Section: Research Paper

SOME CHARACTERIZATIONS OF BETA HAT GENERALIZED CONTINUOUS FUNCTIONS IN GENERALIZED TOPOLOGICAL SPACES

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ABSTRACT
This study introduced and investigated μ-$\hat{\beta}g$-continuous, almost μ-$\hat{\beta}g$-continuous and weakly μ-$\hat{\beta}g$-continuous functions in generalized topological spaces. Properties, characterizations and relationships among these functions are also considered. Hereafter, it has been proven that a μ-continuous function is μ-$\hat{\beta}g$-continuous. Moreover, a μ-$\hat{\beta}g$-continuous function is almost μ-$\hat{\beta}g$-continuous and an almost μ-$\hat{\beta}g$-continuous function is a weakly μ-$\hat{\beta}g$-continuous function.

Keywords: *generalized topological spaces, beta hat generalized, continuous functions*

1. INTRODUCTION

Since pure mathematics gained importance, mathematicians worldwide have introduced various concepts related to sets. Among these, the closed set holds significant importance in the field of topology. Levine [1] introduced generalized closed set, its set properties, closed and open maps, compactness, and normal and separation axioms. More expansions in general topology such as beta hat generalized closed (briefly $\hat{\beta}g$-closed) set, K. Kannan and N. Nagaveni [2]. More so, Császár [3] introduced the concept of generalized topological spaces (briefly GTS) and extended on the μ-$\hat{\beta}g$-closed sets to GTS.

On the other hand, Duangphui et al. [20] defined the concept of $(\mu,\mu')^{(m,n)}$-continuous functions in BGTS and some of their properties are introduced and investigated. Also, Baculta et al. [11] defined the $\mu^{(m,n)}$-rg^*b continuous, almost $\mu^{(m,n)}$-rg^*b continuous and weakly $\mu^{(m,n)}$-rg^*b continuous.

In this paper, beta hat generalized continuous functions are investigated in GTS.

2. On μ-$\hat{\beta}g$-CONTINUOUS FUNCTIONS IN GTS

Here we characterize μ- β g-continuous functions.

Definition 2.1 A function $f: (X, \mu_X) \rightarrow (Y, \mu_Y)$ is said to be:

(i.) *μ-$\hat{\beta}g$-continuous at a point $x \in X$* if for each μ_Y-open set V containing $f(x)$, there exists a μ_X-$\hat{\beta}g$-open set U containing x such that $f(U) \subseteq V$.

(ii.) *μ-$\hat{\beta}g$-continuous* if f is *μ-$\hat{\beta}g$-continuous* at every point $x \in X$.

Example 2.2 Let $X = \{a, b, c\}$ and $Y = \{u, v\}$. Consider the generalized topologies $\mu_X = \{\varnothing, \{a\}, \{a,b\}\}$ and $\mu_Y = \{\varnothing, \{u\}\}$. Thus the μ_X-closed sets in X are X, $\{b,c\}$ and $\{c\}$. On the other hand, the μ_Y-closed sets in Y are Y and $\{v\}$.

Now, consider the following:

set A in X	$c_\mu(A)$	$i_\mu(c_\mu(A))$	$c_\mu(i_\mu(c_\mu(A)))$	μ-open set U *s.t.* $A \subseteq U$
$\varnothing$	$\{c\}$	$\varnothing$	$\{c\}$	all μ-open set
X	X	$\{a,b\}$	X	*none*
$\{a\}$	X	$\{a,b\}$	X	$\{a\}, \{a,b\}$
$\{b\}$	$\{b,c\}$	$\varnothing$	$\{c\}$	$\{a,b\}$
$\{c\}$	$\{c\}$	$\varnothing$	$\{c\}$	*none*
$\{a,b\}$	X	$\{a,b\}$	X	$\{a,b\}$
$\{a,c\}$	X	$\{a,b\}$	X	*none*
$\{b,c\}$	$\{b,c\}$	$\varnothing$	$\{c\}$	*none*

Thus, the μ_X-$\hat{\beta}g$-closed sets in X are $\{X, \{c\}, \{a,c\}, \{b,c\}$. It follows that μ_X-$\hat{\beta}g$-open sets in X are $\varnothing$, $\{a,b\}$, $\{b\}$, $\{a\}$.

