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THIRD ORDER SUBLINEAR NEUTRAL DELAY DIFFERENCE EQUATION

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Abstract

We derive new sufficient conditions for oscillatory and asymptotic behavior of all solutions of third order delay difference equation with a sublinear neutral term. We have compared the third order difference equation into second order delay difference equation with some oscillation characteristics and obtained oscillation results. Examples are provided to illustrate our results.

Keywords: Oscillation, difference equation, third order, sublinear, neutral delay.

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1 Introduction

We examine the oscillatory behavior of third order non linear difference equation with a sublinear neutral term

$$\Delta^2(u(\varsigma)(\Delta x(\varsigma))^\gamma) + p(\varsigma)y^\mu(\varsigma - d_2) = 0,\ \varsigma \geq \varsigma_0 \quad (1)$$

where $x(\varsigma) = y(\varsigma) + \delta(\varsigma)y^\kappa(\varsigma - d_1)$ and ς_0 is a positive integer.

Assumed throughout the entirety of this paper,

1. $\{u(\varsigma)\}$, $\{\delta(\varsigma)\}$ and $\{ \quad p(\varsigma)\}$ are positive real sequences with $\delta(\varsigma) \geq 1$ and $\delta(\varsigma) \to \infty$ as $\varsigma \to \infty$;

2. Each of κ,γ and μ is a ratio of odd positive integers with $0 \quad < \kappa \leq 1$;

3. d_1 and d_2 are positive integers with $d_1 < d_2$ and

4. $\sum_{\varsigma=\varsigma_0}^{\infty} u^{-\frac{1}{\gamma}}(\varsigma) = \infty$.

From the solution of (1) we mean a non trivial sequence $\{y(\varsigma)\}$ defined for all $\varsigma \geq \varsigma_0 - \omega$, $where\ \omega$ = max $\{d_1, d_2\}$, which satisfies (1) for all $\varsigma \geq \varsigma_0$. A non-trivial solution of (1) is neither eventually positive nor eventually negative and non oscillatory otherwise.

In recent years, there has been a lot of interest in figuring out the circumstances that cause difference equation solutions to behave in an oscillatory manner, see [1,2,4,3,5,7,8,11,12,13,14] and the references therein. There are number of oscillation results available for second order difference equations and less number of such results available for third order difference equations even though these type of equations have wide applications in economics, mathematical biology and many other areas in science[6,9,10,15,16,17].

As far as the authors are aware, no research has been done on third order difference equations with a sublinear neutral component, where equation (1) was examined in the instance $u(\varsigma) = 1$, $\gamma = 1$ and $0 < \delta(\varsigma) \leq \delta < 1$.

2 Main Results

In order to make things easier for us, we let

$A(\varsigma)$= $\sum_{s=\varsigma_1}^{\varsigma-1} u^{-\frac{1}{\kappa}}(s)$, for $\varsigma \geq \varsigma_1 \geq \varsigma_0$ and there exist an integers a_1 and a_2 such that $\varsigma + d_1 - d_2 < \varsigma - a_1 < \varsigma - a_2$.

Furthermore, we assume that for every positive integers ϵ and Γ

1. $\eta(\varsigma) = \frac{1}{\delta(\varsigma+d_1)}\left[1 - \frac{A^{\frac{1}{\kappa}}(\varsigma+2d_1)}{A(\varsigma+d_1)} \frac{\epsilon^{\frac{1}{\kappa}-1}}{\delta^{\frac{1}{\kappa}}(\varsigma+2d_1)}\right] > 0$ and

2. $\chi(\varsigma) = \frac{1}{\delta(\varsigma+d_1)}\left[1 - \frac{\Gamma^{\frac{1}{\kappa}-1}}{\delta^{\frac{1}{\kappa}}(\varsigma+2d_1)}\right] > 0$ for sufficiently large ς.

