial differential equations (PDEs). Now a day, much attention has been devoted to studying nonlinear PDEs and methods for numerical solutions to nonlinear problems.

This research article proposes a new Sumudu transform iterative method for efficiently solving time-fractional heat-like and wave-like equations. The Sumudu transform has been extensively used in solving ordinary differential equations due to its ability to convert them into algebraic equations.M.Asiru studied the properties of the Sumudu transform[4, 10, 14, 22], which is used to solve integral equations of convolution type. However, its application to fractional differential equations, especially in the context of time-fractional heat-like and wave-like equations, remains relatively unexplored. Our novel approach utilizes the Sumudu transform to convert the time-fractional differential equations into fractional algebraic equations, thereby reducing the problem to solving a set of algebraic equations. The iterative method is employed to refine the solutions, improving accuracy and convergence. By integrating these techniques, we aim to provide an effective numerical method that is New Sumudu Transform-Iterative method (NSTIM) to handle the complexities of time-fractional heat-like and wave-like equations.

Definition 1.1 *[35] Function $y(\xi,\mu)$ has a caputo fractional derivative defined as,*

$$D_\xi^\beta y(\xi,\mu)=\frac{1}{\Gamma(J-\beta)}\int_0^x(\xi-p)^{(J-\beta-1)}y^{(J)}(p,\mu)dp, J-1<\beta\le J, J\in N \tag{1.1}$$

$d^J \equiv \frac{d^J}{dx^J}$ andj_x^β denote the R-L fractional integral operator of order $\beta>0$ defined as $d^J \equiv \frac{d^J}{dx^J}$ and j_x^β respectively.

$$J_\xi^\beta y(\xi,\mu)=\frac{1}{\Gamma\beta}\int_0^\xi(\xi-p)^{(\beta-1)}y(p,\mu)dp, p>0, k-1<\beta\le k, k\in N". \tag{1.2}$$

Definition 1.2 *[35] The order $\beta\in\mathbb{C}, Re(\beta)>0$ Riemann Liouville fractional integral $I_{p+}^\beta f$ is defined as,*

$$\left({}_pD_q^{-\beta}f\right)(q)=\left(I_{p+}^\beta f\right)(q)=\frac{1}{\Gamma(\beta)}\int_p^q\frac{f(c)}{(q-c)^{1-\beta}}dc, (q>p, Re(\beta)>0". \tag{1.3}$$

Definition 1.3 *[36] The Riemann Liouville fractional derivatives $\left({}_pD_q^\beta y\right)(x)$ of order $\beta\in\mathbb{C}, Re(\beta>0)$ is defined by,*

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$$\left({}_pD_q^{\beta}y\right)(c) = \left(\frac{d}{dc}\right)^n\left(\left(I_{p+}^{n-\beta}y\right)(c)\right) = \frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dc}\right)^n\int_p^c\frac{y(q)dq}{(c-q)^{\beta-n+1}}, (n = Re(\beta)+1; c > p)". \quad (1.4)$$

Definition 1.4 *[36] The mittag-leffler function and its generalazation as*

$$E_\beta(y) = \sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\beta^k+1)}(\beta \in c, re(\beta) > 0), \quad (1.5)$$

$E_{\beta,\beta}$ is Mittag-Leffler function in two parameters.

$$E_{\beta,\beta}(y) = \sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\beta^k+\beta)}\beta, \beta \in C, R(\beta) > 0, R(\beta) > 0". \quad (1.6)$$

Definition 1.5 *[37] The sumudu transform of a function $f(t), t > 0$ is defined as*

$$S[f(t)] = \int_0^{\infty} e^{-t}f(vt)dt, v \in (-T_1, T_2) and f(t) \in A, \quad (1.7)$$

(chaurasia and singh 2010) where

$$A = \left\{f(t)/\exists M, T_1, T_2 > 0, |f(t)| \le Me^{\frac{|t|}{T_j}}, if \ t \in (-1)^j \times [0,\infty)\right\}. \quad (1.8)$$

Definition 1.6 *[37] The Sumudu transform of the Caputo fractional derivatie is defined as (chaurasia and singh 2010)*

$$S[D_\xi^{n\xi}y(\xi,\omega)] = v^{-n\beta}S[y(\xi,\omega)] - \sum_{k=0}^{n-1} v^{-n\beta+k}y^{(k)}(0,\omega), n-1 < n\beta < n. \quad (1.9)$$

2.0 The New Sumudu transform Iterative Method(NSTIM)

To illustrate this new Sumudu Iterative Transform Method[38, 39, 41] we consider a fractional non-linear ,non-homogenous partial differentail equationwith the initial conditions of the form:

$$D_\omega^{n\beta} + Ly(\xi,\omega) + R(y(\xi,\omega)) += g(\xi,\omega), \quad n-1 < n\beta \le n, y(\xi,0) = h(\xi) \quad (2.1)$$

where $D_\omega^{n\beta}$ is the Caputo fractional derivative operator,$D_\omega^{n\beta} = \frac{\partial^{n\beta}}{\partial\omega^{n\beta}}$, L is a linear operator,R is nonlinear operator,$g(\xi,\omega)$ is continuous function.

employing the Sumudu transform to the equation eq .(16) we have,

$$S\left[D_\omega^{n\beta}y(\xi,\omega)\right] + S[L(y(\xi,\omega)) + R(U(\xi,\omega))] = S[g(\xi,\omega], \quad (2.2)$$

using the property of sumudu transformation,we obtian,

$$S[y(\xi,\omega)] - v^{n\beta}\sum_{k=0}^{n-1} y^k(\xi,0) + v^{n\beta}S[Ly(\xi,\omega) + Ry(\xi,\omega)] - [g(\xi,\omega)] = 0. \quad (2.3)$$

employing inverse Sumudu transform to the equation we get,

$$y(\xi,\omega) = S^{-1}[v^{n\beta}\sum_{k=0}^{n-1} y^k(\xi,0)] - S^{-1}[v^{n\beta}S[Ly(\xi,\omega) + Ry(\xi,\omega) - g(\xi,\omega)]]. \quad (2.4)$$

Next assume that,

$$f(\xi,\omega) = S^{-1}[v^{n\beta}\sum_{k=0}^{n-1} y^k(\xi,0) + v^{n\beta}S[g(\xi,\omega)]]; \quad (2.5)$$

$$N(y(\xi,\omega)) = -S^{-1}[v^{n\beta}S[Ry(\xi,\omega)]]; \quad (2.6)$$

ISSN 2063-5346

$$K[y(\xi,\omega)] = -S^{-1}[v^{n\beta}S[Ly(\xi,\omega)]]. \tag{2.7}$$

Thus,equation (10)can be written in the following form

$$y(\xi,\omega) = f(\xi,\omega) + K(y(\xi,\omega)) + N(y(\xi,\omega)). \tag{2.8}$$

Where f is a known function ,K and N are given linear and non linear operator of u respectively. The solution of equation can be written in the series form ,

$$\begin{aligned} y(\xi,\omega) = & \quad (\textstyle\sum_{m=0}^{\infty} y(\xi,\omega)), \\ wehave, & \\ K(\textstyle\sum_{m=0}^{\infty} y(\xi,\omega)) = & \quad \textstyle\sum_{m=0}^{\infty} K(y(\xi,\omega)). \end{aligned} \tag{2.9}$$

The non-linear operator N is decomposed as (see Gejji and Jafari 2006)

$$N(\textstyle\sum_{m=0}^{\infty} y_m \quad) = N(y_0) + \left\{N(\sum_{j=0}^{m} y_j) - N(\sum_{j=0}^{m-1} y_j)\right\} \tag{2.10}$$

Therefore ,equation (11)can be represented in the following form , Defining the recurrence relation

$$\begin{aligned} y_0 = & \quad f, \\ y_1 = & \quad K(y_0) + N(y_0), \\ \ldots\ldots, & \\ y_{m+1} = & \quad K(y_m) + N(y_0 + \ldots + y_m). \end{aligned} \tag{2.11}$$

we have,

$$(y_1 + y_2 + \ldots\ldots + y_{m+1}) = K(y_0 + \ldots. + y_m) + N(y_0 + \ldots. + y_m) \tag{2.12}$$

namely,

$$\textstyle\sum_{m=0}^{\infty} = f + K(\sum_{m=0}^{\infty}) + N(\sum_{m=0}^{\infty}). \tag{2.13}$$

The m-term allying solution of equation (11)is given by,

$$y = y_1 + y_2 + \ldots\ldots + y_{m-1}. \tag{2.14}$$

3.0 Stability and Error Analysis

Theorem 3.1 *Let* $y_p(\xi,\omega)$ *and* $y_n(\xi,\omega)$ *be the members of Banach space H, and the exact solution of (1.1) be* $y(\xi,\omega)$ *. The Series solution* $\sum_{p=0}^{\infty} y_p(\xi,\omega)$ *converges to* $y(\xi,\omega)$*, if* $y_p(\xi,\omega) \le \lambda y_{p-1}(\xi,\omega)$ *for* $\lambda \in (0,1)$*, that is for any* $y > 0, \exists E$ *such that* $||y_{p+n}(\xi,\omega)|| \le y, \forall p, n > E$.