Let $f: (X, \mu_X) \to (Y, \mu_Y)$ be defined by $f(\{a\}) = f(\{b\}) = \{u\}$ and $f(\{c\}) = \{v\}$.

(i.) Consider $a \in X$. Note that $\{u\}$ is the only μ_Y-open set containing $f(\{a\})$, that is $f(\{a\}) = \{u\} \subseteq \{u\}$, and there exists a μ_X-$\hat{\beta}g$-open set $\{a\}$ such that $f(\{a\}) = \{u\} \subseteq \{u\}$. Thus f is a μ-$\hat{\beta}g$-continuous at $a \in X$.

(ii.) Now, let $b \in X$. Observe that $\{u\}$ is the only μ_Y-open set containing $f(\{b\})$, that is $f(\{b\}) = \{u\} \subseteq \{u\}$, and there exists a μ_X-$\hat{\beta}g$-open set $\{b\}$ such that $f(\{b\}) = \{u\} \subseteq \{u\}$. Thus f is a μ-$\hat{\beta}g$-continuous at $b \in X$.

(iii.) Finally, let $c \in X$. Notice that there is no μ_Y-open set containing $f(\{c\} = \{v\}$ and so it is vacuously satisfied. Thus f is a μ-$\hat{\beta}g$-continuous at $c \in X$.

Since, f is *μ-$\hat{\beta}g$-continuous* at points a, b, and c, it follows that f is *μ-$\hat{\beta}g$-continuous* by Definition 1.7.7 (ii).

The next remark follows from Definition 2.1.

Remark 2.3 *Every μ-continuous function is μ-$\hat{\beta}g$-continuous but the converse is not true.*

Theorem 2.4. *For a function $f: (X, \mu) \rightarrow (Y, \nu)$, the following properties are equivalent:*

(i) *f is μ-$\hat{\beta}g$-continuous;*

(ii) *$f^{-1}(V) = \hat{\beta}gi_\mu(f^{-1}(V))$ for every $V \in \nu$;*

(iii) *$f^{-1}(i_\mu(f^{-1}(B)) \subseteq \hat{\beta}gi_\mu(f^{-1}(B))$ for every $B \subseteq Y$, and;*

(iv) *$\hat{\beta}gc_\mu(f^{-1}(F)) = f^{-1}(F)$ for every ν-closed subset F of Y.*

Proof: Let $f: (X, \mu) \rightarrow (Y, \nu)$ be a function and let $x \in X$.

(i) $\Leftrightarrow$ (ii) Let $V \in \nu$ and $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is μ-$\hat{\beta}g$-*continuous* at x, there exists a μ-$\hat{\beta}g$-open set U containing x such that $f(U) \subseteq V$. Hence, $x \in U \subseteq f^{-1}(V)$. This implies that $x \in \hat{\beta}gi_\mu(f^{-1}(V))$. Thus, $f^{-1}(V) \subseteq \hat{\beta}gi_\mu(f^{-1}(V))$. Since $\hat{\beta}gi_\mu(f^{-1}(V)) \subseteq f^{-1}(V)$, (ii) follows.

Conversely, let $x \in X$ and V be a ν-open set in Y with $f(x) \in V$. By (ii), $f^{-1}(V) = \hat{\beta}gi_\mu(f^{-1}(V))$. Since $x \in f^{-1}(V)$, $x \in \hat{\beta}gi_\mu(f^{-1}(V))$. This implies that there exists a μ-$\hat{\beta}g$-open set U with $x \in U \subseteq (f^{-1}(V)$. Thus $f(U) \subseteq V$. Therefore, f is μ-$\hat{\beta}g$-*continuous* at x. Since x is arbitrary, f is μ-$\hat{\beta}g$-*continuous*.