Lemma 2.1 *"Suppose that $(S1) - (S4)$ hold and $\{y(\varsigma)\}$ is an eventually positive solution of (1). Then there exist an integer $\varsigma_1 \geq \varsigma_0$ such that the corresponding sequence $\{x(\varsigma)\}$, meets one of the criteria listed below:"*

1. $x(\varsigma) > 0,\ \Delta x(\varsigma) > 0,\ \Delta(u(\varsigma)(\Delta x(\varsigma))) < 0$ and $\Delta^2(u(\varsigma)(\Delta x(\varsigma))^\gamma) < 0$;

2. $x(\varsigma) > 0,\ \Delta x(\varsigma) < 0,\ \Delta(u(\varsigma)(\Delta x(\varsigma))) > 0$ and $\Delta^2(u(\varsigma)(\Delta x(\varsigma))^\gamma) < 0$, for all $\varsigma \geq \varsigma_1$.

Proof. It is omitted as rather easy.

Lemma 2.2 *Let $(S1) - (S5)$ hold and $\{x(\varsigma)\}$ fulfills Case (C1) of Lemma 2.1 for all $\varsigma \geq \varsigma_1$. Then*

$$x(\varsigma) \geq A(\varsigma)(u(\varsigma)(\Delta x(\varsigma))) \tag{2}$$

and

$$\left\{\frac{x(\varsigma)}{A(\varsigma)}\right\} \quad is \quad non \quad increasing \quad for \quad all \quad \varsigma \geq \varsigma_1. \tag{3}$$

Proof. We assumed $u(\varsigma)(\Delta x(\varsigma)) > 0$ and decreasing, we get

$$x(\varsigma) = x(\varsigma_1) + \sum_{S=\varsigma_1}^{\varsigma-1} \frac{u^{\frac{1}{\gamma}}(s)(\Delta x(s))}{u^{\frac{1}{\gamma}}(s)},$$

$$\geq \quad A(\varsigma)(u^{\frac{1}{\gamma}}(\varsigma)(\Delta x(\varsigma))), \ it \quad proves \quad (2).$$

By (2), we have

$$\Delta\left(\frac{x(\varsigma)}{A(\varsigma)}\right)$$

is non increasing.

Lemma 2.3 *Suppose that $(S1) - (S4)$ hold and $\{y(\varsigma)\}$ is a positive solution of (1) with the corresponding sequence $\{x(\varsigma)\}$ fulfilling case (C1) of Lemma 2.1 $\forall\ \varsigma \geq \varsigma_1 \geq \varsigma_0$, then $\{x(\varsigma)\}$ satisfies the inequality*

$$\Delta^2(u(\varsigma)(\Delta x(\varsigma))^\gamma) + p(\varsigma)\eta^{\frac{\mu}{\kappa}}(\varsigma - d_2)x^{\frac{\mu}{\kappa}}(\varsigma + d_1 - d_2) \leq 0. \tag{4}$$

for all ς.

Proof. Let $\{y(\varsigma)\}$ be a positive solution of (1) and $y(\varsigma) > 0$, $y(\varsigma - d_1) > 0$ and $y(\varsigma - d_2) > 0$ for $\varsigma \geq \varsigma_1$ for some integer $\varsigma_1 \geq \varsigma_0$.

According to the definition of $x(\varsigma)$, we write

$$y^{\kappa}(\varsigma) \geq \frac{1}{\delta(\varsigma+d_1)}\left[x(\varsigma + d_1) - \frac{x^{\frac{1}{\kappa}}(\varsigma+2d_1)}{\delta^{\frac{1}{\kappa}}(\varsigma+2d_1)}\right], \varsigma \geq \varsigma_1. \tag{5}$$

We have $\left\{\frac{x(\varsigma)}{A(\varsigma)}\right\}$ is non increasing and (5) becomes

$$y^{\kappa}(\varsigma) \geq \frac{1}{\delta(\varsigma+d_1)}\left[x(\varsigma + d_1) - \frac{x^{\frac{1}{\kappa}}(\varsigma+d_1)}{\delta^{\frac{1}{\kappa}}(\varsigma+2d_1)}\frac{A^{\frac{1}{\kappa}}(\varsigma+2d_1)}{A^{\frac{1}{\kappa}}(\varsigma+d_1)}\right]. \tag{6}$$

Additionally, a constant $\epsilon > 0$ so that $\frac{x(\varsigma+d_1)}{A(\varsigma+d_1)} \leq \epsilon$. By this criteria and (6), we get

$$\begin{aligned} y^{\kappa}(\varsigma) &\geq \frac{x(\varsigma+d_1)}{\delta(\varsigma+d_1)}\left[1 - \frac{\epsilon^{\frac{1}{\kappa}-1}}{\delta^{\frac{1}{\kappa}}(\varsigma+2d_1)}\frac{A^{\frac{1}{\kappa}}(\varsigma+2d_1)}{A^{\frac{1}{\kappa}}(\varsigma+d_1)}\right], \\ &\geq \eta(\varsigma)x(\varsigma + d_1). \end{aligned}$$

Applying the last inequality in (1), we can get (4). The proof is completed.