Proof. Let $u_p(\xi,\omega) = y_0(\xi,\omega) + y_1(\xi,\omega) + y_2(\xi,\omega) + \cdots + y_p(\xi,\omega)$ be the sequence of p^{th} partial sum of series $\sum_{p=0}^{\infty} y_p(\xi,\omega)$. Now consider

$$\begin{aligned} ||u_{p+1}(\xi,\omega) - u_p(\xi,\omega)|| \quad & = ||y_{p+1}(\xi,\omega)|| \\ & \le \lambda ||y_p(\xi,\omega)|| \\ & \le \lambda^2 ||y_{p-1}(\xi,\omega)|| \\ & \le \lambda^3 ||y_{p-2}(\xi,\omega)|| \\ & \vdots \\ & \le \lambda^{p+1} ||y_0(\xi,\omega)||. \end{aligned} \tag{3.1}$$

for $\forall n, p \in E$

Consider,

$$\begin{aligned}
||u_p(\xi,\omega)-u_n(\xi,\omega)|| &= ||y_{p+n}(\xi,\omega)|| \\
&= ||(u_p(\xi,\omega)-u_{p-1}(\xi,\omega)) \\
&+(u_{p-1}(\xi,\omega)-u_{p-2}(\xi,\omega)) \\
&+(u_{p-2}(\xi,\omega)-u_{p-3}(\xi,\omega)) \\
&+\cdots+(u_{n+1}(\xi,\omega)-u_n(\xi,\omega))|| \\
&\le ||(u_p(\xi,\omega)-u_{p-1}(\xi,\omega))|| \\
&+||(u_{p-1}(\xi,\omega)-u_{p-2}(\xi,\omega))|| \\
&+||(u_{p-2}(\xi,\omega)-u_{p-3}(\xi,\omega))|| \\
&+\cdots+||(u_{n+1}(\xi,\omega)-u_n(\xi,\omega))|| \\
&\le \lambda^p||y_0(\xi,\omega)|| \\
&+\lambda^{p-1}||y_0(\xi,\omega)|| \\
&+\lambda^{p-2}||y_0(\xi,\omega)|| \\
&+\cdots+\lambda^{p-1}||y_0(\xi,\omega)|| \\
&= ||y_0(\xi,\omega)||(\lambda^p+\lambda^{p-1}+\cdots+\lambda^{p+1}) \\
&= ||y_0(\xi,\omega)||(\tfrac{1-\lambda^{p-n}}{1-\lambda})\lambda^{n+1}
\end{aligned} \tag{3.2}$$

Since $0<\lambda<1$, and $y_0(\xi,\omega)$ is bounded, so assume that,

$$y=||y_0(\xi,\omega)||(\tfrac{1-\lambda^{p-n}}{1-\lambda})\lambda^{n+1},$$

we get the desired result. Also $\sum_{p=0}^{\infty} y_p(\xi,\omega)$ is a cauchy sequence in H, which imples that there exists $y_0(\xi,\omega)\in H$ such that $\lim_{p\to\infty} y_p(\xi,\omega)=y(\xi,\omega)$, Hence prove.

Theorem 3.2 *Let* $\sum_{p=0}^{q} y_p(\xi,\omega)$ *be the finite and allying solution of* $y(\xi,\omega)$. *If* $||y_{p+1}(\xi,\omega)||\le\lambda||y_0(\xi,\omega)||$ *for* $\lambda\in(0,1)$, *then the maximum absolute error is*

$$||y(\xi,\omega)-\sum_{p=0}^{q} y_p(\xi,\omega)||\le\frac{\lambda^{q+1}}{1-\lambda}||y_0(\xi,\omega)||.$$

Proof.

$$\begin{aligned}
||y(\xi,\omega)-\sum_{p=0}^{q} y_p(\xi,\omega)|| &= ||\sum_{p=0}^{\infty} y_p(\xi,\omega)|| \\
&\le \sum_{p=q+1}^{\infty} ||y_p(\xi,\omega)|| \\
&\le \sum_{p=q+1}^{\infty} \lambda^q||y_0(\xi,\omega)|| \\
&\lambda^{q+1}(1+\lambda+\lambda^2+\cdots)||y_0(\xi,\omega)|| \\
&\le \frac{\lambda^{q+1}}{1-\lambda}||y_0(\xi,\omega)||
\end{aligned} \tag{3.3}$$

hence prove

4.0 Numerical Examples:

Example 4.1 *[40] We acknowlege the following one dimensional time fractional heat equation :*

$$D_\omega^\beta y(\xi,\omega) = \frac{1}{2}\xi^2\frac{\partial^2 y}{\partial\xi^2}, 0<\beta\le 1 \tag{4.1}$$

Subject to the initial condition

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$$y(\xi,0)=\xi^2. \tag{4.2}$$

employing Sumudu transform on the equation (4.1) and using the initial condition of equation (4.2) we get,

$$S[y(\xi,\omega)]=\xi^2+\frac{1}{2u^{-\beta}}\xi^2 S[\frac{\partial^2 y}{\partial\xi^2}] \tag{4.3}$$

employing inverse Sumudu transform of the equation (4.3) we get,

$$\begin{aligned} y(\xi,\omega)= &\quad S^{-1}[\xi^2]+S^{-1}[\frac{1}{2u^{-\beta}}\xi^2 S[\frac{\partial^2 y}{\partial\xi^2}]],\\ namely, &\\ y(\xi,\omega)= &\quad \xi^2+S^{-1}[\frac{1}{2u^{-\beta}}\xi^2 S[\frac{\partial^2 y}{\partial\xi^2}]]. \end{aligned} \tag{4.4}$$

According to the NSTIM,we have,

$$\begin{aligned} y_0= &\quad \xi^2,\\ K[y(\xi,\omega)]= &\quad S^{-1}[\frac{1}{2u^{-\beta}}\xi^2 S[\frac{\partial^2 y}{\partial\xi^2}]]. \end{aligned} \tag{4.5}$$

By iteration ,the following results are obtained

$$\begin{aligned} y_0(\xi,\omega)= &\quad \xi^2,\\ y_1(\xi,\omega)= &\quad S^{-1}[\frac{1}{2u^{-\beta}}\xi^2 S[\frac{\partial^2 y_0}{\partial\xi^2}]],\\ = &\quad \xi^2\frac{\omega^\beta}{\Gamma(\beta+1)}. \end{aligned} \tag{4.6}$$

$$\begin{aligned} y_2(\xi,\omega)= &\quad S^{-1}[\frac{1}{2u^{-\beta}}\xi^2 S[\frac{\partial^2(y_0+y_1)}{\partial\xi^2}]]\\ &\quad -S^{-1}[\frac{1}{2u^{-\beta}}y^2 S[\frac{\partial^2 y_0}{\partial\xi^2}]]\\ = &\quad \xi^2[\frac{\omega^{2\beta}}{\Gamma(2\beta+1)}+\frac{\omega^\beta}{\Gamma(2\beta+1)}]-(\xi^2\frac{\omega^\beta}{\Gamma(2\beta+1)})\\ = &\quad \xi^2\frac{\omega^{2\beta}}{\Gamma(2\beta+1)}. \end{aligned} \tag{4.7}$$

Therefore, solution of the problem is given by,

$$\begin{aligned} y(\xi,\omega)= &\quad y_0(\xi,\omega)+y_1(\xi,\omega)+\ldots\\ y(\xi,\omega)= &\quad \xi^2[1+\frac{\omega^\beta}{\Gamma(\beta+1)}+\frac{\omega^\beta}{\Gamma(2\beta+1)}+\ldots]\\ = &\quad \xi^2 E_\beta(\omega^\beta). \end{aligned} \tag{4.8}$$

Where - $E_\beta(\omega^\beta)$ is mittage leffer function defined by (1.5).

Setting $\beta=1$,equation (4.1) becomes the following heat equation of 1-dimension,

$$\begin{aligned} y(\xi,\omega) &= \frac{1}{2}x^2\frac{\partial^2 y}{\partial\xi^2},\\ \text{with accurate solution} &\\ y(\xi,\omega) &= \xi^2 e^\omega. \end{aligned} \tag{4.9}$$

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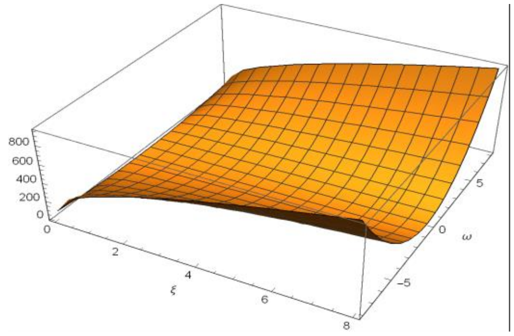
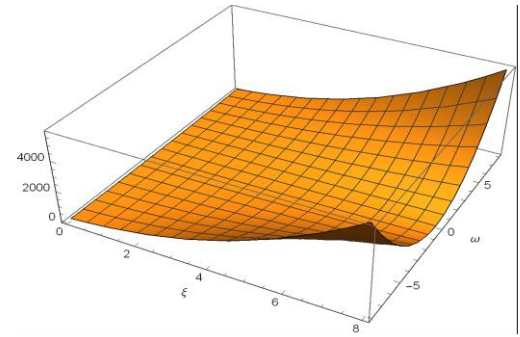
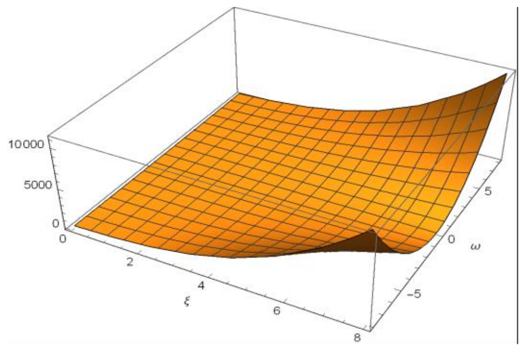