(ii) $\Longrightarrow$ (iii) Let $B \subseteq Y$. Since $i_\nu(B)$ is a ν - open set in Y, by (ii) we have $f^{-1}(i_\nu(B)) = \hat{\beta}gi_\mu(f^{-1}(i_\nu(B))) \subseteq \hat{\beta}gi_\mu(f^{-1}(B))$. Therefore, $f^{-1}(i_\nu(B)) \subseteq \hat{\beta}gi_\mu(f^{-1}(B))$.

(iii) $\Longrightarrow$ (iv) Let F be a ν-closed subset of Y. Then,

$$
\begin{aligned}
X \setminus f^{-1}(F) &= f^{-1}(Y \setminus F) \\
&= f^{-1}(i_\nu(Y \setminus F)) \\
&\subseteq \beta g i_\mu(f^{-1}(Y \setminus F)) \\
&= \beta g i_\mu(X \setminus f^{-1}(F)) \\
&= X \setminus \beta g c_\mu(f^{-1}(F))
\end{aligned}
$$

Thus, $\hat{\beta}gc_\mu(f^{-1}(F)) \subseteq f^{-1}(F)$. Hence, $\hat{\beta}gc_\mu(f^{-1}(F)) = f^{-1}(F)$.

(iv) $\Longrightarrow$ (ii) Let $V \in \nu$. Then $Y \setminus V$ is ν-closed set in Y. By (iv), $\hat{\beta}gc_\mu(f^{-1}(Y \setminus V)) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V) = X \setminus \hat{\beta}gi_\mu(f^{-1}(V))$. This implies that $f^{-1}(V) = \hat{\beta}gi_\mu(f^{-1}(V))$.

Theorem 2.5 *Let f: (X, μ) → (Y, ν) be a function. If for each μ_Y-open set U of Y, $f^{-1}(U)$ is μ_X-$\hat{\beta}g$-open in X, then f is μ-$\hat{\beta}g$-continuous.*

Proof: Let $x \in X$ and V be any μ_Y-open set in Y such that f(x) ∈ V. By assumption, $f^{-1}(V)$ is μ_X-$\hat{\beta}g$-open in X with $x \in f^{-1}(V)$. Take $O = f^{-1}(V)$. Then $x \in O$ and $f(O) \subseteq V$. Therefore, f is μ-$\hat{\beta}g$-continuous.

Definition 2.6 A function f: (X, μ_X) → (Y, μ_Y) is said to be:

(i.) *almost μ-$\hat{\beta}g$-continuous at a point* $x \in X$ if for each μ_Y-open set V containing $f(x)$, there exists a μ_X-$\hat{\beta}g$-open set U containing x such that $f(U) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$.

(ii.) *almost μ-$\hat{\beta}g$-continuous* if f is almost μ-$\hat{\beta}g$-*continuous* at every point $x \in X$.

Example 2.7 Illustrating this in the example below.

(i.) Consider $a \in X$. Note that $\{u\}$ is the only μ_Y-open set containing $f(\{a\})$, that is $f(\{a\}) = \{u\} \subseteq \{u\}$, and there exists a μ_X-$\hat{\beta}g$-open set $\{a\}$ such that $f(\{a\}) = \{u\} \subseteq \{u\} = i_{\mu_Y}(c_{\mu_Y}(\{u\}))$. Thus f is almost μ-$\hat{\beta}g$-continuous at $a \in X$.

(ii.) Now, let $b \in X$. Observe that $\{u\}$ is the only μ_Y-open set containing $f(\{b\})$, that is $f(\{b\}) = \{u\} \subseteq \{u\}$, and there exists a μ_X-$\hat{\beta}g$-open set $\{b\}$ such that $f(\{b\}) = \{u\} \subseteq \{u\} = i_{\mu_Y}(c_{\mu_Y}(\{u\}))$. Thus f is almost μ-$\hat{\beta}g$-continuous at $b \in X$.

