PEBBLING IN GOLDBERG SNARK GRAPH

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Abstract

A pebbling move on a connected graph $G = (V, E)$ is the removal of two pebbles from one vertex and placing one pebble on one of its adjacent vertex. The pebbling number $f(G)$ is the least number of pebbles required in moving one pebble to an arbitrary vertex by a sequence of pebbling moves. In this paper, we have determined the pebbling number of an n – dimensional Goldberg Snark G_n for $n \geq 3$.

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1. Introduction

Graph theory is a branch of mathematics that has experienced tremendous growth and impact among researchers due to its applications in Science and Engineering. Many real world networks can be modeled as a graph or a network. Design and use of multistage interconnection networks have recently drawn considerable attention due to the availability of inexpensive, powerful, microprocessors and memory chips. A multistage interconnection network is usually modeled as a graph in which the vertices correspond to processors / nodes and the edges corresponds to connections / communication links. Graph theory has a close association with combinatorics which are needed to count, enumerate or represent possible solution. Combinatorial optimization is concerned with deducing optimal solution in a finite solution space. Although, certain practical problems such as finding shortest or cheapest route trips, internet data packet routing, planning, scheduling and time tabling which appears to be NP- complete but the literature has a vast number of problems which could be solved in polynomial time. To accomplish this, there are certain combinatorial games available such as pebbling, peg solitaire, chip firing and checker jumping.

Graph pebbling is a network optimization model for the transportation of resources that are consumed in the transit. The concept of pebbling has its applications in reduction of memory traffic in computers, register allocation problem and transportation of resources that are consumed in the transit. The pebbling steps analyze the cost in loss of pebbles and it has been the subject of deep and extensive research in the context of proving lower bounds for computation on graphs. In 1956, Erdos initiated the study of zero sum sequences. On the subject of this study, Lemke and Klietman proved the conjecture of Erdos. It was Lagarias and Saks who suggested graph pebbling as a tool for solving the number theoretical conjecture. Chung [3] was the first to introduce graph pebbling into literature where she obtained the pebbling number of hypercube.

For a connected graph G, a pebbling configuration is the distribution of pebbles on the vertices of G. A pebbling move consists of removing two pebbles from a vertex and placing one pebble on the adjacent vertex. We say, one pebble is moved to any arbitrarily chosen target vertex say v, if one can repeatedly apply pebbling move so that in the resulting distribution v has at least one pebble. The pebbling number $f(G)$ is the minimum number of pebbles that ensures that every vertex of the graph G can be pebbled, regardless of the initial configuration of pebbles. In case, one pebble is placed on all the vertices of the graph G except the target vertex then there is no pebbling move which means that $f(G) \geq n(G)$, where $n(G)$ is the number of vertices of G [3] . For $w, v \in V(G)$, if w is at distance d from v and $2^d - 1$ pebbles are placed on w, then no pebble is moved to v which leads to the fact that $f(G) \geq 2^d$, where d is the diameter of G. Thus, we can say that $f(G) \geq max\{n(G), 2^d\}$ [3]. A transmitting subgraph of a graph G is a path $v_0, v_1, v_2, \ldots, v_n$ in which one pebble is transmitted from v_0 to v_n with the distribution of at least two pebbles in v_0 and at least one pebble on each of the other vertices in the path, except possibly v_n. With this distribution of pebbles one can transmit a pebble from v_0 to v_n [8].

There are some known graphs for which the pebbling number is computed such as path P_n on n vertices, complete graph K_n, hypercube Q_n [3], product graph $C_5 \times C_5$ [8], fan graph F_n and wheel graph W_n [6], complete bipartite graph $K_{s,t}$ [5], graphs with diameter 2 [4], cycle [17], Jahangir graph $J_{2,\,m}$, $m \geq 8$ [16], Flower Snark graph [2] , power of paths [1], n - star graph [15], split graph [12]. Computing bounds for pebbling is always an interesting topic of research. Kenter et.al [11] have found the pebbling bounds on product graph pebbling. In recent past years, graph pebbling has evolved as wide topic of research with its new variations. To list a few, Generalized Optimal cover pebbling [10], Monophonic pebbling [13], Non - Split Domination cover pebbling [14] and many more.

