Applications of $\alpha(gg)^*$-Closed Sets in Topological Spaces

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ABSTRACT

The object of this paper is to introduce and study topological properties of $\alpha(gg)^*$-closure, $\alpha(gg)^*$-interior, $\alpha(gg)^*$-derived sets, $\alpha(gg)^*$- border, $\alpha(gg)^*$-frontier and $\alpha(gg)^*$-exterior using the concept of $\alpha(gg)^*$-closed sets.

Keywords and phrases: $\alpha(gg)^*$-closure, $\alpha(gg)^*$-interior, $\alpha(gg)^*$-derived sets, $\alpha(gg)^*$- border, $\alpha(gg)^*$-frontier, $\alpha(gg)^*$-exterior.

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1. INTRODUCTION

In 1837, John Bennedict Listing introduced the term topology. Topology is relatively a new branch of mathematics, most of the research in topology has been done since 1900. It is widely used in many branches of mathematics. General topology usually considers local properties of spaces and is closely related to analysis. In 1965, Njastad[6] introduced the notion of α-open sets in topological spaces. In 1970, Norman Levine[5] introduced the concept of generalized closed sets(briefly g-closed sets). In 2022, authors introduced the class of $\alpha(gg)^*$-closed sets[1]. For these sets, we introduce the notion of $\alpha(gg)^*$-closure, $\alpha(gg)^*$-interior, $\alpha(gg)^*$-derived sets, $\alpha(gg)^*$-border, $\alpha(gg)^*$-frontier and $\alpha(gg)^*$-exterior and discussed some of their properties. Further, we investigated their relations with each other.

2. PRELIMINARIES

Throughout this paper (X,τ) (simply X) represents the topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space (X,τ), $Cl(A)$, $Int(A)$ denote the closure and interior of A, respectively. In this section we recall some basic definitions.

Definition 2.1. [6] A subset A of a topological space X is called α-open if $A \subseteq intclint(A)$ and α-closed if $clintcl(A) \subseteq A$.

Definition 2.2. [3] A subset A of a topological space X is called regular-open if $A = intcl(A)$ and regular-closed if $clint(A) = A$.

Definition 2.3. [7] A subset A of a topological space X is called regular semi-open if there is a regular open set U such that $U \subseteq A \subseteq cl(U)$.

Definition 2.4. [5] A subset A of a topological space X is called generalized-closed (briefly g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Definition 2.5. [2] A subset A of a topological space X is called generalization of generalized closed (briefly gg-closed) if $gcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semi-open.

Definition 2.6. [4] A subset A of a topological space X is called generalization of generalized star closed (briefly $(gg)^*$-closed) if $rcl(A) \subseteq U$ whenever $A \subseteq U$ and U is gg-open.

Definition 2.7. [1] A subset A of a topological space X is called alpha generalization of generalized star closed (briefly $\alpha(gg)^*$-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(gg)^*$-open.

Lemma 2.8. [1]

(i) Every closed set is $\alpha(gg)^*$-closed.
(ii) Every α-closed set is $\alpha(gg)^*$-closed.

3. $\boldsymbol{\alpha(gg)^*}$-CLOSURE

Definition 3.1. The **alpha generalization of generalized star closure** of a subset A (briefly $\alpha(gg)^*cl(A)$) in (X,τ) is defined as follows:

$$\alpha(gg)^*cl(A) = \cap\{F: A \subseteq F \text{ and } F \text{ is } \alpha(gg)^*\text{- closed in } (X,\tau)\}$$

(or)

$$\alpha(gg)^*cl(A) = \cap\{F: A \subseteq F, F \in \alpha(GG)^*C(X)\}$$

Theorem 3.2. If A and B are subsets of space (X,τ) then
(i) $\alpha(gg)^*cl(X) = X$, $\alpha(gg)^*cl(\emptyset) = \emptyset$
(ii) $A \subseteq \alpha(gg)^*cl(A)$
(iii) If B is any $\alpha(gg)^*$-closed set containing A, then $\alpha(gg)^*cl(A) \subseteq B$.
(iv) If $A \subseteq B$ then $\alpha(gg)^*cl(A) \subseteq \alpha(gg)^*cl(B)$
(v) $\alpha(gg)^*cl(A \cup B)= \alpha(gg)^*cl(A) \cup \alpha(gg)^*cl(B)$

Proof.

(i) By Definition: 3.1, the result is evident.

(ii)By Definition: 3.1, it is obvious that $A \subseteq \alpha(gg)^*cl(A)$.

(iii) Let B be any $\alpha(gg)^*$closed set containing A. Since $\alpha(gg)^*cl(A)$ is the intersection of all $\alpha(gg)^*$-closed set containing A, $\alpha(gg)^*cl(A)$ is contained in every $\alpha(gg)^*$-closed set containing A. Therefore, in particular $\alpha(gg)^*cl(A) \subseteq B$.

(iv) Let A and B be subsets of (X,τ) such that $A \subseteq B$. By Definition: 3.1, $\alpha(gg)^*cl(B) = \cap \{F: B \subseteq F$ and F is $\alpha(gg)^*$-closed in $(X,\tau)\}$. That is, $\alpha(gg)^*cl(B) = \cap \{F: B \subseteq F, F \in \alpha(GG)^*WC(X)\}$. Since $B \subseteq F$ and F is $\alpha(gg)^*$-closed, F is a $\alpha(gg)^*$-closed set containing B. Then by (iii), $\alpha(gg)^*cl(B) \subseteq F$. Since $A \subseteq B, A \subseteq B \subseteq F$ and F is $\alpha(gg)^*$-closed, by (iii), $\alpha(gg)^*cl(A) \subseteq F$. Now, $\alpha(gg)^*cl(A) = \cap \{F: B \subseteq F \; and \; F \in \alpha(GG) * C(X)\} = \alpha(gg)^*cl(B)$. Therefore, we get $\alpha(gg)^*cl(A) \subseteq \alpha(gg)^*cl(B)$.