(iii.) Finally, let $c \in X$. Notice that there is no μ_Y-open set containing $f(\{c\} = v$ and so it is vacuously satisfied. Thus f is almost μ-$\hat{\beta}g$-continuous at $c \in X$.

Since, f is almost μ-$\hat{\beta}g$-*continuous* at points *a, b,* and *c*, it follows that f is almost *μ-$\hat{\beta}g$-continuous*.

Theorem 2.8 *If f: (X, μ_X) → (Y, μ_Y) is μ-$\hat{\beta}g$-continuous, then f is almost μ-$\hat{\beta}g$-continuous.*

Proof: Let $x \in X$ and V be a μ_Y-open set with $f(x) \in V$. Since f is μ-$\hat{\beta}g$-*continuous* at x, there exists a μ_X-$\hat{\beta}g$-open set U with $x \in U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V = i_{\mu_X}(V) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$. Therefore, by Definition 4.1.6, f is almost μ-$\hat{\beta}g$-continuous.

Theorem 2.9 *For a function f: (X, μ_X) → (Y, μ_Y), the following properties are equivalent:*

(i.) *f is almost μ-$\hat{\beta}g$-continuous at $x \in X$;*

(ii.) $x \in \hat{\beta}g i_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))))$ *for very $V \in \mu_Y$ containing* f(x);

(iii.) *$x \in \hat{\beta}gi_{\mu_X}(f^{-1}(V))$ for every μ-regular open subset V of Y containing f(x);*

(iv.) *For every μ-regular open subset V containing f(x), there exists μ_X-$\hat{\beta}g$-open set U containing x such that f(U) ∈ V.*

Proof: Let $x \in X$ and $f: (X, \mu_X) \to (Y, \mu_Y)$ be a function.

(i) ⟹ (ii) Let $V \in \mu_Y$ containing $f(x)$. Then $x \in f^{-1}(V)$. Since f is almost μ-$\hat{\beta}g$-continuous at x, there exists a μ_X-$\hat{\beta}g$-open set U containing x such that $f(U) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$. Therefore, $x \in U \subseteq f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V)))$. This implies that $x \in \hat{\beta}gi_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))))$.

(ii) ⟹ (iii) Let V be any μ-regular open subset Y containing $f(x)$. Then $f(x) \in V = i_{\mu_Y}(c_{\mu_Y}(V))$. Since V is μ_Y-open, by (ii), we have

$$x \in \hat{\beta}gi_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V)))) = \hat{\beta}gi_{\mu_X}(f^{-1}(i_{\mu_Y}(V)).$$

(iii) ⟹ (iv) Let V be any μ-regular open subset Y containing $f(x)$. Then by (iii), $x \in \hat{\beta}gi_{\mu_X}(f^{-1}(V))$. Thus, there exists a μ_X-$\hat{\beta}g$-open set U with $x \in U \subseteq f^{-1}(V)$. Hence, $f(U) \subseteq V$.

(iv) ⟹ (i) Let $V \in \mu_Y$ with $f(x) \in V \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$. Since $i_{\mu_Y}(c_{\mu_Y}(V))$ is μ-regular open, by (iv) there exists a μ_X-$\hat{\beta}g$-open set U containing x such that $f(U) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$. Therefore, f is almost μ-$\hat{\beta}g$-continuous at $x \in X$.

Theorem 2.10 *Let $f:(X, \mu_X) \to (Y, \mu_Y)$ be a function. Then the following properties are equivalent:*

(i) *f is almost μ-$\hat{\beta}g$-continuous;*

(ii) *$f^{-1}(V) \subseteq \hat{\beta}gi_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))))$ for every $V \in \mu_Y$;*

(iii) *$\hat{\beta}gc_{\mu_X}(f^{-1}(c_{\mu_Y}(i_{\mu_Y}(F)))) \subseteq f^{-1}(F)$ for every μ_Y-closed subset F of Y;*