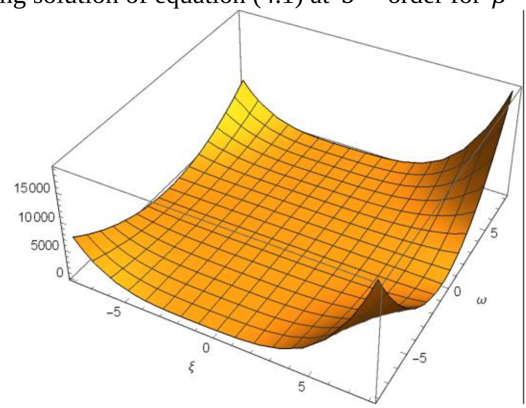
(a) allying solution of equation (4.1) at 5^{th} order for $\beta = 0.2$



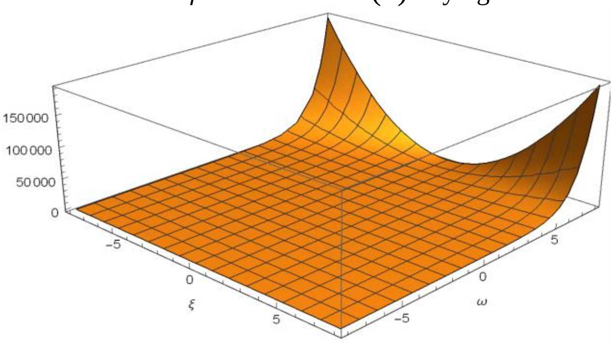
(b)allying solution of equation (4.1) at 5^{th} order for $\beta = 0.6$



(c) allying solution of equation (4.1) at 5^{th} order for $\beta = 0.8$



(d) allying solution of equation (4.1) at 5^{th} order for $\beta = 1$



(e) accurate solution of equation (4.1) for $\beta = 1$

Figure 1

Remark:1 The linear fractional one-dimensional heat equation shown above. The allying solutions of the linear fractional one-dimensional heat equation at different values for β=0.2,0.6,0.8,1 and the accurate solution for β=1 are shown in Figures 1,2,3,4,5 respectively. The solution is straight forward to discover that it is constantly dependent on the values of time-fractional derivatives.

Example 4.2 *[40] We acknowlege the following two- dimensional time fractional heat equation :*

$$D_{\omega}^{\beta} y(\xi, \phi, \omega) \quad = \frac{\partial^2 y}{\partial \xi^2} + \frac{\partial^2 y}{\partial \phi^2}, 0 < \beta \leq 1 \tag{4.10}$$

Subject to the initial condition

$$y(\xi,\phi,0)=sin(\xi)sin(\phi). \quad (4.11)$$

employing Sumudu transform on the equation (4.10) and using the initial condition of equation (4.11) we get,

$$S[y(\xi,\phi,\omega)]=sin(\xi)sin(\phi)+\frac{1}{u^{-\beta}}S[\frac{\partial^2 y}{\partial\xi^2}+\frac{\partial^2 y}{\partial\phi^2}] \quad (4.12)$$

employing inverse Sumudu transform of the equation (4.12) we get,

$$y(\xi,\phi,\omega)=S^{-1}[sin(\xi)sin(\phi)]+S^{-1}[\frac{1}{u^{-\beta}}S[\frac{\partial^2 y}{\partial\xi^2}+\frac{\partial^2 y}{\partial\phi^2}]],$$
$$namely,$$
$$y(\xi,\phi,\omega)=sin(\xi)sin(\phi)+S^{-1}[\frac{1}{u^{-\beta}}S[\frac{\partial^2 y}{\partial\xi^2}+\frac{\partial^2 y}{\partial\phi^2}]], \quad (4.13)$$

According to the NSTIM,we have

$$y_0= sin(\xi)sin(\phi),$$
$$K[y(\xi,\phi,\omega)]= S^{-1}[\frac{1}{u^{-\beta}}S[\frac{\partial^2 y}{\partial\xi^2}+\frac{\partial^2 y}{\partial\phi^2}]]. \quad (4.14)$$

By iterative method ,the following result are obtained

$$y_0(\xi,\phi,\omega)=y(\xi,\phi,0)=sin(\xi)sin(\phi),$$
$$y_1(\xi,\phi,\omega)=S^{-1}[\frac{1}{u^{-\beta}}S[\frac{\partial^2 y}{\partial\xi^2}+\frac{\partial^2 y}{\partial\phi^2}]], \quad (4.15)$$
$$=-2sin(\xi)sin(\phi)\frac{\omega^\beta}{\Gamma(\beta+1)},$$

$$y_2(\xi,\phi,\omega)= S^{-1}[\frac{1}{u^{-\beta}}S[\frac{\partial^2(y_0+y_1)}{\partial\xi^2}+\frac{\partial^2(y_0+y_1)}{\partial\phi^2}]]-S^{-1}[\frac{1}{u^{-\beta}}S[\frac{\partial^2(y_0)}{\partial\xi^2}+\frac{\partial^2(y_0)}{\partial\phi^2}]]$$
$$= sin(\xi)sin(\phi)(\frac{(-2)^2\omega^{2\beta}}{\Gamma(2\beta+1)}-\frac{2\omega^{2\beta}}{\Gamma(\beta+1)})+2sin(\xi)sin(\phi)\frac{\omega^\beta}{\Gamma(\beta+1)} \quad (4.16)$$
$$= (-2)^2 sin(\xi)sin(\phi)\frac{\omega^{2\beta}}{\Gamma(2\beta+1)},$$

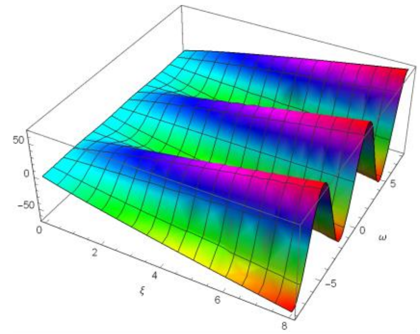
Therefore, solution of the problem is given by,

$$y(\xi,\phi,\omega)= y_0(\xi,\phi,\omega)+y_1(\xi,\phi,\omega)+\ldots$$
$$y(\xi,\phi,\omega)= sin(\xi)sin(\phi)[1+\frac{(-2\omega^\beta)}{\Gamma\beta+1}+1+\frac{(-2\omega^\beta)^2}{\Gamma 2\beta+1}+\ldots] \quad (4.17)$$
$$= sin(\xi)sin(\phi)E_\beta(-2\omega^\beta).$$

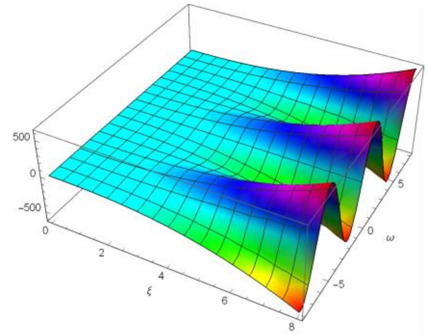
Where - $E_\beta(\omega^\beta)$ is mittage leffer function defined by (1.5).

Setting $\beta=1$,equation (4.10) become the heat equation of 2-dimensional,

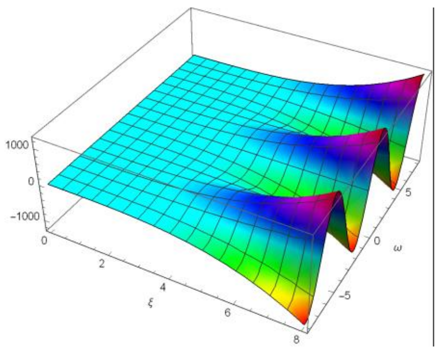
$$y(\xi,\phi,\omega)= \frac{\partial^2 y}{\partial\xi^2}+\frac{\partial^2 y}{\partial\phi^2}$$
with accurate solution
$$y(\xi,\phi,\omega)= \xi^2 e^\omega. \quad (4.18)$$



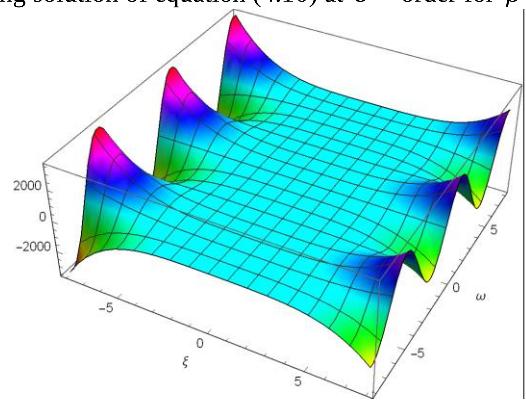
(a) allying solution of equation (4.10) at 5^{th} order for $\beta = 0.2$



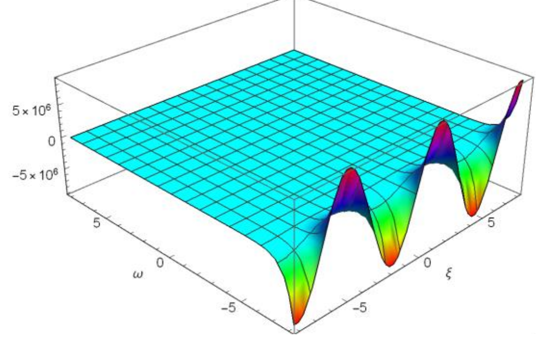
(b)allying solution of equation (4.10) at 5^{th} order for $\beta = 0.6$



(c) allying solution of equation (4.10) at 5^{th} order for $\beta = 0.8$



(d) allying solution of equation (4.10) at 5^{th} order for $\beta = 1$



(e) accurate solution of equation (4.10) for $\beta = 1$

Figure 2

Remark:2 The linear fractional one-dimensional heat equation shown above. The allying solutions of the linear fractional one-dimensional heat equation at different values for β=0.2,0.6,0.8,1 and the accurate solution for β=1 are shown in Figures 6,7,8,9,10 respectively. TThe solution is straight forward to discover that it is constantly dependent on the values of time-fractional derivatives.