(v) Let A and B be subsets of X. It is clear that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. From (iv), $\alpha(gg)^*cl(A) \subseteq \alpha(gg)^*cl(A \cup B)$ and$\alpha(gg)^*cl(B) \subseteq \alpha(gg)^*cl(A \cup B)$.
Hence $\alpha(gg)^*cl(A) \cup \alpha(gg)^*cl(B) \subseteq \alpha(gg)^*cl(A \cup B) \longrightarrow (1)$
Next to prove that $\alpha(gg)^*cl(A \cup B) \subseteq \alpha(gg)^*cl(A) \cup \alpha(gg)^*cl(B)$. Suppose that $x \notin \alpha(gg)^*cl(A) \cup \alpha(gg)^*cl(B)$ then there exists $\alpha(gg)^*$-closed sets A_1 and B_1 such that $A \subseteq A_1 \; and \; B \subseteq B_1$ also $x \notin A_1 \cup B_1$. Hereby, we have $A \cup B \subseteq A_1 \cup B_1$ and $A_1 \cup B_1$ is a $\alpha(gg)^*$-closed set such that $x \notin A_1 \cup B_1$. Then by (iii), $\alpha(gg)^*cl(A \cup B) \subseteq A_1 \cup B_1$. Since $x \notin A_1 \cup B_1$, $x \notin \alpha(gg)^*cl(A \cup B)$.Therefore, we get $\alpha(gg)^*cl(A \cup B) \subseteq \alpha(gg)^*cl(A) \cup \alpha(gg)^*cl(B) \longrightarrow (2)$
Combining (1) and (2), we get $\alpha(gg)^*cl(A \cup B) = \alpha(gg)^*cl(A) \cup \alpha(gg)^*cl(B)$.

Lemma 3.3. If $A \subseteq X$ is $\alpha(gg)^*$-closed set then $\alpha(gg)^*cl(A) = A$.

Proof. Let $A \subseteq X$ be $\alpha(gg)^*$-closed set. To prove that $\alpha(gg)^*cl(A) = A$. By Theorem 4.2 (ii), $A \subseteq \alpha(gg)^*cl(A) \longrightarrow (1)$
Clearly, $A \subseteq A$. Since $A \subseteq A$ and A is $\alpha(gg)^*$-closed, by Theorem 4.2(iii), we get $\alpha(gg)^*cl(A) \subseteq A \longrightarrow (2)$
Combining (1) and (2) we get $\alpha(gg)^*cl(A) = A$.

Lemma: 3.4. For every $x \in X, x \in \alpha(gg)^*cl(A)$ if and only if $K \cap A \neq \emptyset$ for every $\alpha(gg)^*$-open set K containing the point x.

Proof. Let $x \in \alpha(gg)^*cl(A)$. To prove that $K \cap A \neq \emptyset$, for every $\alpha(gg)^*$-open set K containing the point x. Suppose that there exists an $\alpha(gg)^*$-open set K containing the point x such that $K \cap A = \emptyset$. Since $K \cap A = \emptyset$, $A \subseteq X\backslash K$. Then by Theorem 3.2(iv), $\alpha(gg)^*cl(A) \subseteq \alpha(gg)^*cl(X\backslash K)$. Since K is $\alpha(gg)^*$-open, $X\backslash K$ is $\alpha(gg)^*$-closed. Then by Lemma 3.3, $\alpha(gg)^*cl(X\backslash K) = X\backslash K$. Since $\alpha(gg)^*cl(A) \subseteq \alpha(gg)^*cl(X\backslash K) = X\backslash K$, $\alpha(gg)^*cl(A) \subseteq X\backslash K$**.** Thus $x \notin \alpha(gg)^*cl(A)$, which is a contradiction to our assumption. Hence $K \cap A \neq \emptyset$ for every $\alpha(gg)^*$-open set K containing the point x. Conversely , assume that $K \cap A \neq \emptyset$ for every $\alpha(gg)^*$-open set K containing the point x. To prove that $\in \alpha(gg)^*cl(A)$. Suppose that $x \notin \alpha(gg)^*cl(A)$. Then by Definition 3.1, there exists an $\alpha(gg)^*$-closed set F containing A such that $x \notin F$ and so $x \in X\backslash F$.Since $A \subseteq F$ and $x \in X\backslash F$, $(X\backslash F) \cap A = \emptyset$, which is a contradiction to our assumption. Thus, $x \in \alpha(gg)^*cl(A)$.

Lemma 3.5. If A and B are the subsets of a space (X,τ). Then $\alpha(gg)^*cl(A) = \alpha(gg)^*cl(\alpha(gg)^*cl(A))$.

Proof. From Theorem 3.2(ii), $A \subseteq \alpha(gg)^*cl(A)$. This implies that $\alpha(gg)^*cl(A) \subseteq \alpha(gg)^*cl(\alpha(gg)^*cl(A))$. Now, let us take $x \in \alpha(gg)^*cl(\alpha(gg)^*cl(A))$. Suppose that K

is an $\alpha(gg)^*$-open set containing x. Then by Lemma 3.4, $K \cap \alpha(gg)^*cl(A) \neq \emptyset$. Let $y \in K \cap \alpha(gg)^*cl(A)$. Then $y \in K$ and $y \in \alpha(gg)^*cl(A)$. Again by using Lemma 3.4, $K \cap A \neq \emptyset$ and so $x \in \alpha(gg)^*cl(A)$. Thus, $\alpha(gg)^*cl(\alpha(gg)^*cl(A)) \subseteq \alpha(gg)^*cl(A)$. Therefore, $\alpha(gg)^*cl(A) = \alpha(gg)^*cl(\alpha(gg)^*cl(A))$.