(iv) *$\hat{\beta}gc_{\mu_X}(f^{-1}(c_{\mu_Y}(i_{\mu_Y}(c_{\mu_Y}(B))))) \subseteq f^{-1}(c_{\mu_Y}(B)$ for every subset B of Y;*

(v) *$f^{-1}(i_{\mu_Y}(B)) \subseteq \hat{\beta}gi_{\mu_X}(f^{-1}i_{\mu_Y}(c_{\mu_Y}(i_{\mu_Y}(B)))))$ for every subset B of Y;*

(vi) *$f^{-1}(V) = \hat{\beta}gi_{\mu_X}(f^{-1}(V))$ for every μ-regular open subset V of Y.*

(vii) *$f^{-1}(F) = \hat{\beta}gc_{\mu_X}(f^{-1}(F))$ for every μ-regular closed subset F of Y.*

Proof: Let $f: (X, \mu_X) \to (Y, \mu_Y)$ be a function.

(i) ⟹ (ii) Let V be a μ_Y-open set in Y and $x \in f^{-1}(V)$. Since f is almost μ-$\hat{\beta}g$-continuous, there exists a μ_X-$\hat{\beta}g$-open set U containing x such that $f(U) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$. This implies that $x \in \hat{\beta}gi_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))))$. Therefore, $f^{-1}(V) \subseteq \hat{\beta}gi_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))))$.

(ii) ⟹ (iii) Let F be any μ_Y-closed set. Then $Y \setminus F$ is μ_Y-open. By (ii),

$$f^{-1}(Y \setminus F) \subseteq \hat{\beta} g i_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(Y \setminus F))))$$

$$= \hat{\beta} g i_{\mu_X}(f^{-1}(Y \setminus (c_{\mu_Y}(i_{\mu_Y}(F)))))$$

$$= X \setminus \hat{\beta} g c_{\mu_X}(f^{-1}(c_{\mu_Y}(i_{\mu_Y}(F)))).$$

Hence, $X \setminus f^{-1}(F) \subseteq X \setminus \hat{\beta} g c_{\mu_X}(f^{-1}(c_{\mu_Y}(i_{\mu_Y}(F))))$. It follows that $\hat{\beta} g c_{\mu_X}(f^{-1}(c_{\mu_Y}(i_{\mu_Y}(F)))) \subseteq f^{-1}(F)$.

(iii) $\Longrightarrow$ (iv) Let B be any subset of Y. Since $c_{\mu_Y}(B)$ is a μ_Y-closed subset of Y, by (iii), $\hat{\beta} g c_{\mu_X}(f^{-1}(c_{\mu_Y}(i_{\mu_Y}(c_{\mu_Y}(B))))) \subseteq f^{-1}(c_{\mu_Y}(B))$.

(iv) $\Longrightarrow$ (v) Let B be any subset of Y. Then,

$$f^{-1}(i_{\mu_Y}(B) = f^{-1}(Y \setminus (c_{\mu_Y}(Y \setminus B)))$$

$$= X \setminus f^{-1}(c_{\mu_Y}(Y \setminus B))$$

$$\subseteq X \setminus \hat{\beta} g c_{\mu_X}(f^{-1}(c_{\mu_Y}(i_{\mu_Y}(c_{\mu_Y}(Y \setminus B)))))$$

$$= \hat{\beta} g i_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(i_{\mu_Y}(B))))).$$

(v) $\Longrightarrow$ (vi) Let V be any μ-regular open subset of Y. Then V is μ_Y-open in Y. Hence, $V = i_{\mu_Y}(V)$. Since V is μ-regular open,

$$V = i_{\mu_Y}(c_{\mu_Y}(V)) = i_{\mu_Y}(c_{\mu_Y}(i_{\mu_Y}(V))).$$

By (v),

$$f^{-1}(i_{\mu_Y}(V) = f^{-1}(V)$$

$$\subseteq \hat{\beta} g i_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(i_{\mu_Y}(V)))))$$

$$= \hat{\beta} g i_{\mu_X}(f^{-1}(V))$$

$$\subseteq f^{-1}(V).$$

Therefore, $f^{-1}(V) = \hat{\beta} g i_{\mu_X}(f^{-1}(V))$.