Example 4.3 *[40] We acknowlege the following Three- dimensional time fractional heat equation :*

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$$D_{\omega}^{\beta}y(\xi,\phi,\delta,\omega) = \xi^4\phi^4\delta^4+\frac{1}{36}\left(\frac{\xi^2\partial^2 y}{\partial\xi^2}+\frac{\phi^2\partial^2 y}{\partial\phi^2}+\frac{\delta^2\partial^2 y}{\partial\delta^2}\right), 0<\beta\leq 1 \quad (4.19)$$

Subject to the initial condition

$$y(\xi,\phi,\delta,0)=0. \quad (4.20)$$

employing Sumudu transform on the equation (4.19) and using the initial conditions of equation (4.20) we get,

$$S[y(\xi,\phi,\delta,\omega)]=\frac{1}{u^{-\beta}}S[\xi^4\phi^4\delta^4]+\frac{\xi^2}{36}S\left(\frac{1}{u^{-\beta}}y_{\xi\xi}\right)+\frac{\phi^2}{36}S\left(\frac{1}{u^{-\beta}}y_{\phi\phi}\right)+\frac{\delta^2}{36}S\left(\frac{1}{u^{-\beta}}y_{\delta\delta}\right) \quad (4.21)$$

employing inverse Sumudu transform of the equation (4.21) we get,

$$y(\xi,\phi,\delta,\omega)= S^{-1}\left[\frac{1}{u^{-\beta}}S[\xi^4\phi^4\delta^4]+\frac{\xi^2}{36}S\left(\frac{1}{u^{-\beta}}y_{\xi\xi}\right)+\frac{\phi^2}{36} S\left(\frac{1}{u^{-\beta}}y_{\phi\phi}\right)+\frac{\delta^2}{36}S\left(\frac{1}{u^{-\beta}}y_{\delta\delta}\right)\right]$$

namely,

$$y(\xi,\phi,\delta,\omega)=\begin{bmatrix}\xi^4\phi^4\delta^4\frac{\omega^{\beta}}{\Gamma(\beta+1)}+S^{-1}[\frac{\xi^2}{36}S(\frac{1}{u^{-\beta}}y_{\xi\xi}) \\ +\frac{\phi^2}{36}S(\frac{1}{u^{-\beta}}y_{\phi\phi})+\frac{\delta^2}{36}S(\frac{1}{u^{-\beta}}y_{\delta\delta})]\end{bmatrix} \quad (4.22)$$

According to the NSTIM,we have

$$y_0 = \xi^4\phi^4\delta^4\frac{\omega^{\beta}}{\Gamma(\beta+1)},$$

$$K[y(\xi,\phi,\delta,\omega)]= S^{-1}\left[\frac{1}{36(u^{-\beta})}\xi^2S[\frac{\partial^2 y_0}{\partial\xi^2}]+\frac{1}{36(u^{-\beta})}\phi^2S[\frac{\partial^2 y_0}{\partial\phi^2}]+\frac{1}{36(u^{-\beta})}\delta^2S[\frac{\partial^2 y_0}{\partial\delta^2}]\right]. \quad (4.23)$$

By iterative method ,the following result are obtained

$$y_0(\xi,\phi,\delta,\omega)= \xi^4\phi^4\delta^4\frac{t^{\beta}}{\Gamma(\beta+1)},$$

$$y_1(\xi,\phi,\delta,\omega)= S^{-1}\left[\frac{1}{36(u^{-\beta})}\xi^2S[\frac{\partial^2 y_0}{\partial\xi^2}]+\frac{1}{36(u^{-\beta})}\phi^2S[\frac{\partial^2 y_0}{\partial\phi^2}]+\frac{1}{36(u^{-\beta})}\delta^2S[\frac{\partial^2 y_0}{\partial\delta^2}]\right] \quad (4.24)$$

$$= \xi^4\phi^4\delta^4\frac{\omega^{2\beta}}{\Gamma(2\beta+1)},$$

$$\begin{aligned} y_2(\xi,\phi,\delta,\omega)= & S^{-1}\begin{bmatrix}\frac{1}{36(u^{-\beta})}\xi^2S[\frac{\partial^2(y_0+y_1)}{\partial\xi^2}]+\frac{1}{36(u^{-\beta})}\phi^2S[\frac{\partial^2(y_0+y_1)}{\partial\phi^2}] \\ +\frac{1}{36(u^{-\beta})}\delta^2S[\frac{\partial^2(y_0+y_1)}{\partial\delta^2}]\end{bmatrix} \\ & -S^{-1}\begin{bmatrix}\frac{1}{36(u^{-\beta})}\xi^2S[\frac{\partial^2 y_0}{\partial\xi^2}]+\frac{1}{36(u^{-\beta})}\phi^2S[\frac{\partial^2 y_0}{\partial\phi^2}] \\ +\frac{1}{36(u^{-\beta})}\delta^2S[\frac{\partial^2 y_0}{\partial\delta^2}]\end{bmatrix} \quad (4.25) \\ = & \left(\xi^4\phi^4\delta^4\frac{\omega^{(2\beta)}}{\Gamma(2\beta+1)}+\xi^4\phi^4\delta^4\frac{\omega^{(3\beta)}}{\Gamma(2\beta+1)}\right)-\left(\xi^4\phi^4\delta^4\frac{\omega^{(2\beta)}}{\Gamma(2\beta+1)}\right) \\ = & \xi^4\phi^4\delta^4\frac{\omega^{3\beta}}{\Gamma(3\beta+1)}, \end{aligned}$$

Therefore, solution of the problem is given by,

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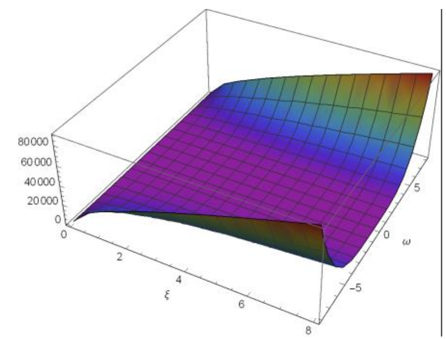
$$y(\xi,\phi,\delta,\omega) = y_0(\xi,\phi,\delta,\omega) + y_1(\xi,\phi,\delta,\omega) + \ldots$$
$$y(\xi,\phi,\delta,\omega) = \xi^4\phi^4\delta^4\left[\frac{(\omega^{\beta})}{\Gamma\beta+1} + \frac{(\omega^{2\beta})}{\Gamma 2\beta+1} + \frac{(\omega^{3\beta})}{\Gamma 3\beta+1} + \ldots\right] \tag{4.26}$$
$$= \xi^4\phi^4\delta^4[E_\beta(\omega^\beta) - 1].$$

Where - $E_\beta(t^\beta)$ is mittage leffer function defined by (1.5).
Setting $\beta = 1$,equation(4.19)beome the heat equation of the 3-dimensional,

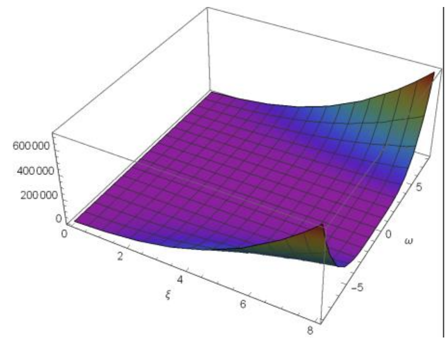
$$y(\xi,\phi,\delta,\omega) = \xi^4\phi^4\delta^4 + \frac{1}{36}\left(\frac{\xi^2\partial^2 y}{\partial\xi^2} + \frac{\phi^2\partial^2 y}{\partial\phi^2} + \frac{\delta^2\partial^2 u}{\partial\delta^2}\right) \tag{4.27}$$

with accurate solution

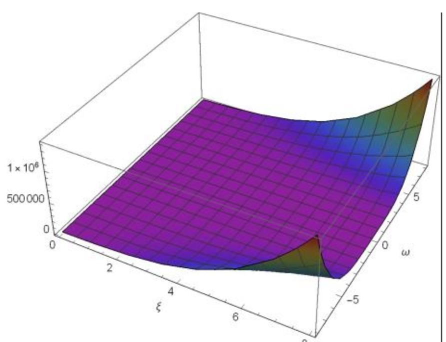
$$y(\xi,\phi,\delta,\omega) = \xi^4\phi^4\delta^4(e^\omega - 1).$$