Theorem 3.6. If A and B are subsets of a space X then $\alpha(gg)^*cl(A \cap B) \subseteq \alpha(gg)^*cl(A) \cap \alpha(gg)^*cl(B)$.

Proof. Let A and B be subsets of a space X. It is clear that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then by Theorem 3.2(iv), the proof follows.

Remark 3.7. The converse of the above theorem need not be true in general as seen from the following example.

Example 3.8. Let $X = \{a.b.c,d\}$ with topology $\tau = \{\emptyset, \{a\}, \{c,d\}, \{a,c,d\}, X\}$. Let $A = \{a,b\}$, $B = \{c,d\}$ then $A \cap B = \{\emptyset\}$. Here $\alpha(gg)^*cl(A) = \{a,b\}, \alpha(gg)^*cl(B) = \{b,c,d\}$ and $\alpha(gg)^*cl(A) \cap \alpha(gg)^*cl(B) = \{b\}$.But $\alpha(gg)^*cl(A \cap B) = \{\emptyset\}$. Therefore, $\alpha(gg)^*cl(A) \cap \alpha(gg)^*cl(B) \nsubseteq \alpha(gg)^*cl(A \cap B)$.

Theorem 3.9. Let A be a subset of X. Then
(i)$\alpha(gg)^*cl(A) \subseteq cl(A)$
(ii) $\alpha(gg)^*cl(A) \subseteq \alpha cl(A)$

Proof.
(i) By the definition of closure, $Cl(A) = \cap \{F : A \subseteq F, F \in C(X)\}$. Since every closed set is $\alpha(gg)^*$-closed, $A \subseteq F \in C(X) \Longrightarrow A \subseteq F \in \alpha(GG)^*C(X)$**.** Since F is a $\alpha(gg)^*$-closed set containing A, by Theorem 4.2(iii), $\alpha(gg)^*cl(A) \subseteq F$. This implies that $\alpha(gg)^*cl(A) \subseteq \cap \{F : A \subseteq F, F \in C(X)\} = Cl(A)$.Thus, $\alpha(gg)^*cl(A) \subseteq cl(A)$.

(ii) From the definition of α-closure, $\alpha cl(A) = \cap \{F : A \subseteq F, F \in \alpha C(X)\}$. Since every α-closed set is $\alpha(gg)^*$-closed, $A \subseteq F \in \alpha C(X) \Longrightarrow A \subseteq F \in G^*\alpha WC(X)$. Since F is a $\alpha(gg)^*$-closed set containing A, by Theorem 4.2(iii), $\alpha(gg)^*cl(A) \subseteq F$. This implies that $\alpha(gg)^*cl(A) \subseteq \cap \{F : A \subseteq F, F \in \alpha C(X)\} = \alpha cl(A)$. Thus, $\alpha(gg)^*cl(A) \subseteq \alpha cl(A)$.

Remark 3.10. The converse of the above theorem need not be true in general as seen from the following example.

Example 3.11. Let $X = \{a.b.c,d\}$ with topology $\tau = \{\emptyset, \{a\}, \{c,d\}, \{a,c,d\}, X\}$.

(i)$Let\ A = \{c\}$.Then $cl(A) = \{b,c,d\}$ and $\alpha(gg)^*cl(A) = \{b,c\}$.
Thus $cl(A) \nsubseteq \alpha(gg)^*cl(A)$.

(ii)Also, here $\alpha(gg)^*cl(A) = \{b,c\}$ and $\alpha cl(A) = \{b,c,d\}$. Thus, $\alpha cl(A) \nsubseteq \alpha(gg)^*cl(A)$.

4. $\boldsymbol{\alpha(gg)^*}$-INTERIOIR

Definition 4.1. For a subset A of (X,τ) α(gg)*-interior of A is defined as $\alpha(gg)^*int(A) = \cup\{G: G \subseteq A$ and $G \in \alpha(GG)^*O(X)\}$.

Theorem 4.2. Let A and B be subsets of a space X. Then

(i) $\alpha(gg)^*int(X) = X, \alpha(gg)^*int(\emptyset) = \emptyset$.
(ii) $\alpha(gg)^*int(A) \subseteq A$.
(iii) If B is any $\alpha(gg)^*$-open set contained in A, then $B \subseteq \alpha(gg)^*int(A)$.
(iv) If $A \subseteq B$ then $\alpha(gg)^*int(A) \subseteq \alpha(gg)^*int(B)$.
(v) $\alpha(gg)^*int(A) = \alpha(gg)^*int(\alpha(gg)^*int(A))$.
(vi) $\alpha(gg)^*int(A \cap B) = \alpha(gg)^*int(A) \cap \alpha(gg)^*int(B)$.

Proof.

By the Definition 4.1, (i) and (ii) are obvious.

(iii) Let B be any $\alpha(gg)^*$-open set contained in A. Let $x \epsilon B$, where B is an $\alpha(gg)^*$-open set that is contained in A. Therefore, x is an $\alpha(gg)^*$interior of A. That is, $x \epsilon\ \alpha(gg)^*int(A)$. Hence $B \subseteq \alpha(gg)^*int(A)$.

The proofs of (iv) and (vi) are similar to the proofs of Theorem: 3.2 (iv), (v) and proof of (v) is similar to that of Lemma:3.5.

Theorem 4.3. If a subset A of X is $\alpha(gg)^*$-open then $\alpha(gg)^*int(A) = A$.

Proof. Let a subset A of X is $\alpha(gg)^*$-open. To prove that $\alpha(gg)^*int(A) = A$. From Theorem 4.2 (ii), $\alpha(gg)^*int(A) \subseteq A$. Also, since A is an $\alpha(gg)^*$open set contained in A, by Theorem 4.2 (iii), $A \subseteq \alpha(gg)^*int(A)$. Thus, we get $\alpha(gg)^*int(A) = A$.