The study of snark graphs were initiated in early 1880's from a classical problem in Graph Theory namely the Four-Colour theorem. The equivalence of Four-Colour theorem with the fact that every bridgeless cubic graph is 3-colourable highlights the importance of the family of snark graphs. The Petersen graph is considered as the smallest snark graph. However, in 1975 Isaacs found an infinite family of snarks namely Flower snark J_n [9]. Further in 1981, Goldberg added his contribution to the infinite family of snarks which is named after him as Goldberg snark [7]. The Goldberg snark is also referred as Loupekine in the literature. Driven by its physical importance and motivated by the interesting counter examples on snark graphs available in the literature, we have obtained the pebbling number of Goldberg snark G_n for $n \geq 3$.

2. Goldberg Snark Graph

Goldberg snark graphs are recursive structures generated by the basic block graph B_n. The vertex and edge set of B_n is defined as $V(B_n) = \{a_n, b_{n0}, b_{n1}, c_{n0}, c_{n1}, u_n, v_n, w_n\}$, $E(B_n) = \{a_n v_n, v_n w_n, v_n u_n, u_n b_{n0}, b_{n0} b_{n1}, b_{n1} w_n, w_n c_{n0}, u_n c_{n1}, c_{n0} c_{n1}\}$. The graph in Figure 1 is the basic block graph B_1.For every such block graph we add a set of link edges E_{nj} where $E_{nj} = \{c_{n1}c_{j0}, b_{n1} b_{j0}, a_n a_j\}$, $j = n+1$. The graph thus obtained is referred as link graph L_n where the vertex and edge set of L_n are $V(L_n) = V(B_n) \cup V(B_{n+1})$ and $E(L_n) = E(B_n) \cup E(B_{n+1}) \cup E(B_{n(n+1)})$ respectively. See Figure 2.

For n odd, $n \geq 3$ graph G_n is obtained from n copies of B_1. The vertex set of G_n is $V(G_n) = V(B_1) \cup V(B_2) \cup ... \cup V(B_n)$ such that $|V(G_n)| = 8n$. The three cycles of G_n are $\{a_1, a_2, . . . , a_n\}$ forms a n – cycle, $\{b_{10}, b_{11}, b_{20}, b_{21}, . . . , b_{n0}, b_{n1}\}$ and $\{c_{10}, c_{11}, c_{20}, c_{21}, . . . , c_{n0}, c_{n1}\}$ forms a $2n$ – cycle. The Goldberg snark G_3 shown in Figure 3 is obtained as the union of basic block graphs B_1, B_2 and B_3.

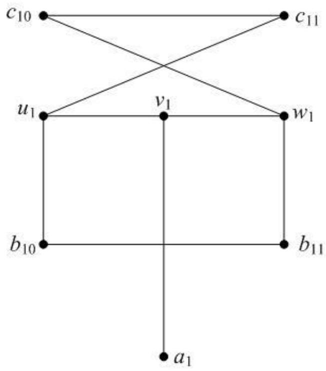
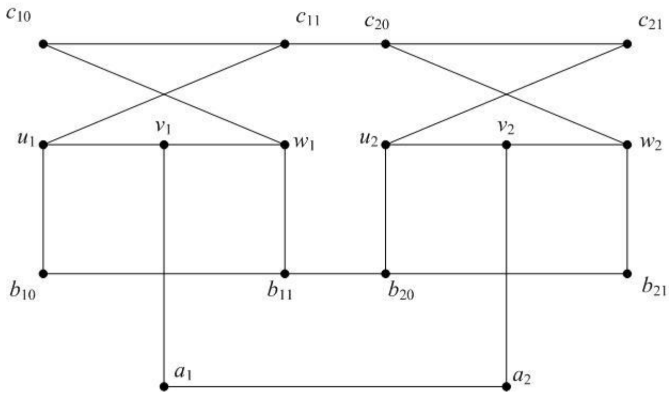


Figure 1: Basic block graph B_1

Figure 2: Link Graph L_1



Lemma 2.1: For a basic block graph B_1, $f(B_1) = 8$.