(vi) $\Longrightarrow$ (vii) Let F be any μ-regular closed subset of Y. Then $X \setminus F$ is a μ-regular open subset of Y. By (vi),

$$f^{-1}(Y \setminus F) = \hat{\beta} g i_{\mu_X}(f^{-1}(Y \setminus F)).$$

Thus, $X \setminus f^{-1}(F) = \hat{\beta} g i_{\mu_X}(X \setminus f^{-1}(F)) = X \setminus \hat{\beta} g c_{\mu_X}(f^{-1}(F))$. Therefore, $f^{-1}(F) = \hat{\beta} g c_{\mu_X}(f^{-1}(F))$.

(vii) $\Longrightarrow$ (i) Let $x \in X$ and V be any μ_Y- open set in Y with $f(x) \in V$. Then, $V = i_{\mu_Y}(V) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$. Since $i_{\mu_Y}(c_{\mu_Y}(V))$ is μ-regular open, by (vii), $f^{-1}(Y \setminus (i_{\mu_Y}(c_{\mu_Y}(V)))) = \hat{\beta} g c_{\mu_X}(f^{-1}(Y \setminus (i_{\mu_Y}(c_{\mu_Y}(V)))))$. Thus,

$$X \setminus f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))) = \beta g c_{\mu_X}(X \setminus f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))))$$

$$= X \setminus \beta g i_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V)))).$$

It follows that $f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))) = \hat{\beta} g i_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))))$. Since $f(x) \in V \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$, $x \in f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))) = \hat{\beta} g i_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))))$. Hence, there exists a μ_X-$\hat{\beta} g$-open set O with $x \in O \subseteq f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))$. This implies that $f(O) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$. Therefore, the theorem follows.

Definition 2.11 A function $f: (X, \mu_X) \to (Y, \mu_Y)$ is said to be:

(i.) *weakly μ-$\hat{\beta} g$-continuous at a point $x \in X$* if for each μ_Y-open set V containing $f(x)$, there exists a μ_X-$\hat{\beta} g$-open set U containing x such that $f(U) \subseteq c_{\mu_Y}(V)$.

(ii.) *weakly μ-$\hat{\beta} g$-continuous* if f is weakly *μ-$\hat{\beta} g$-continuous* at every point $x \in X$.

Example 2.12 To illustrate,

(i.) Consider $a \in X$. Note that $\{u\}$ is the only μ_Y-open set containing $f(\{a\})$, that is $f(\{a\}) = \{u\} \subseteq \{u\}$, and there exists a μ_X-$\hat{\beta} g$-open set $\{a\}$ such that $f(\{a\}) = \{u\} \subseteq Y = c_{\mu_Y}(\{u\})$. Thus f is weakly μ-$\hat{\beta} g$-continuous at $a \in X$.

(ii.) Now, let $b \in X$. Observe that $\{u\}$ is the only μ_Y-open set containing $f(\{b\})$, that is $f(\{b\}) = \{u\} \subseteq \{u\}$, and there exists a μ_X-$\hat{\beta} g$-open set $\{b\}$ such that $f(\{b\}) = \{u\} \subseteq Y = c_{\mu_Y}(\{u\})$. Thus f is weakly μ-$\hat{\beta} g$-continuous at $b \in X$.

(iii.) Finally, let $c \in X$. Notice that there is no μ_Y-open set containing $f(\{c\} = v$ and so it is vacuously satisfied. Thus f is weakly μ-$\hat{\beta} g$-continuous at $c \in X$.