Theorem 4.4. If A and B are subsets of a space X then $\alpha(gg)^*int(A) \cup \alpha(gg)^*int(B) \subseteq \alpha(gg)^*int(A \cup B)$.

Proof. For subsets A and B of a space X, it is clear that $A \subseteq A \cup B$, then by Theorem 4.2(iv), $\alpha(gg)^*int(A) \subseteq \alpha(gg)^*int(A \cup B)$. Similarly, $B \subseteq A \cup B$ implies $\alpha(gg)^*int(A) \subseteq \alpha(gg)^*int(A \cup B)$.Then $\alpha(gg)^*\text{int}(A) \cup \alpha(gg)^*int(B) \subseteq \alpha(gg)^*int(A \cup B)$.

Remark 4.5. The converse of the above theorem need not be true as seen from the following example.

Example 4.6. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, X\}$. Let $A = \{a. b\}$ and $B = \{c, d\}$. Then $\alpha(gg)^*int(A) = \{a\}$ and $\alpha(gg)^*int(B) = \{c, d\}$ and $\alpha(gg)^*int(A) \cup \alpha(gg)^*int(B) = \{a, c, d\}$.Here $A \cup B = X$.Then $\alpha(gg)^*int(A \cup B) = X$.Therefore, $\alpha(gg)^*int(A \cup B) \nsubseteq \alpha(gg)^*int(A) \cup \alpha(gg)^*int(B)$.

Theorem 4.7. If A is a subset of X then

(i) $int(A) \subseteq \alpha(gg)^*int(A)$
(ii) $\alpha int(A) \subseteq \alpha(gg)^*int(A)$.

Proof:

(i) Let $x \in int(A) \Rightarrow x \in \cup \{G: G$ is open, $G \subseteq A\}$. Since every open set is $\alpha(gg)^*$-open, $x \in \cup \{G: G$ is $\alpha(gg)^*$-open, $G \subseteq A\} \Rightarrow x \in \alpha(gg)^*int(A)$. Thus, $x \in int(A)$ implies $x \in \alpha(gg)^*int(A)$. That is, $int(A) \subseteq \alpha(gg)^*int(A)$.

(ii) Let $x \in \alpha int(A) \Rightarrow x \in \cup \{G: G$ is α-open, $G \subseteq A\}$. Since every α-open set is $\alpha(gg)^*$-open, $x \in \cup \{G: G$ is $\alpha(gg)^*$-open, $G \subseteq A\} \Rightarrow x \in \alpha(gg)^*int(A)$. Thus, $x \in \alpha int(A)$ implies $x \in \alpha(gg)^*int(A)$.That is, $\alpha int(A) \subseteq \alpha(gg)^*int(A)$.

Remark 4.8. Containment relation can be proper in the above theorem as seen from the following example.

Example 4.9. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, X\}$. If $A = \{a, c\}$ then $int(A) = \{a\}, \alpha int(A) = \{a\}$ and $\alpha(gg)^*int(A) = \{a. c\}$. Therefore, $int(A) \subset \alpha(gg)^*int(A)$ and $\alpha int(A) \subset \alpha(gg)^*int(A)$.

Theorem 4.10.

Let A be a subset of X. Then

(i) $X\setminus \alpha(gg)^*int(A) = \alpha(gg)^*cl(X\setminus A)$
(ii) $X\setminus\alpha(gg)^*cl(X\setminus A) = \alpha(gg)^*int(A)$
(iii) $X\setminus \alpha(gg)^*int(X\setminus A) = \alpha(gg)^*cl(A)$
(iv) $X\setminus\alpha(gg)^*cl(A) = \alpha(gg)^*int(X\setminus A)$.

Proof.

(i) Let $x \in X\setminus \alpha(gg)^*int(A)$. Here x is not in $\alpha(gg)^*int(A)$. That is, for every $\alpha(gg)^*$open set G containing x such that $G \nsubseteq A$.This implies that every $\alpha(gg)^*$open set G containing x intersects $(X\setminus A)$. That is, $G \cap (X\setminus A) \neq \emptyset$. Then by Lemma 3.4, $x \in \alpha(gg)^*cl(X\setminus A)$.Therefore, $X\setminus \alpha(gg)^*int(A) \subseteq \alpha(gg)^*cl(X\setminus A)$.Now, let $x \in \alpha(gg)^*cl(X\setminus A)$. Then every x belongs to $\alpha(gg)^*$open set G containing x intersects $X\setminus A$. That is, $G \cap (X\setminus A) \neq \emptyset$. Then for every $\alpha(gg)^*$open set G containing x such that $G \nsubseteq A$. From this, it is clear that x is not in $\alpha(gg) * int(A)$. That is, $x \in X\setminus \alpha(gg)^*int(A)$ and hence we get $\alpha(gg)^*cl(X\setminus A) \subseteq X\setminus\alpha(gg)^*int(A)$. Thus, we get $X\setminus \alpha(gg)^*int(A) = \alpha(gg)^*cl(X\setminus A)$.

(ii) Taking the complement of (i), we get $X \setminus (X \setminus \alpha(gg)^*int(A)) = X \setminus \alpha(gg)^*cl(X \setminus A)$. Thus, $\alpha(gg)^*int(A) = X\setminus\alpha(gg)^*cl(X\setminus A)$.

(iii) Replacing A by $X \setminus A$ in (i), we get $X \setminus (\alpha(gg)^*int(X \setminus A)) = \alpha(gg)^*cl(X \setminus(X \setminus A))$. Thus, $X \setminus \alpha(gg)^*int(X \setminus A) = \alpha(gg)^*cl(A)$.