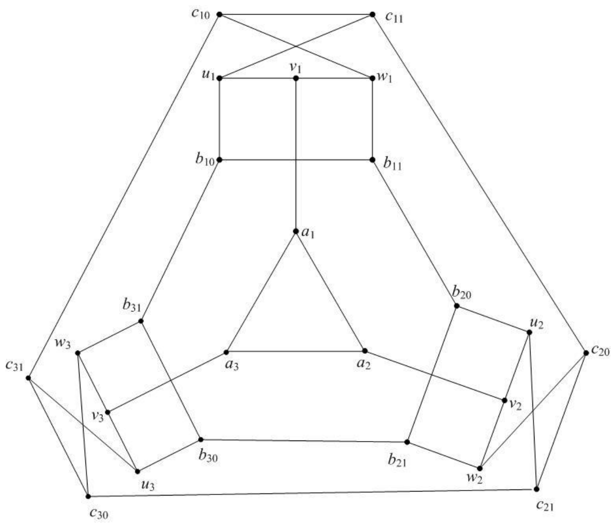
Proof: Consider the block B_1 with vertices a_1, b_{10}, b_{11}, u_1, v_1, w_1, c_{10}, c_{11}. Assume the vertex c_{11} as the target vertex. Consider a distribution of eight pebbles on the vertices of B_1. For the distribution, $p(a_1) = 3$, $p(b_{10}) = 1$, $p(b_{11}) = 1$, $p(u_1) = 1$, $p(w_1) = 1$ the target is not pebbled. Hence, consider a distribution of eight pebbles. Let $p(a_1) = 2$ and assume that there is one pebble on the remaining vertices except the target vertex. The possible pebbling paths for reaching the target with one pebble are $\{a_1, v_1, u_1, c_{11}\}$, $\{a_1, v_1, w_1, c_{10}, c_{11}\}$, $\{a_1, v_1, u_1, b_{10}, b_{11}, w_1, c_{10}, c_{11}\}$ $\{a_1, v_1, w_1, b_{11}, b_{10}, u_1, c_{11}\}$. In case, if there are zero pebbles on all the vertices then place eight pebbles on the vertex initiating the pebbling move in such a way that the target vertex is pebbled.

Notation 2.1: The vertex set of G_n is partitioned into four disjoint subsets S_1, S_2, S_3 and S_4 where the vertex set $S_1 = \{a_1, a_2, \ldots, a_n\}$, $S_2 = \{b_{10}, b_{11}, b_{20}, b_{21}, \ldots, b_{n0}, b_{n1}\}$, $S_3 = \{u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_n, v_n, w_n\}$ and $S_4 = \{c_{10}, c_{11}, c_{20}, c_{21}, \ldots, c_{n0}, c_{n1}\}$. Let p_i, $p(x_i)$ and $p^{(i)}$, $i = 1, 2, 3, 4$ denote the number of pebbles distributed over each vertex of S_i, number of pebbles initially placed on a particular vertex x_i and the total number of pebbles on the set S_i respectively.

Theorem 2.1: For a Goldberg snark graph G_3, $f(G_3) = 24$.

Proof: The graph G_3 contains three basic block graphs B_1, B_2 and B_3. By Lemma 2.1, $f(B_1) = 8$. Fix some vertex say w_1 as the target vertex. Excluding the trivial possibilities, assume $p(v_1) = 0$, $p(b_{11}) = 0$, $p(c_{10}) = 0$, $p(c_{11}) < 4$, $p(u_1) < 4$, $p(b_{10}) < 4$, $p(a_1) < 4$. The total number of pebbles considered on B_1 is at most five. The remaining three pebbles removed from B_1 will be utilized in later case. If eight pebbles are distributed on each block B_i, for $i \in \{2, 3\}$ then it is possible to move one pebble to any vertex of B_i. For if, $p(a_2) = 1$ and $p(a_3) = 1$ then the target is pebbled using the three pebbles that was excluded from B_1 through the transmitting path $\{a_3, a_2, a_1, v_1, w_1\}$. In a similar manner, the target is pebbled from any arbitrary vertex of G_3. Hence, $f(G_3) = 24$.

Figure 3: Goldberg snark graph G_3



Theorem 2.1: The pebbling number of Goldberg snark graph G_n for $n \geq 5$ is $f(G_n) = 8n + 1$.

Proof: We consider four possible cases by fixing the target vertex in the sets S_1, S_2, S_3 and S_4.