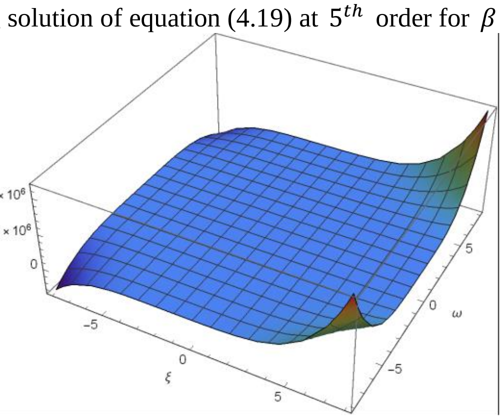
(a) allying solution of equation (4.19) at 5^{th} order for $\beta = 0.2$



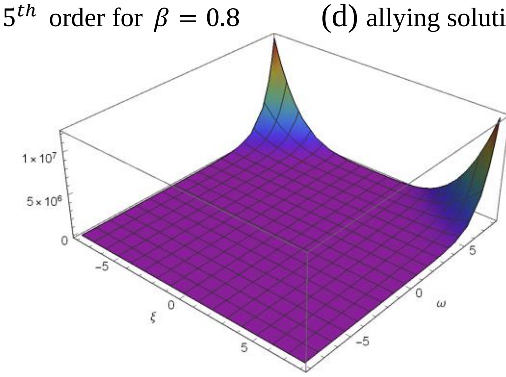
(b)allying solution of equation (4.19) at 5^{th} order for $\beta = 0.6$



(c) allying solution of equation (4.19) at 5^{th} order for $\beta = 0.8$



(d) allying solution of equation (4.19) at 5^{th} order for $\beta = 1$



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(e) accurate solution of equation (4.19) for $\beta = 1$

Figure 3

Remark:3 The linear fractional one-dimensional heat equation shown above. The allying solutions of the linear fractional one-dimensional heat equation at different values for β =0.2,0.6,0.8,1 and the accurate solution for β =1 are shown in Figures 11,12,13,14,15 respectively.The solution is straight forward to discover that it is constantly dependent on the values of time-fractional derivatives.

Example 4.4 *[40] We acknowlege the following one dimensional time fractional wave equation :*

$$D_\omega^\beta y(\xi,\omega) \quad = \frac{1}{2}\xi^2 \frac{\partial^2 y}{\partial \xi^2}, 1 < \beta \leq 2 \tag{4.28}$$

Subject to the initial condition

$$y(\xi,0) = \xi, y_\omega(\xi,0) = \xi^2. \tag{4.29}$$

employing Sumudu transform on the equation (4.28) and using the initial conditions of equation (4.29) we get,

$$S[y(\xi,\omega)] = \xi + \frac{\xi^2}{u^{-1}} + \frac{\xi^2}{2u^{-\beta}} S[\frac{\partial^2 y}{\partial \xi^2}] \tag{4.30}$$

employing inverse Sumudu transform of the equation (4.30) we get,

$$\begin{aligned} y(\xi,\omega) = &\quad S^{-1}[\xi + \frac{\xi^2}{u^{-1}} + \frac{\xi^2}{2u^{-\beta}} S[\frac{\partial^2 y}{\partial \xi^2}]]. \\ & namely, \\ y(\xi,\omega) = &\quad \xi + \xi^2\omega + S^{-1}[\frac{\xi^2}{2u^{-\beta}} + S[\frac{\partial^2 y}{\partial \xi^2}]]. \end{aligned} \tag{4.31}$$

According to the NSTIM,we have

$$\begin{aligned} y_0(\xi,\omega) = &\quad \xi + \xi^2\omega \\ K[y(\xi,\omega)] = &\quad S^{-1}[\frac{\xi^2}{2u^{-\beta}} + S[\frac{\partial^2 y}{\partial \xi^2}]]. \end{aligned} \tag{4.32}$$

By iterative method ,the following result are obtained :

$$\begin{aligned} y_0(\xi,\omega) = &\quad \xi + \xi^2\omega, \\ y_1(\xi,\omega) = &\quad S^{-1}[\frac{\xi^2}{2u^{-\beta}} S[\frac{\partial^2 y_0}{\partial \xi^2}]] \\ = &\quad \xi^2 \frac{\omega^{\beta+1}}{\Gamma(\beta+2)}, \end{aligned} \tag{4.33}$$

$$\begin{aligned} y_2(\xi,\omega) = &\quad S^{-1}[\frac{\xi^2}{2u^{-\beta}} S[\frac{\partial^2 (y_0+y_1)}{\partial \xi^2}]] - S^{-1}[\frac{\xi^2}{2u^{-\beta}} S[\frac{\partial^2 y_0}{\partial \xi^2}]] \\ = &\quad \xi^2[\frac{\omega^{2\beta+1}}{\Gamma(2\beta+2)} + \frac{\omega^{\beta+1}}{\Gamma(2\beta+2)}] - (\xi^2 \frac{\omega^{\beta+1}}{\Gamma(2\beta+2)}) \\ = &\quad \xi^2[\frac{\omega^{2\beta+1}}{\Gamma(2\beta+2)}], \end{aligned} \tag{4.34}$$

Therefore, solution of the problem is given by,

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$$y(\xi,\omega) = \quad y_0(\xi,\omega) + y_1(\xi,\omega) + \ldots$$
$$y(\xi,\omega) = \quad \xi + \xi^2[\omega + \frac{\omega^{\beta+1}}{\Gamma\beta+2} + \frac{\omega^{2\beta+1}}{\Gamma 2\beta+2} + \ldots] \qquad (4.35)$$
$$= \qquad \xi + \xi^2 \omega E_{\beta,2}(\omega^\beta).$$

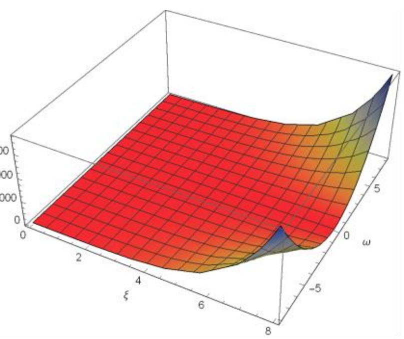
Where - $E_\beta(\omega^\beta)$ is mittage leffer function defined by (1.5).

Setting $\beta = 2$,equation (4.28) becomes the wave equation of order 1-dimensional ,

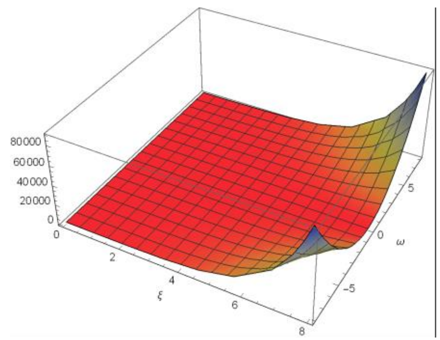
$$\frac{\partial y}{\partial t} = \frac{1}{2}\xi^2 \frac{\partial^2 y}{\partial \xi^2}, \qquad (4.36)$$

with accurate solution

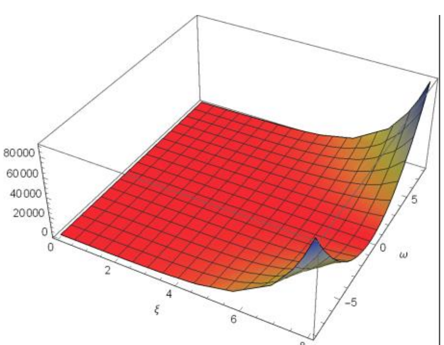
$$y(\xi,\omega) = \xi + \xi^2 sinh\omega.$$



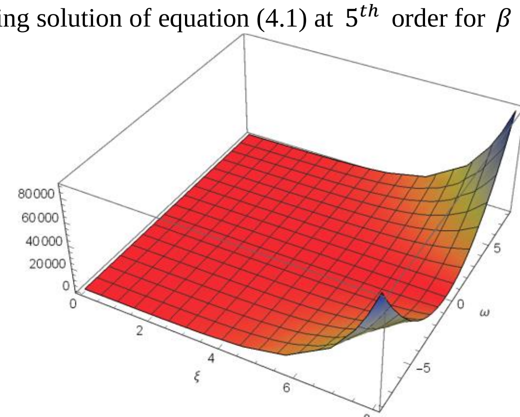
(a) allying solution of equation (4.1) at 5^{th} order for $\beta = 0.2$



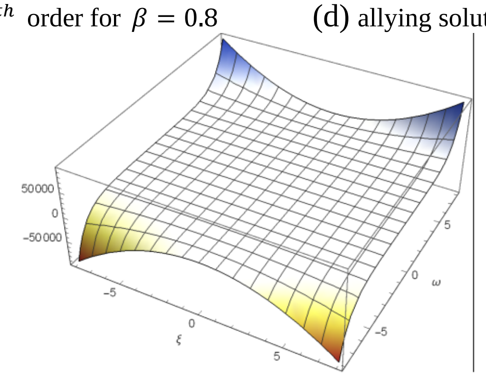
(b)allying solution of equation (4.1) at 5^{th} order for $\beta = 0.6$



(c) allying solution of equation (4.1) at 5^{th} order for $\beta = 0.8$



(d) allying solution of equation (4.1) at 5^{th} order for $\beta = 1$



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(e) accurate solution of equation (4.1) for $\beta = 1$

Figure 4

Remark:4 The linear fractional one-dimensional heat equation shown above. The allying solutions of the linear fractional one-dimensional heat equation at different values for β =0.2,0.6,0.8,1 and the accurate solution for β =2 are shown in Figures 16,17,18,19,20 respectively. The solution is straight forward to discover that it is constantly dependent on the values of time-fractional derivatives.