(iv) Taking the complement of (iii), we get $X \setminus (X \setminus\alpha(gg)^*int(X \setminus A)) = X \setminus\alpha(gg)^*cl(A)$. Thus, $\alpha(gg)^*int(X \setminus A) = X \setminus \alpha(gg)^*cl(A)$.

5. $\boldsymbol{\alpha(gg)^*}$-DERIVED SETS

Definition 5.1. Let A be a subset of a topological space X, then a point $x \in A$ is said to be an $\alpha(gg)^*$- limit point of A, if for each $\alpha(gg)^*$- open set U containing x, $U \cap (A\backslash\{x\}) \neq \emptyset$.

The set of all $\alpha(gg)^*$- limit points of A is called $\alpha(gg)^*$-derived set of A and is denoted by $\alpha(gg)^*D(A)$.

Theorem 5.2. For $A, B \subseteq (X,\tau)$, the following results hold:

(i) $\alpha(gg)^*D(A) \subseteq D(A)$, where $D(A)$ is the derived set of A.

(ii) $A \subseteq B \Rightarrow \alpha(gg)^*D(A) \subseteq \alpha(gg)^*D(B)$.

(iii)$\alpha(gg)^*D(A) \cup \alpha(gg)^*D(B) \subseteq \alpha(gg)^*D(A \cup B)$ and $\alpha(gg)^*D(A \cap B) \subseteq \alpha(gg)^*D(A) \cap \alpha(gg)^*D(B)$.

(iv) $\alpha(gg)^*D(\alpha(gg)^*D(A))\backslash A \subseteq \alpha(gg)^*D(A)$.

(v) $\alpha(gg)^*D(A \cup \alpha(gg)^*D(A)) \subseteq A \cup \alpha(gg)^*D(A)$.

Proof:

(i) It is sufficient to observe that every open set is $\alpha(gg)^*$-open

(ii) Let $x \in \alpha(gg)^*D(A)$ and U be an $\alpha(gg)^*$- open set U containing x, then $U \cap (A\backslash\{x\}) \neq \emptyset$.Since $A \subseteq B, U \cap (B\backslash\{x\}) \neq \emptyset \Rightarrow x \in \alpha(gg)^*D(B)$.Therefore, $\alpha(gg)^*D(A) \subseteq \alpha(gg)^*D(B)$.

(iii) It follows from (ii).

(iv) Let $x \in \alpha(gg)^*D\big(\alpha(gg)^*D(A)\big)\backslash A$. Then there exists an $\alpha(gg)^*$- open set U containing x such that $U \cap (\alpha(gg)^*D(A)\{x\}) \neq \emptyset$. Let $y \in U \cap (\alpha(gg)^*D(A)\{x\})$. Then $y \in U$ and $y \in \alpha(gg)^* D(A)$, which implies that $U \cap (A\{y\}) \neq \emptyset$. Let $w \in U \cap (A\{y\})$.Then $w \in U$ and $w \in A$.Since $w \in A$ and $x \notin A, w \neq x$. So, $U \cap (A\backslash\{x\}) \neq \emptyset \Rightarrow x \in \alpha(gg)^* D(A)$. Therefore, $\alpha(gg)^*D(\alpha(gg)^*D(A))\backslash A \subseteq \alpha(gg)^*D(A)$.

(v) Let $x \in \alpha(gg)^*D(A \cup \alpha(gg)^*D(A))$. If x is in A the result follows. But if $x \in \alpha(gg)^*D(A \cup \alpha(gg)^*D(A))\backslash A$, then for any $\alpha(gg)^*$- open set U containing x, $U \cap (A \cup \alpha(gg)^*D(A)) \backslash\{x\}) \neq \emptyset$. Thus, $U \cap (A\backslash\{x\}) \neq \emptyset$ or $U \cap (\alpha(gg)^*D(A)) \backslash\{x\}) \neq \emptyset$. Now from (iv), $U \cap (A\backslash\{x\}) \neq \emptyset$. Hence, $x \in \alpha(gg)^*D(A)$. Therefore, in both the cases $\alpha(gg)^*D(A \cup \alpha(gg)^*D(A)) \subseteq A \cup \alpha(gg)^*D(A)$.

Theorem: 5.3 For a subset A in X, if A is $\alpha(gg)^*$- closed then $\alpha(gg)^*D(A) \subseteq A$.

Proof. Let A be an $\alpha(gg)^*$- closed set in X. If $x \notin A$, then $x \in X\backslash A$. Since $X\backslash A$ is $\alpha(gg)^*$-open, x is not an $\alpha(gg)^*$- limit point of A. This means that $x \notin \alpha(gg)^*D(A)$, as $(X\backslash A) \cap (A\backslash\{x\}) = \emptyset$. Therefore, $\alpha(gg)^*D(A) \subseteq A$.

Remark 5.4. The converse of the above theorem need not be true in general as seen from the example given below.

Example: 5.5 Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}, X\}$. If $A = \{a, b\}$ then $\alpha(gg)^*D(A) = \emptyset$. Therefore, $\emptyset \subseteq \{a, b\}$. That is, $\alpha(gg)^*D(A) \subseteq A$, but $A = \{a, b\}$ is not an $\alpha(gg)^*$- closed set.

Theorem: 5.6. For a subset A in X, if G is an $\alpha(gg)^*$- closed subset of A then $\alpha(gg)^*D(A) \subseteq G$.

Proof. By Theorem 5.2(ii), $A \subseteq G \Rightarrow \alpha(gg)^*D(A) \subseteq \alpha(gg)^*D(G)$. Since G is $\alpha(gg)^*$-closed, from Theorem: 5.3, $\alpha(gg)^*D(A) \subseteq G$. Therefore, $\alpha(gg)^*D(A) \subseteq G$.

6. $\alpha(gg)^*$-BORDER

Definition 6.1. For a subset A of a topological space X, $\alpha(gg)^*$-border of A is defined as

$\alpha(gg)^*Bd(A) = A \setminus \alpha(gg)^*Int(A)$.