Case 1: Let $a_1 \in S_1$ be the target vertex. Suppose $p(a_2) = 2$ or $p(a_n) = 2$ then the proof is trivial. Hence, assume that $p(a_2) = 1$ and $p(a_n) = 1$.

Case 1.1: $p_1 > 1$

In this case we place at least two pebbles on the vertices of S_1 in such a way that there exist a vertex say a_i for which $p(a_i) \geq 2$, $2 \leq i \leq n - 3$ will initiate the pebbling move. The transmitting path in this case would be $\{a_i, a_{i-1}, \ldots, a_2, a_1\}$. On placing two pebbles on the $n - 3$ vertices of S_1 and one pebble on a_2 and a_n the total number of pebbles required to pebble the target vertex is $p^{(1)} \geq 2(n - 3) + 1 + 1 = 2n - 4$.

Case 1.2: $p_1 \leq 1$

Here, we consider the case where the vertices of S_1 are either distributed with one pebble or no pebble. Now S_1 has inadequate pebbles to initiate the pebbling move. In order to pebble the target vertex pebbles are extracted from either S_2 or S_3 or S_4. The vertex $a_i \in S_1$ is adjacent to the vertex $v_i \in S_3$. Initially assume $p(v_1) < 2$ otherwise the solution is trivial.

Case 1.2.1: $p_3 \geq 2$

In this case, we consider $p(u_i) = 2$, $p(v_i) = 2$, $p(w_i) = 2$ for $i \in \{2, 3, \ldots, n\}$. The vertex v_i is adjacent to the vertices u_i and w_i. After a pebbling move which is initially considered in the set S_3 the vertex v_i receives at least two pebbles. That is one pebble from the vertex u_i and the other pebble from the vertex w_i. Hence, after a pebbling move $p(v_i) = 4$. Since a_i is adjacent to v_i after a pebbling move every vertex a_i will have at least two pebbles and as a consequence the target is pebbled through the transmitting path $\{v_i, a_i, a_{i+1}, a_{i+2}, \ldots, a_n, a_1\}$ or $\{v_i, a_i, a_{i-1}, a_{i-2}, \ldots, a_2, a_1\}$.

There is one pebble on $(n - 1)$ vertices of S_1 and minimum two pebbles on $3(n - 1)$ vertices of S_3. Thus, the number of pebbles required in this case is $p^{(1)} + p^{(3)} \geq 2(3(n - 1)) + n - 1 = 7n - 7$.

Case 1.2.2: $p_3 \leq 1$

By Lemma 2.1 the pebbling number of each block of G_n is eight. On assuming that there are no pebbles on the vertices of S_3 we need to have eight pebbles distributed on the vertices of S_2 and S_4. If $p(b_{i0}) = 2$, $p(b_{i1}) = 2$, $p(c_{i0}) = 2$, $p(c_{i1}) = 2$ then after a pebbling move there will be two pebbles on the vertices u_i and w_i. Now these vertices in turn contribute two pebbles to the vertex v_i so that one pebble is moved to a_i. It is to note that, there is no vertex to trigger the pebbling move in the vertex set of S_1 as $p(a_i) = 1$. Hence, for a distribution of eight pebbles on each B_i are not sufficient. Therefore, we require one more pebble to initiate the pebbling move. Thus the total number of pebbles required is $p^{(1)} + p^{(2)} + p^{(3)} + p^{(4)} \geq 8n + 1$.

Case 2: $b_{1i} \in S_2$ either $i = 0$ or 1 as the target vertex.
Fix b_{10} as the target and the proof is similar if any vertex of S_2 is chosen as the target vertex. Without loss of generality, assume that $p(u_1) = 0$, $p(b_{11}) = 0$ and $p(b_{n1}) = 0$.

Case 2.1: $p_2 \geq 2$
The vertices of S_2 forms a $2n$ – cycle and if minimum two pebbles are placed on the vertices of S_2 then the transmitting path to pebble the target vertex b_{10} is $\{b_{i0}, b_{i1}, b_{(i+1)0}, b_{(i+1)1}, \ldots, b_{n0}, b_{n1}, b_{10}\}$. Excluding the vertices with zero pebbles in S_2 place two pebbles on the $(2n – 3)$ vertices of S_2. In this case, the number of pebbles required is $p^{(2)} \geq 2(2n – 3) = 4n – 6$.