Since, f is weakly *μ-$\hat{\beta} g$-continuous* at points a, b, and c, it follows that f is weakly *μ-$\hat{\beta} g$-continuous.*

Theorem 2.13 *If $f : (X,\ \mu_X) \rightarrow (Y,\ \mu_Y)$ is almost μ-$\hat{\beta}g$-continuous, then f is* weakly *μ-$\hat{\beta}g$-continuous.*

Proof: Let f be almost μ-$\hat{\beta}g$-continuous. Let $x \in X$ and V be a μ_Y-open set in Y containing $f(x)$. Since f is almost μ-$\hat{\beta}g$-continuous, there exists a μ_X-$\hat{\beta}g$-open set U containing x such that $f(U) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$. Since $i_{\mu_Y}(c_{\mu_Y}(V)) \subseteq c_{\mu_Y}(V)$, it follows that there exists a μ_X-$\hat{\beta}g$-open set U containing x such that $f(U) \subseteq c_{\mu_Y}(V)$. Therefore, f is weakly μ-$\hat{\beta}g$-continuous.

Theorem 2.14 *For a function $f: (X,\ \mu_X) \rightarrow (Y,\ \mu_Y)$, the following properties are equivalent:*

(i.) *f is weakly μ-$\hat{\beta}g$-continuous;*

(ii.) *$f^{-1}(V) \subseteq \hat{\beta}gi_{\mu_X}(f^{-1}(c_{\mu_X}(V)))$ for every μ_Y-open subset V of Y ;*

(iii.) *$\hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(F))) \subseteq f^{-1}(F)$ for every μ_Y-closed subset F of Y ;*

(iv.) *$\hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(A)))) \subseteq f^{-1}(c_{\mu_Y}(A))$ for every subset A of Y ;*

(v.) *$f^{-1}(i_{\mu_Y}(V)) \subseteq \hat{\beta}gi_{\mu_X}(f^{-1}(c_{\mu_Y}(i_{\mu_Y}(A))))$ for every subset A of Y ;*

(vi.) *$\hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(V))) \subseteq f^{-1}(c_{\mu_Y}(V))$ for every μ_Y-open subset V of Y.*

Proof: Let $f: (X,\ \mu_X) \rightarrow (Y,\ \mu_Y)$ be a function.

(i) $\Longrightarrow$ (ii) Let V be any μ_Y-open subset of Y. If $f^{-1}(V) = \emptyset$, then we are done. Let $x \in f^{-1}(V)$. Since f is weakly μ-$\hat{\beta}g$-continuous, there exists a μ_X-$\hat{\beta}g$-open set U containing x such that $f(U) \subseteq c_{\mu_Y}(V)$. This implies that $x \in f^{-1}(c_{\mu_Y}(V))$. Therefore, $x \in \hat{\beta}gi_{\mu_X}(f^{-1}(c_{\mu_Y}(V)))$ and (ii) holds.

(ii) $\Longrightarrow$ (iii) Let F be a μ_Y-closed subset of Y. Then $Y \setminus F$ is a μ_Y-open set subset of Y. By (ii),

$$X \setminus f^{-1}(F) = f^{-1}(Y \setminus F) \subseteq \beta gi_{\mu_X}(f^{-1}(c_{\mu_Y}(Y \setminus F)))$$

$$= \beta gi_{\mu_Y}(f^{-1}(Y \setminus i_{\mu_Y}(F)))$$

$$= \beta gi_{\mu_Y}(X \setminus f^{-1}(i_{\mu_Y}(F)))$$

$$= X \setminus \beta gc_{\mu_X}(f^{-1}(i_{\mu_Y}(F))).$$

Thus,

$$f^{-1}(Y \setminus c_{\mu_Y}(V)) \subseteq \beta gi_{\mu_X}(f^{-1}(c_{\mu_Y}(Y \setminus c_{\mu_Y}(V))))$$

$$= \beta gi_{\mu_X}(f^{-1}(Y \setminus i_{\mu_X}(c_{\mu_Y}(V)))$$

$$= \beta gi_{\mu_X}(X \setminus (f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))))).$$

Hence, $X \setminus f^{-1}(c_{\mu_Y}(V)) \subseteq X \setminus \hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V))))$. This implies that $\hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V)))) \subseteq f^{-1}(c_{\mu_Y}(V))$. Since $V \subseteq c_{\mu_Y}(V)$, we have $i_{\mu_Y}(V) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$. Therefore, $\hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(V))) \subseteq f^{-1}(c_{\mu_Y}(V))$.