Theorem 6.2. If $A \subseteq (X, \tau)$ then

(i) $\alpha(gg)^*Bd(A) \subseteq Bd(A)$, where $Bd(A)$ is the border of A
(ii)$A = \alpha(gg)^*int(A) \cup \alpha(gg)^*Bd(A)$
(iii) $\alpha(gg)^*int(A) \cap \alpha(gg)^*Bd(A) = \emptyset$
(iv)A is $\alpha(gg)^*$-open$\Leftrightarrow \alpha(gg)^*Bd(A) = \emptyset$
(v) $\alpha(gg)^*Bd(\alpha(gg)^*int(A)) = \emptyset$
(vi) $\alpha(gg)^*Bd(A) = A \cap \alpha(gg)^*cl(X \setminus A)$
(vii)$\alpha(gg)^*Bd(A) = \alpha(gg)^*D(X \setminus A)$

Proof:

(i) $\alpha(gg)^*Bd(A) = A \setminus \alpha(gg)^*int(A) \subseteq A \setminus int(A) = Bd(A)$. Therefore, $\alpha(gg)^*Bd(A) \subseteq Bd(A)$.

(ii) $\alpha(gg)^*int(A) \cup \alpha(gg)^*Bd(A) = \alpha(gg)^*int(A) \cup (A \setminus \alpha(gg)^*int(A)) = A$.

(iii) $\alpha(gg)^*int(A) \cap \alpha(gg)^*Bd(A) = \alpha(gg)^*int(A) \cap (A \setminus \alpha(gg)^*int(A)) = \emptyset$.

(iv)Suppose that A is $\alpha(gg)^*$-open in X. Then by Theorem: 4.3, $A = \alpha(gg)^*int(A)$. Therefore, $\alpha(gg)^*Bd(A) = A \setminus \alpha(gg)^*int(A) = \alpha(gg)^*int(A) \setminus \alpha(gg)^*int(A) = \emptyset$. Conversely, suppose that $\alpha(gg)^*Bd(A) = \emptyset$. Then $A \setminus \alpha(gg)^*int(A) = \emptyset \Rightarrow \alpha(gg)^*int(A) = A$. Hence by Theorem: 4.3, A is $\alpha(gg)^*$-open.

(v)Since $\alpha(gg)^*int(A)$ is $\alpha(gg)^*$-open, from (iv), $\alpha(gg)^*Bd(\alpha(gg)^*int(A)) = \emptyset$.

(vi) $\alpha(gg)^*Bd(A) = A \setminus \alpha(gg)^*int(A) = A \setminus (X \setminus \alpha(gg)^*cl(X \setminus A)) = A \cap \alpha(gg)^*cl(X \setminus A)$.

(vii) $\alpha(gg)^*Bd(A) = A \setminus \alpha(gg)^*int(A) = A \setminus (A \setminus \alpha(gg)^*D(X \setminus A)) = \alpha(gg)^*D(X \setminus A)$.

Theorem 6.3. If $A \subseteq X$ is $\alpha(gg)^*$-open then

(i) $\alpha(gg)^*int(\alpha(gg)^*Bd(A)) = \emptyset$
(ii) $\alpha(gg)^*Bd(\alpha(gg)^*Bd(A)) = \alpha(gg)^*Bd(A)$.

Proof. Let A be an $\alpha(gg)^*$-open subset of X. (i) $\alpha(gg)^*int(\alpha(gg)^*Bd(A)) = \alpha(gg)^*int(A\backslash \alpha(gg)^*int(A)) = \alpha(gg)^*int(A\backslash A) = \alpha(gg)^*int(\emptyset) = \emptyset$.

(ii) $\alpha(gg)^*Bd(\alpha(gg)^*Bd(A)) = \alpha(gg)^*Bd(A\backslash \alpha(gg)^*int(A)) = (A\backslash \alpha(gg)^*int(A))\backslash \alpha(gg)^*int(A\backslash \alpha(gg)^* int(A)) = (A\backslash \alpha(gg)^*int(A))\backslash \alpha(gg)^*int(\alpha(gg)^*Bd(A)) = (A\backslash\alpha(gg)^*int(A))\backslash \emptyset = \alpha(gg)^*Bd(A) \backslash \emptyset = \alpha(gg)^*Bd(A)$.

7. $\alpha(gg)^*$-FRONTIER

Definition 7.1. For a subset A of a topological space X, α(gg)*-frontier of A is defined as

$\alpha(gg)^*Fr(A) = \alpha(gg)^*Cl(A)\backslash \alpha(gg)^*Int(A)$.

Theorem:7.2. For a subset A of a topological space (X,τ),

(i) $\alpha(gg)^*Fr(A) \subseteq Fr(A)$, where $Fr(A)$ is the frontier of A.

(ii) $\alpha(gg)^*cl(A) = \alpha(gg)^*int(A) \cup \alpha(gg)^*Fr(A)$

(iii) $\alpha(gg)^*int(A) \cap \alpha(gg)^*Fr(A) = \emptyset$

(iv)$\alpha(gg)^*Bd(A) \subseteq \alpha(gg)^*Fr(A)$

(v) $\alpha(gg)^*Fr(A) \cup \alpha(gg)^*Bd(A) = \alpha(gg)^*cl(A)$

(vi) If A is $\alpha(gg)^*$-closed then $\alpha(gg)^*Fr(A) = \alpha(gg)^*Bd(A)$

(vii) $\alpha(gg)^*Fr(A) = \alpha(gg)^*cl(A) \cap \alpha(gg)^*cl(X\backslash A)$

(viii) $\alpha(gg)^*Fr(A) = \alpha(gg)^*Fr(X\backslash A)$

(ix) $\alpha(gg)^*Fr(A)$ is $\alpha(gg)^*$-closed

(x) $\alpha(gg)^*Fr(\alpha(gg)^*Fr(A)) \subseteq \alpha(gg)^*Fr(A)$

(xi) $\alpha(gg)^*Fr(\alpha(gg)^*int(A)) \subseteq \alpha(gg)^*Fr(A)$

(xii) $\alpha(gg)^*Fr(\alpha(gg)^*cl(A)) \supseteq \alpha(gg)^*Fr(A)$

(xiii) $\alpha(gg)^*int(A) = A\backslash \alpha(gg)^*Fr(A)$

(xiv) $\alpha(gg)^*Fr(X) = \emptyset = \alpha(gg)^*Fr(\emptyset)$

(xv)$\alpha(gg)^*cl(\alpha(gg)^*Fr(X\backslash A)) = \alpha(gg)^*Fr(A)$.