Case 2.2: $p_2 \leq 1$
Pebbling move within the set S_2 is not possible as p_2 has either one or zero pebble. Due to insufficient pebbles, extract pebbles from S_3 and S_4 which is discussed in the following subcases.

Case 2.2.1: Extraction of pebbles from S_3.
If $p_3 \geq 4$ then at least two pebbles are moved to the vertices of S_2 such that $p_2 \geq 2$ and the target can be pebbled as in Case 2.1.
In case, $p_3 < 4$ then we have to extract pebbles from S_4. Here we need $p^{(2)} + p^{(3)} \geq 4(3(n – 1)) + 2n – 1 = 14n – 13$ pebbles for pebbling the target vertex.

Case 2.2.2: Extraction of pebbles from S_4.
With $p_4 \geq 4$, after a pebbling move at least two pebbles are placed on the vertices u_i and w_i. Thereafter, at least one pebble is moved to the vertices of S_2. Further, to facilitate the pebbling move in the set S_2 place one pebble on any vertex of S_2 so that the target is pebbled as in Case 2.1. It is to note that $p_4 < 4$ is not possible by Lemma 2.1. Hence, the number of pebbles required is $p^{(2)} + p^{(4)} \geq 8n + 1$.

Case 3: Let u_1 or v_1 or w_1 be the target vertex.
Without loss of generality fix $w_1 \in S_3$ as the target. The proof is similar if any vertex u_1 or v_1 is chosen as the target vertex. Initially assume $p(u_1) < 4$ and $p(v_1) = 0$ otherwise the solution is trivial.

Case 3.1: $p_3 \geq 4$
The vertices of S_3 are adjacent to the vertices of S_1, S_2 and S_4. With $p_3 \geq 4$ it is evident that two pebbles can be moved to every vertex either in S_1 or S_2 or S_4. The target is thus pebbled through the transmitting paths $\{v_i, a_i, a_{i+1}, a_{i+2}, \ldots, a_n, a_1, v_1, w_1\}$ or $\{w_i, b_{i1}, b_{i0}, b_{(i-1)1}, b_{(i-1)0}, \ldots, b_{(i-2)1}, b_{(i-2)0}, \ldots, b_{11}, w_1\}$ or $\{c_{i0}, c_{i1}, c_{(i+1)0}, c_{(i+1)1}, c_{(i+2)0}, c_{(i+2)1}, \ldots, c_{n0}, c_{n1}, c_{10}, w_1\}$. In this case, we require $p^{(3)} \geq 4(3(n – 1)) = 12n – 12$ pebbles to move a pebble to the target vertex.

Case 3.2: $p_3 < 4$
By Lemma 2.1, it is obvious that there should exists at least eight pebbles on each block B_i. Excluding the pebbles considered on S_3, the number of pebbles on each B_i should be at least five. But with five pebbles only one pebble is placed on the vertex a_i. Hence with $p_1 = 1$, the target vertex cannot be pebbled. We need an additional pebble to initiate the pebbling move such that the target is pebbled. In this case we require $p^{(1)} + p^{(2)} + p^{(3)} + p^{(4)} \geq 8n + 1$ pebbles.

Case 4: Let c_{10} or $c_{11} \in S_4$ be the target vertex.

As the vertices of S_2 and S_4 forms a $2n$ – cycle and it is adjacent to the vertices of S_3 the methodology for proving is same as in the Case 2.

All the possibilities of pebbling the target vertex in the sets S_1, S_2, S_3 and S_4 are discussed above. Hence, we conclude that the least possibility of pebbling the target vertex from the four cases. As $f(G_n)$ cannot be less than $8n$, the least possibility of $\{2n - 4, 7n - 7, 8n + 1, 4n - 6, 14n - 13, 12n - 12\}$ is $8n + 1$. Hence, we conclude that $f(G_n) = 8n + 1$.

3. Conclusion

The family of snarks falls under bridgeless cubic graphs. Motivated by its topological structure, in this paper we have determined the pebbling number of Goldberg snark G_n. The problem is open to find the pebbling number for other graphs in the snark family and find a bound for the pebbling number of cubic graphs.

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