Example 4.5 *[40] We acknowlege the following two- dimensional time fractional wave equation :*

$$D_{\omega}^{\beta} y(\xi,\phi,\omega) = \frac{1}{12}[\xi^2 \frac{\partial^2 y}{\partial \xi^2} + \phi^2 \frac{\partial^2 y}{\partial \phi^2}], 0 < \beta \leq 2 \tag{4.37}$$

Subject to the initial condition

$$y(\xi,\phi,0) = \xi^4, y_{\omega}(\xi,\phi,0) = \phi^4. \tag{4.38}$$

employing Sumudu transform on the equation (4.37) and using the initial conditions of equation (4.38) we get,

$$S[y(\xi,\phi,\omega)] = \xi^4 + \frac{\phi 4}{u^{-1}} + \frac{\xi^2}{12u^{-\beta}} S[\frac{\partial^2 y}{\partial \xi^2}] + \frac{\phi^2}{12u^{-\beta}} S[\frac{\partial^2 y}{\partial \phi^2}] \tag{4.39}$$

employing inverse Sumudu transform of the equation (4.39) we get,

$$y(\xi,\phi,\omega) = S^{-1}[\xi^4 + \frac{\phi 4}{u^{-1}} + \frac{\xi^2}{12u^{-\beta}} S[\frac{\partial^2 y}{\partial \xi^2}] + \frac{\phi^2}{12u^{-\beta}} S[\frac{\partial^2 y}{\partial \phi^2}]].$$

$$namely, \tag{4.40}$$

$$y(\xi,\phi,\omega) = \xi^4 + \phi^4\omega + S^{-1}[\frac{\xi^2}{12u^{-\beta}} S[\frac{\partial^2 y}{\partial \xi^2}] + \frac{\phi^2}{12u^{-\beta}} S[\frac{\partial^2 y}{\partial \phi^2}]]$$

According to the NSTIM,we have

$$\begin{aligned} y_0(\xi,\phi,\omega) &= \xi^4 + \phi^4\omega, \\ K[y(\xi,\phi,\omega)] &= S^{-1}[\frac{\xi^2}{12u^{-\beta}} S[\frac{\partial^2 y}{\partial \xi^2}] + \frac{\phi^2}{12u^{-\beta}} S[\frac{\partial^2 y}{\partial \phi^2}]]. \end{aligned} \tag{4.41}$$

By iterative method ,the following result are obtained

$$\begin{aligned} y_0(\xi,\phi,\omega) &= \xi^4 + \phi^4\omega, \\ y_1(\xi,\phi,\omega) &= S^{-1}[\frac{\xi^2}{12u^{-\beta}} S[\frac{\partial^2 y}{\partial \xi^2}] + \frac{\phi^2}{12u^{-\beta}} S[\frac{\partial^2 y}{\partial \phi^2}]] \\ &= \xi^4 \frac{\omega^{\beta}}{\Gamma(\beta+1)} + \phi^4 \frac{\omega^{\beta+1}}{\Gamma(\beta+2)}, \end{aligned} \tag{4.42}$$

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$$
\begin{aligned}
y_2(\xi,\phi,\omega) = \quad & S^{-1}[\frac{\xi^2}{12u^{-\beta}}S[\frac{\partial^2(y_0+y_1)}{\partial\xi^2}] + \frac{\xi^2}{12u^{-\beta}}S[\frac{\partial^2(y_0+y_1)}{\partial\phi^2}]] \\
& -S^{-1}[\frac{\xi^2}{12u^{-\beta}}S[\frac{\partial^2(y_0)}{\partial\xi^2}] + \frac{\xi^2}{12u^{-\beta}}S[\frac{\partial^2(y_0)}{\partial\phi^2}]] \\
= \quad & \xi^4(\frac{\omega^\beta}{\Gamma(\beta+1)} + \frac{\omega^{2\beta}}{\Gamma(2\beta+1)}) + \phi^4(\frac{\omega^{\beta+1}}{\Gamma(\beta+2)} + \frac{\omega^{2\beta+1}}{\Gamma(2\beta+2)}) - \xi^4(\frac{\omega^\beta}{\Gamma(\beta+1)}) - \phi^4(\frac{\omega^{\beta+1}}{\Gamma(\beta+2)}) \\
= \quad & \xi^4\frac{\omega^{2\beta}}{\Gamma(2\beta+1)} + \phi^4\frac{\omega^{2\beta+1}}{\Gamma(2\beta+2)},
\end{aligned}
\tag{4.43}
$$

Therefore, solution of the problem is given by,

$$
\begin{aligned}
y(\xi,\phi,\omega) = \quad & y_0(\xi,\phi,\omega) + y_1(\xi,\phi,\omega) + \ldots \\
y(\xi,\phi,\omega) = \quad & x^4[1 + \frac{t^\beta}{\Gamma\beta+1} + \frac{t^{2\beta}}{\Gamma2\beta+1} + \ldots] + y^4[t + \frac{t^{(\beta+1)}}{\Gamma\beta+2} + \frac{t^{2\beta+1}}{\Gamma2\beta+2} + \ldots] \\
= \quad & \xi^4E_\beta(\omega^\beta) + \omega\phi^4E_\beta(\omega^\beta).
\end{aligned}
\tag{4.44}
$$

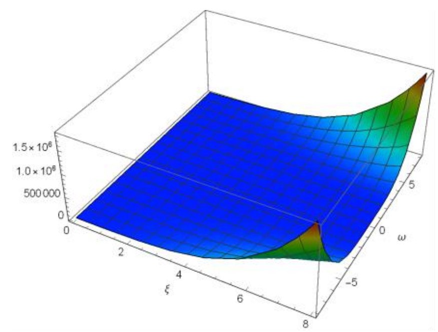
Where - $E_\beta(\omega^\beta)$ is mittage leffer function defined by (1.5)

Setting $\beta = 2$,equation(4.37) becomes wave equation of order 2-dimensional ,

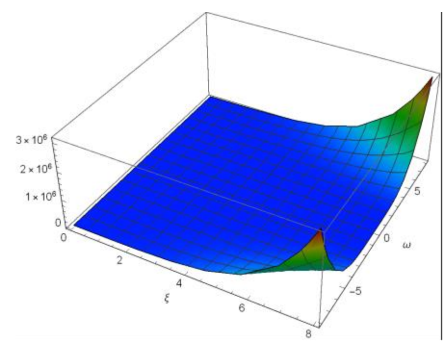
$$
\frac{\partial y}{\partial t} = \frac{1}{12}[\xi^2\frac{\partial^2 y}{\partial\xi^2} + \phi^2\frac{\partial^2 y}{\partial\phi^2}]
\tag{4.45}
$$

with accurate solution

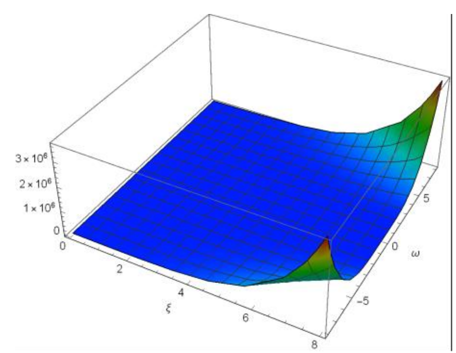
$$
y(\xi,\phi,\omega) = \xi^4 cosh\omega + \phi^4 sinh\omega.
$$



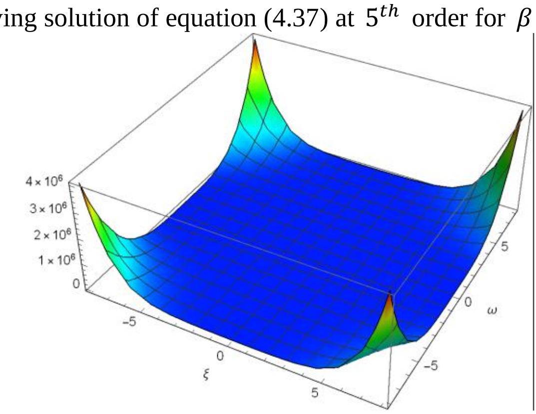
(a) allying solution of equation (4.37) at 5^{th} order for $\beta = 0.2$



(b)allying solution of equation (4.37) at 5^{th} order for $\beta = 0.6$



(c) allying solution of equation (4.37) at 5^{th} order for $\beta = 0.8$



(d) allying solution of equation (4.37) at 5^{th} order for $\beta = 1$

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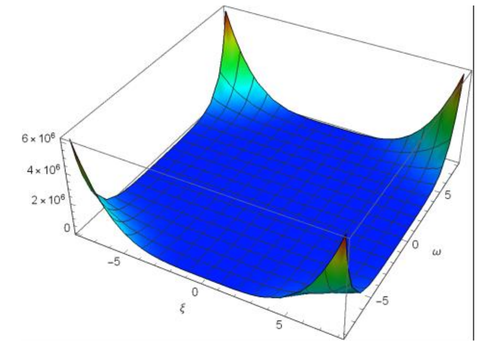

(e) accurate solution of equation (4.37) for $\beta = 1$

Figure 5

Remark:5 The linear fractional one-dimensional heat equation shown above. The allying solutions of the linear fractional one-dimensional heat equation at different values for β =0.2,0.6,0.8,1 and the accurate solution for β =2 are shown in Figures 21,22,23,24,25 respectively.The solution is straight forward to discover that it is constantly dependent on the values of time-fractional derivatives.