Proof:

(i) $\alpha(gg)^*Fr(A) = \alpha(gg)^*cl(A)\backslash \alpha(gg)^*int(A) \subseteq cl(A)\backslash int(A) = Fr(A)$.Thus, $\alpha(gg)^*Fr(A) \subseteq Fr(A)$.

(ii) $\alpha(gg)^*int(A) \cup \alpha(gg)^*Fr(A) = \alpha(gg)^*int(A) \cup (\alpha(gg)^*cl(A)\backslash \alpha(gg)^*int(A)) = \alpha(gg)^*cl(A)$

(iii) $\alpha(gg)^*int(A) \cap \alpha(gg)^*Fr(A) = \alpha(gg)^*int(A) \cap (\alpha(gg)^*cl(A)\setminus \alpha(gg)^*int(A)) = \emptyset$.

(iv)Let $x \in \alpha(gg)^*Bd(A). Then\ x \in A\setminus \alpha(gg)^*int(A) \subseteq \alpha(gg)^*cl(A)\setminus \alpha(gg)^*int(A) = \alpha(gg)^*Fr(A)$.Therefore, $\alpha(gg)^*Bd(A) \subseteq \alpha(gg)^*Fr(A)$.

(v) $\alpha(gg)^*Fr(A) \cup \alpha(gg)^*Bd(A) = (\alpha(gg)^*cl(A)\setminus \alpha(gg)^*int(A)) \cup (A\setminus \alpha(gg)^*int(A)) = A \cup \alpha(gg)^*cl(A) = \alpha(gg)^*cl(A)$

(vi) It follows from Lemma: 3.3

(vii)$\alpha(gg)^*Fr(A) = \alpha(gg)^*cl(A)\setminus \alpha(gg)^*int(A) = \alpha(gg)^*cl(A)\setminus(X\setminus \alpha(gg)^*cl(X\setminus A)) = \alpha(gg)^*cl(A) \cap \alpha(gg)^*cl(X\setminus A)$.

(viii) Replacing A by $X\setminus A$ in (vii), we get $\alpha(gg)^*Fr(X\setminus A) = \alpha(gg)^*cl(X\setminus A)\setminus \alpha(gg)^*cl(A) = \alpha(gg)^*Fr(A)$.

(ix) $\alpha(gg)^*cl\big(\alpha(gg)^*Fr(A)\big) = \alpha(gg)^*cl\big(\alpha(gg)^*cl(A) \cap \alpha(gg)^*cl(X\setminus A)\big) \subseteq \alpha(gg)^*cl\ (\alpha(gg)^*cl(A) \cap \alpha(gg)^*cl(\alpha(gg)^*cl(X\setminus A)) = \alpha(gg)^*cl(A) \cap \alpha(gg)^*cl(X\setminus A) = \alpha(gg)^*Fr(A)$.Hence $\alpha(gg)^*cl(\alpha(gg)^*Fr(A)) \subseteq \alpha(gg)^*Fr(A)$. Now by Theorem 3.2(ii), $\alpha(gg)^*Fr(A) \subseteq \alpha(gg)^*cl(\alpha(gg)^*Fr(A))$.Therefore, $\alpha(gg)^*cl(\alpha(gg)^*Fr(A)) = \alpha(gg)^*Fr(A)$.Thus, $\alpha(gg)^*Fr(A)$ is $\alpha(gg)^*$-closed.

(x)The proof follows from Definition: 7.1.

(xi) It follows from Definition: 7.1 and Theorem: 4.2(v).

(xii) It follows from Definition: 7.1 and Lemma: 3.5

(xiii)$A\setminus \alpha(gg)^*Fr(A) = A\setminus(\alpha(gg)^*cl(A)\setminus\alpha(gg)^*int(A)) = A \cap (X\setminus \alpha(gg)^*cl(A) \cup \alpha(gg)^*int(A)) = (A \cap (X\setminus \alpha(gg)^*cl(A)) \cup (A \cap \alpha(gg)^*int(A)) = \emptyset \cup (A \cap \alpha(gg)^*int(A)) = \alpha(gg)^*int(A)$.

(xiv)The proof is evident

(xv) $\alpha(gg)^*cl(\alpha(gg)^*Fr(X\setminus A)) = \alpha(gg)^*Fr(X\setminus A) = \alpha(gg)^*Fr(A)$

Theorem 7.3. If $A \subseteq X$ and A is both $\alpha(gg)^*$-closed and $\alpha(gg)^*$-open then $\alpha(gg)^*Fr(A) = \emptyset$.

Proof. The proof follows from Lemma: 3.3 and Theorem:4.3.