(vi) $\Longrightarrow$ (i) Let $x \in X$ and V be a μ_Y-open set in Y containing $f(x)$. Then $V = i_{\mu_Y}(V) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$. By (vi),

$$x \in f^{-1}(V) \subseteq f^{-1}(i_{\mu_Y}(c_{\mu_Y}(V)))$$

$$= X , \ (f^{-1}(c_{\mu_Y}(Y , \ c_{\mu_Y}(V)))$$

$$\subseteq X , \ \beta gc_{\mu_X}(f^{-1}(i_{\mu_Y}(Y , \ c_{\mu_Y}(V))))$$

$$= X , \ \beta gc_{\mu_X}(f^{-1}(Y , \ c_{\mu_Y}(V)))$$

$$= \beta gi_{\mu_X}(f^{-1}(c_{\mu_Y}(V))).$$

Thus, there exist a μ_X-$\hat{\beta}g$-open set U with $x \in U$ and $f(U) \subseteq c_{\mu_Y}(V)$. Therefore, f is weakly μ-$\hat{\beta}g$-continuous.

Theorem 2.15 *Let $f: (X, \mu_X) \to (Y, \mu_Y)$ be a function. Then the following are equivalent:*

(i) *f is weakly μ-$\hat{\beta}g$-continuous;*

(ii) *$\hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(F))) \subseteq f^{-1}(F)$ for every μ_Y-regular closed subset F of Y;*

(iii) *$\hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(G)))) \subseteq f^{-1}(c_{\mu_Y}(G))$ for every μ_Y-β-open subset G of Y;*

(iv) *$\hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(G)))) \subseteq f^{-1}(c_{\mu_Y}(G))$ for every μ_Y-semiopen subset G of Y.*

Proof: Let $f: (X, \mu_X) \to (Y, \mu_Y)$ be a function.

(i) $\Longrightarrow$ (ii) Follows from Theorem 2.15 (iii).

(ii) $\Longrightarrow$ (iii) Let G be μ_Y-β-open subset of Y. The $G \subseteq ((c_{\mu_Y}(i_{\mu_Y}(c_{\mu_Y}(G)))$. It follows that $c_{\mu_Y}(G) \subseteq c_{\mu_Y}(c_{\mu_Y}(i_{\mu_Y}(c_{\mu_Y}(G)))) = (c_{\mu_Y}(i_{\mu_Y}(c_{\mu_Y}(G)))$. Now, $(i_{\mu_Y}(c_{\mu_Y}(G))$ is a μ_Y-regular closed subset of Y. By (ii), we have

$$\hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(G)))) \subseteq f^{-1}(c_{\mu_Y}(G)).$$

(iii) $\Longrightarrow$ (iv) Let G be μ_Y-semiopen set in Y. Then G is μ_Y-β-open. By (iii),

$$\hat{\beta}gc_{\mu_X}(f^{-1}(i_{\mu_Y}(c_{\mu_Y}(G)))) \subseteq f^{-1}(c_{\mu_Y}(G)).$$

(iv) $\Longrightarrow$ (i) Let V be any μ_Y-open subset of Y. Then V is μ_Y-semiopen. By Theorem 2.15 (iv), f is weakly μ-$\hat{\beta}g$-continuous.

From Remark 2.3, Theorem 2.5, Theorem 2.8, and Theorem 2.14, we have the following implications but the converses are not true.

μ-continuous $\Rightarrow$ μ-$\hat{\beta}g$-continuous

$\Downarrow$

almost μ-$\hat{\beta}g$-continuous

$\Downarrow$

weakly μ-$\hat{\beta}g$-continuous

(The symbol $\Rightarrow$ means an implication).

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