Example 4.6 *[40] We acknowlege the following Three- dimensional time fractional wave equation :*

$$D_{\omega}^{\beta} y(\xi,\phi,\delta,\omega) \quad = \xi^2+\phi^2+\delta^2+\frac{1}{2}\left(\frac{\xi^2\partial^2 y}{\partial\xi^2}+\frac{\phi^2\partial^2 y}{\partial\phi^2}+\frac{\delta^2\partial^2 y}{\partial\delta^2}\right), 1<\beta\leq 2 \qquad (4.46)$$

Subject to the initial condition

$$y(\xi,\phi,\delta,0)=0, y_{\omega}(\xi,\phi,\delta,0)=\xi^2+\phi^2-\delta^2. \qquad (4.47)$$

employing Sumudu transform on the equation (4.46) and using the initial conditions of equation (4.47) we get,

$$S[y(\xi,\phi,\delta,\omega)]=\begin{bmatrix}\left(\frac{\xi^2+\phi^2-\delta^2}{u^{-1}}\right)+\left(\frac{\xi^2+\phi^2+\delta^2}{u^{-\beta}}\right)\\ +\frac{\xi^2}{2u^{-\beta}}S\left(\frac{\partial^2 y}{\partial\xi^2}\right)+\frac{\phi^2}{2u^{-\beta}}S\left(\frac{\partial^2 y}{\partial\phi^2}\right)+\frac{\delta^2}{2u^{-\beta}}S\left(\frac{\partial^2 y}{\partial\delta^2}\right)\end{bmatrix} \qquad (4.48)$$

employing inverse Sumudu transform of the equation (4.48) we get,

$$\begin{aligned} y(\xi,\phi,\delta,\omega)= &\quad S^{-1}\left(\frac{\xi^2+\phi^2-\delta^2}{u^{-1}}\right)+S^{-1}\left[\frac{1}{u^{-\beta}}S(\xi^2+\phi^2+\delta^2)\right.\\ + &\quad \left.\frac{\xi^2}{2u^{-\beta}}S\left(\frac{\partial^2 y}{\partial\xi^2}\right)+\frac{\phi^2}{2u^{-\beta}}S\left(\frac{\partial^2 y}{\partial\phi^2}\right)+\frac{\delta^2}{2u^{-\beta}}S\left(\frac{\partial^2 y}{\partial\delta^2}\right)\right]\end{aligned} \qquad (4.49)$$

namely

$$\begin{aligned} y(\xi,\phi,\delta,\omega)= &\quad \omega(\xi^2+\phi^2-\delta^2)+(\xi^2+\phi^2+\delta^2)\frac{\omega^{\beta}}{\Gamma(\beta+1)}\\ + &\quad S^{-1}\left[\frac{\xi^2}{2u^{-\beta}}S\left(\frac{\partial^2 y}{\partial\xi^2}\right)+\frac{\phi^2}{2u^{-\beta}}S\left(\frac{\partial^2 y}{\partial\phi^2}\right)+\frac{\delta^2}{2u^{-\beta}}S\left(\frac{\partial^2 y}{\partial\delta^2}\right)\right]\end{aligned}$$

According to the NSTIM,we have

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$$y_0(\xi,\phi,\delta,\omega) = \omega(\xi^2+\phi^2-\delta^2)+(\xi^2+\phi^2+\delta^2)\frac{\omega^\beta}{\Gamma(\beta+1)}$$
$$K[y(\xi,\phi,\delta,\omega)] = S^{-1}[\frac{\xi^2}{2u^{-\beta}}S(\frac{\partial^2 y}{\partial\xi^2})+\frac{\phi^2}{2u^{-\beta}}S(\frac{\partial^2 y}{\partial\phi^2})+\frac{\delta^2}{2u^{-\beta}}S(\frac{\partial^2 y}{\partial\delta^2})] \quad (4.50)$$

By iterative method ,the following result are obtained

$$y_0(\xi,\phi,\delta,\omega) = \omega(\xi^2+\phi^2-\delta^2)+(\xi^2+\phi^2+\delta^2)\frac{\omega^\beta}{\Gamma(\beta+1)},$$
$$y_1(\xi,\phi,\delta,\omega) = S^{-1}[\frac{\xi^2}{2u^{-\beta}}S(\frac{\partial^2 y}{\partial\xi^2})+\frac{\phi^2}{2u^{-\beta}}S(\frac{\partial^2 y}{\partial\phi^2})+\frac{\delta^2}{2u^{-\beta}}S(\frac{\partial^2 y}{\partial\delta^2})] \quad (4.51)$$
$$= \omega(\xi^2+\phi^2-\delta^2)\frac{\omega^{\beta+1}}{\Gamma(\beta+2)}+(\xi^2+\phi^2+\delta^2)\frac{\omega^{2\beta}}{\Gamma(2\beta+1)},$$

$$y_2(\xi,\phi,\delta,\omega) = S^{-1}\begin{bmatrix}\frac{\xi^2}{2u^{-\beta}}S(\frac{\partial^2(y_0+y_1)}{\partial\xi^2})+\frac{\phi^2}{2u^{-\beta}}S(\frac{\partial^2(y_0+y_1)}{\partial\phi^2})\\ +\frac{\delta^2}{2u^{-\beta}}S(\frac{\partial^2(y_0+y_1)}{\partial\delta^2})\end{bmatrix}$$
$$- S^{-1}\left[\frac{\xi^2}{2u^{-\beta}}S(\frac{\partial^2 y_0}{\partial\xi^2})+\frac{\phi^2}{2u^{-\beta}}S(\frac{\partial^2 y_0}{\partial\phi^2})+\frac{\delta^2}{2u^{-\beta}}S(\frac{\partial^2 y_0}{\partial\delta^2})\right] \quad (4.52)$$
$$= (\xi^2+\phi^2-\delta^2)\frac{\omega^{2\beta+1}}{2\Gamma(\beta+2)}+(\xi^2+\phi^2+\delta^2)\frac{\omega^{3\beta}}{\Gamma(3\beta+1)}$$

Therefore, solution of the problem is given by,

$$y(\xi,\phi,\delta,\omega) = y_0(\xi,\phi,\delta,\omega)+y_1(\xi,\phi,\delta,\omega)+\ldots$$
$$y(\xi,\phi,\delta,\omega) = (\xi^2+\phi^2-\delta^2)[\frac{(\omega^{\beta+1})}{\Gamma\beta+2}+\frac{(\omega^{2\beta+1})}{\Gamma2\beta+2}+\frac{(\omega^{3\beta+1})}{\Gamma3\beta+2}+\ldots]$$
$$+ (\xi^2+\phi^2+\delta^2)[\frac{(\omega^{\beta})}{\Gamma\beta+1}+\frac{(\omega^{2\beta})}{\Gamma2\beta+1}+\frac{(\omega^{3\beta})}{\Gamma3\beta+1}+\ldots] \quad (4.53)$$
$$= (\xi^2+\phi^2-\delta^2)[E_{\beta,2}(\omega^\beta)]+(\xi^2+\phi^2-\delta^2)[E_\beta(\omega^\beta)-1].$$

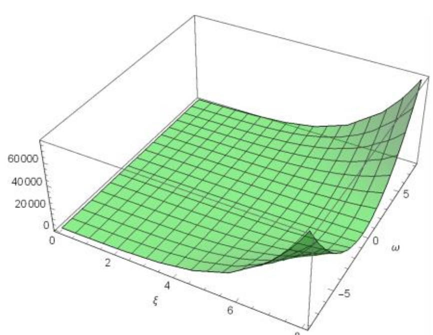
Where - $E_\beta(\omega^\beta)$ is mittage leffer function defined by (1.5)

Setting $\beta=2$,equation (4.46) becomes wave equation of order 3-dimensional ,

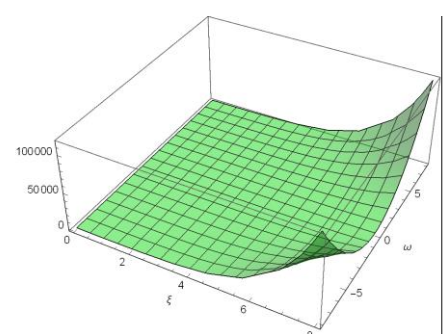
$$\frac{\partial y}{\partial\omega} = \xi^2+\phi^2+\delta^2+\frac{1}{2}(\frac{\xi^2\partial^2 y}{\partial\xi^2}+\frac{\phi^2\partial^2 y}{\partial\phi^2}+\frac{\delta^2\partial^2 y}{\partial\delta^2})$$

withaccuratesolution (4.54)

$$y(\xi,\phi,\delta,\omega) = (\xi^2+\phi^2)e^\omega+\delta^2 e^{-\omega}-(\xi^2+\phi^2+\delta^2).$$