Theorem: 7.4. Let $A, B \subseteq X$. Then

(i) $\alpha(gg)^*Fr(A \cap B) = \alpha(gg)^*Fr(A) \cap \alpha(gg)^*Fr(B)$
(ii) $\alpha(gg)^*Fr(A \cup B) = \alpha(gg)^*Fr(A) \cup \alpha(gg)^*Fr(B)$

Proof:

(i) Let $x \in \alpha(gg)^*Fr(A \cap B) \Rightarrow x \in \alpha(gg)^*cl(A \cap B) \setminus \alpha(gg)^*int(A \cap B) \Rightarrow x \in \alpha(gg)^*cl(A \cap B) \Rightarrow x \in \alpha(gg)^*cl(A) \cap \alpha(gg)^*cl(B) \Rightarrow x \in$

$\alpha(gg)^*cl(A)$ and $\alpha(gg)^*cl(B) \Rightarrow x \in \alpha(gg)^*cl(A) \setminus \alpha(gg)^*int(A)$ and $x \in \alpha(gg)^*cl(B) \setminus \alpha(gg)^*int(B) \Rightarrow x \in \alpha(gg)^*Fr(A)$ and $x \in \alpha(gg)^*Fr(B) \Rightarrow x \in \alpha(gg)^*Fr(A) \cap \alpha(gg)^*Fr(B)$.Therefore, $\alpha(gg)^*Fr(A \cap B) \subseteq \alpha(gg)^*Fr(A) \cap \alpha(gg)^*Fr(B)$. Similarly, we can prove that $\alpha(gg)^*Fr(A) \cap \alpha(gg)^*Fr(B) \subseteq \alpha(gg)^*Fr(A \cap B)$.Thus $\alpha(gg)^*Fr(A \cap B) = \alpha(gg)^*Fr(A) \cap \alpha(gg)^*Fr(B)$.

(ii) Let $x \in \alpha(gg)^*Fr(A \cup B) \Rightarrow x \in \alpha(gg)^*cl(A \cup B) / \alpha(gg)^*int(A \cup B) \Rightarrow x \in \alpha(gg)^*cl(A \cup B) \Rightarrow x \in \alpha(gg)^*cl(A) \cup \alpha(gg)^*cl(B) \Rightarrow x \in \alpha(gg)^*cl(A)$ or $\alpha(gg)^*cl(B) \Rightarrow x \in \alpha(gg)^*cl(A) \setminus \alpha(gg)^*int(A)$ or $x \in \alpha(gg)^*cl(B) \setminus \alpha(gg)^*int(B) \Rightarrow x \in \alpha(gg)^*Fr(A)\ or\ x \in \alpha(gg)^*Fr(B) \Rightarrow x \in \alpha(gg)^*Fr(A) \cup \alpha(gg)^*Fr(B)$.Therefore, $\alpha(gg)^*Fr(A \cup B) \subseteq \alpha(gg)^*Fr(A) \cup \alpha(gg)^*Fr(B)$. Similarly, we can prove that $\alpha(gg)^*Fr(A) \cup \alpha(gg)^*Fr(B) \subseteq \alpha(gg)^*Fr(A \cup B)$.Thus $\alpha(gg)^*Fr(A \cup B) = \alpha(gg)^*Fr(A) \cup \alpha(gg)^*Fr(B)$.

8. $\boldsymbol{\alpha(gg)^*}$-EXTERIOR:

Definition: 8.1. For a subset A of a topological space X, $\alpha(gg)^*$-exterior of A is defined as $\alpha(gg)^*Ext(A) = \alpha(gg)^*Int(X \setminus A)$.

Theorem: 8.2. If $A, B \subseteq (X, \tau)$ then

(i) $\alpha(gg)^*Ext(A)$ is $\alpha(gg)^*$-open

(ii) $\alpha(gg)^*Ext(X) = \emptyset$; $\alpha(gg)^*Ext(\emptyset) = X$

(iii)$\alpha(gg)^*Ext(A) = X \setminus \alpha(gg)^*cl(A)$

(iv) $\alpha(gg)^*Ext(\alpha(gg)^*Ext(A)) = \alpha(gg)^*int(\alpha(gg)^*cl(A))$

(v) $A \subseteq B \Rightarrow \alpha(gg)^*Ext(A) \supseteq \alpha(gg)^*Ext(B)$

(vi) $\alpha(gg)^*Ext(A \cap B) \supseteq \alpha(gg)^*Ext(A) \cup \alpha(gg)^*Ext(B)$

(vii) $\alpha(gg)^*Ext(A \cup B) = \alpha(gg)^*Ext(A) \cap \alpha(gg)^*Ext(B)$

(viii) $\alpha(gg)^*Ext(A) = \alpha(gg)^*Ext(X \setminus \alpha(gg)^*Ext(A))$

(ix) $\alpha(gg)^*cl(A) \supseteq \alpha(gg)^*Ext(\alpha(gg)^*Ext(A))$

(x) $X = \alpha(gg)^*int(A) \cup \alpha(gg)^*Ext(A) \cup \alpha(gg)^*Fr(A)$

(xi) $\alpha(gg)^*int(A) \cap \alpha(gg)^*Fr(A) \cap \alpha(gg)^*Ext(A) = \emptyset$.

Proof:

(i) The proof follows from Definition: 8.1 and Theorem:4.2(v).

(ii) The proof is evident.

(iii) It follows from Theorem: 4.10(iv).

(iv) It follows from (iii) and Definition: 8.1.

(v) Since $A \subseteq B, X \setminus B \subseteq X \setminus A$. Now, $\alpha(gg)^* Ext(B) = \alpha(gg)^* int(X \setminus B) \subseteq \alpha(gg)^* int(X \setminus A) = \alpha(gg)^* Ext(A)$.

(vi) It follows from Definition: 8.1 and Theorem: 4.4.

(vii) It follows from Definition: 8.1 and Theorem: 4.2 (vi).

(viii) The proof is evident.

(ix) The proof follows from Definition: 8.1 and Theorem: 4.10(iii).

(x) It follows from Theorem: 7.2(ii) and (iii).

(xi) It follows from Theorem: 7.2(iii).

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