(a) allying solution of equation (4.46) at 5^{th} order for $\beta=0.2$



(b)allying solution of equation (4.46) at 5^{th} order for $\beta=0.6$

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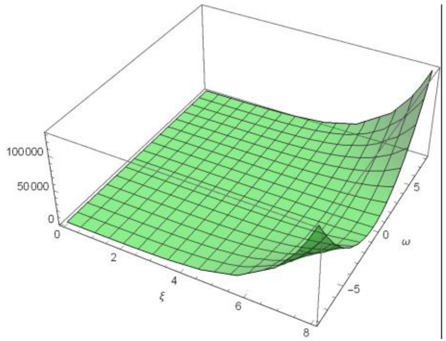
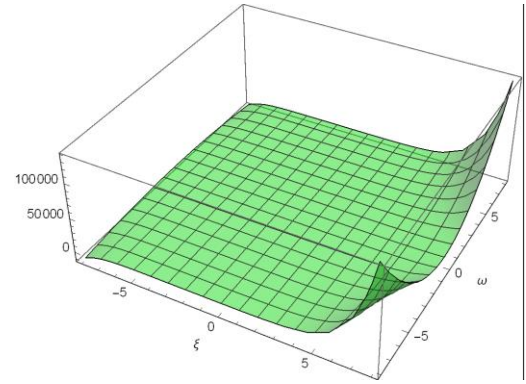
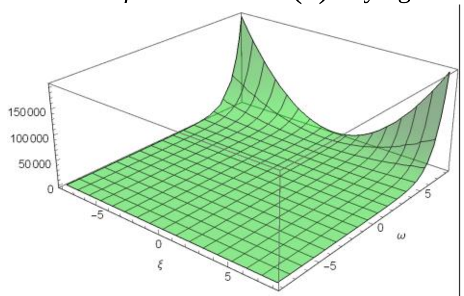


(c) allying solution of equation (4.46) at 5^{th} order for $\beta = 0.8$



(d) allying solution of equation (4.46) at 5^{th} order for $\beta = 1$



(e) accurate solution of equation (4.46) for $\beta = 1$

Figure 6

Remark:6 The linear fractional one-dimensional heat equation shown above. The allying solutions of the linear fractional one-dimensional heat equation at different values for β =0.2,0.6,0.8,1 and the accurate solution for β =2 are shown in Figures 26,27,28,29,30 respectively. The solution is straight forward to discover that it is constantly dependent on the values of time-fractional derivatives.

Table 1: analyze the solution with one, the 5th order allying solution of equation(4.1) and either side of the accurate solution for $\beta = 1$.

$\beta = 1$				
ξ	ω	y(NSTIM)	y(accurate)	$\lvert y_{NSTIM} - y_{accurate} \rvert$
.2	0.3	0.0539935	0.0539944	9×10^{-07}
.4	0.5	0.26375	0.263795	5×10^{-05}
.6	0.7	0.724381	0.724951	6×10^{-04}
.8	0.9	1.57046	1.57415	4×10^{-03}

Table 2: analyze the solution with one, the 5th order allying solution of equation(4.10) and the either side of the accurate solution for $\beta = 1$.

$\beta = 1$				
ξ	ω	y(NSTIM)	y(accurate)	$\lvert y_{NSTIM} - y_{accurate} \rvert$

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.2	0.3	0.109149	0.109032	1×10^{-04}
.4	0.5	0.146032	0.143259	3×10^{-03}
.6	0.7	0.159643	0.139239	2×10^{-02}
.8	0.9	0.204733	0.118578	9×10^{-02}

Table 3: analyze the solution with one, the 5th order allying solution of equation(4.19) and the either side of accurate solution for $\beta = 1$.

$\beta = 1$				
ξ	ω	y(NSTIM)	y(accurate)	$\lvert y_{NSTIM} - y_{accurate} \rvert$
.2	0.3	0.00055977	0.000559774	2×10^{-09}
.4	0.5	0.0166067	0.0166073	6×10^{-07}
.6	0.7	0.131359	0.131382	2×10^{-05}
.8	0.9	0.597507	0.597853	3×10^{-04}

Table 4: analyze the solution with one,the 5th order allying solution of equation(4.28) and the either side of accurate solution for $\beta = 2$.

		$\beta = 2$		
ξ	ω	y(NSTIM)	y(accurate)	$\lvert y_{NSTIM} - y_{accurate} \rvert$
.2	0.3	0.212181	0.212181	0
.4	0.5	0.483375	0.483375	0
.6	0.7	0.87309	0.87309	0
.8	0.9	1.45697	1.45697	0

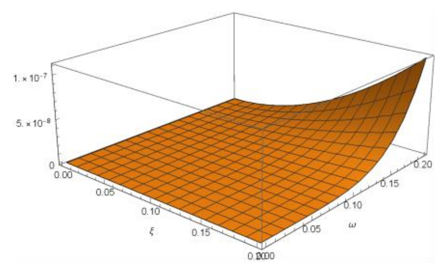
Table 5: analyze the solution with one,the 5th order allying solution of equation(4.37) and the either side of accurate solution for $\beta = 2$.

$\beta = 2$				
ξ	ω	y(NSTIM)	y(accurate)	$\lvert y_{NSTIM} - y_{accurate} \rvert$
.2	0.3	0.306193	0.363853	0
.4	0.5	0.752517	0.752517	0
.6	0.7	1.3787	1.3787	0
.8	0.9	2.39375	2.39375	0

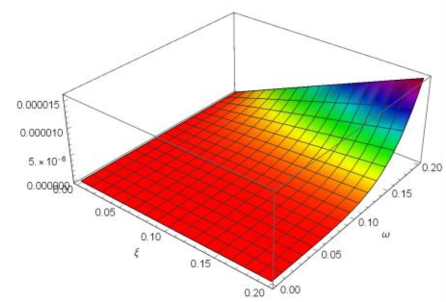
ISSN 2063-5346

Table 6: analyze the solution with one,the 5th order allying solution of equation(4.46) and the either side of accurate solution for $\beta = 2$.

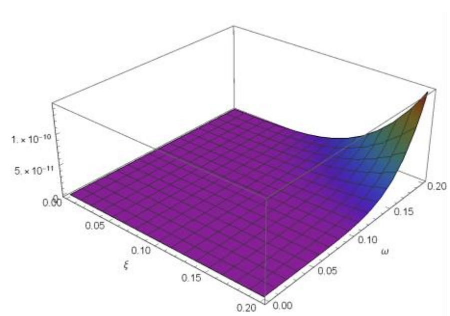
		$\beta = 2$		
ξ	ω	y(NSTIM)	y(accurate)	$\lvert y_{NSTIM} - y_{accurate} \rvert$
.2	0.3	0.363853	0.363853	0
.4	0.5	0.752517	0.752517	0
.6	0.7	1.3787	1.3787	0
.8	0.9	2.39375	2.39375	0



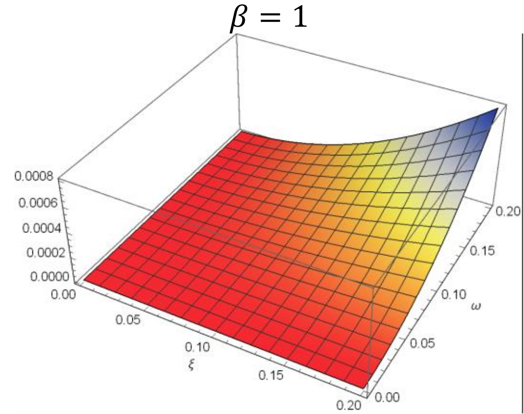
(a) $\beta = 1$



$\beta = 1$



$\beta = 1$



$\beta = 2$

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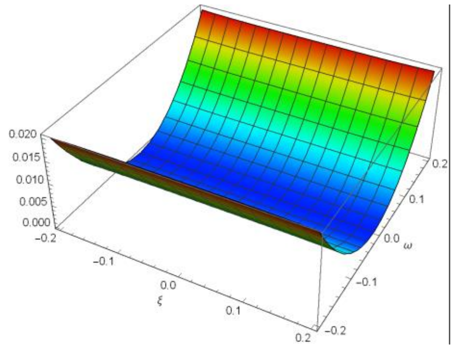

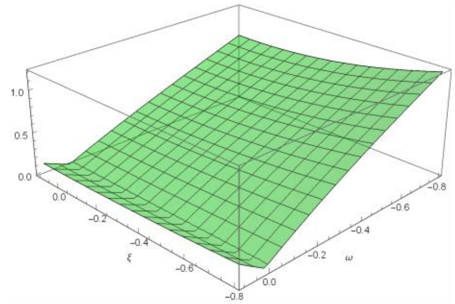

$\beta = 2$ $\beta = 2$

Figure 7: The absolute error $|y_{NSTIM} - y_{accurate}|$ of equation (4.46) at 5^{th} order for $\beta = 1, 2$

Remark:7 Figures 31,32, and 33 depict the absolute error between allying and accurate solutions for β=1, whereas Figures 34,35 and 36 depict the absolute error between allying and accurate solutions for β=2. By comparison, it is clear that by computing additional terms, the efficiency and accuracy of this method (NSTIM)can be greatly improved. We use a few terms in this post. The precision of the estimated solution will be substantially enhanced if we employ additional terms. As a result, the recommended method for solving the linear differential equation is precise and efficient.

Conclusion

The new Sumudu transform iterative approach was successfully employed in this research to get an allying solution for the time-fractional heat-like and wave-like equations. The New Sumudu Transform Iterative Method (NSTIM) combines the New Iterative Method (NIM) and the Sumudu transform to achieve accurate and allying analytical solutions for time-fractional heat and wave equations. The numerical findings reveal that the Sumudu transform iterative method is more efficient and accurate than previous methods, requiring less calculation.

Conflict of interests

The authors declare that there is no conflict of interests.

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