Lemma 2.4 *Suppose that* $(S1) - (S4)$ *and* $(S6)$ *hold. If* $\{y(\varsigma)\}$ *is an eventually positive solution of (1) with the corresponding sequence* $\{x(\varsigma)\}$ *fulfilling Case (C2) of Lemma 2.1* $\forall$ $\varsigma \geq \varsigma_1 \geq \varsigma_0$, *then* $\{x(\varsigma)\}$ *satisfies the inequality*

$$\Delta^2(u(\varsigma)(\Delta x(\varsigma))^{\gamma}) + p(\varsigma)\chi^{\frac{\mu}{\kappa}}(\varsigma - d_2)x^{\frac{\mu}{\kappa}}(\varsigma + d_1 - d_2) \leq 0. \tag{7}$$

for large ς.

Proof. Let $\{y(\varsigma)\}$ be a positive solution of (1) and $y(\varsigma) > 0$, $y(\varsigma - d_1) > 0$ and $y(\varsigma - d_2) > 0$ for $\varsigma \geq \varsigma_1$ for some integer $\varsigma_1 \geq \varsigma_0$. Continuing with the proof of the Lemma 2.3, we see that (5) holds $\forall$ $\varsigma \geq \varsigma_1$. Since $\{x(\varsigma)\}$ is decreasing, It proves by reference to (5) that

$$y^{\kappa}(\varsigma) \geq \frac{x(\varsigma+d_1)}{\delta(\varsigma+d_1)}\left(1 - \frac{x^{\frac{1}{\kappa}-1}(\varsigma+d_1)}{\delta^{\frac{1}{\kappa}}(\varsigma+2d_1)}\right). \tag{8}$$

Also there exist a constant $\Gamma > 0$ and $x(\varsigma) \leq \Gamma$ and $\frac{1}{\kappa} > 1$. we see that

$$x^{\frac{1}{\kappa}-1}(\varsigma + d_1) \leq \Gamma^{\frac{1}{\kappa}-1}, forall \varsigma \geq \varsigma_1.$$

Applying this in (8), we obtain

$$y^{\kappa}(\varsigma) \geq \chi(\varsigma)x(\varsigma + d_1). \tag{9}$$

Substituting (9) in (1) gives (7). Hence the proof.

Theorem 2.5 *Suppose that* $(S1) - (S6)$ *hold. If for all sufficiently large integer* $\varsigma_1 \geq \varsigma_0$ *and for some* $\varsigma_2 \geq \varsigma_1$,

$$\sum_{\varsigma=\varsigma_2}^{\infty} p(\varsigma)\eta^{\frac{\mu}{\kappa}}(\varsigma - d_2) = \infty \tag{10}$$

and

$$\sum_{\varsigma=\varsigma_1}^{\infty} p(\varsigma)\chi^{\frac{\mu}{\kappa}}(\varsigma-d_2)=\infty, \tag{11}$$

then every solution of (1) is either oscillatory or $\lim\limits_{\varsigma\to\infty} y(\varsigma)$=0.

Proof. Consider $\{y(\varsigma)\}$ to be non oscillatory solution of (1). Without sacrificing generality, we can assume $y(\varsigma)>0$, $y(\varsigma-d_1)>0$ and $y(\varsigma-d_2)>0$ for $\varsigma\geq\varsigma_1$ for some $\varsigma_1\geq\varsigma_0$. Considering the Lemma 2.1, $\{x(\varsigma)\}$ will meet case (C1) or case (C2) for all $\varsigma\geq\varsigma_1$.

Case (C1): According to the Lemma 2.3, we see the above inequality (4) holds for every $\varsigma\geq\varsigma_1$. Since $\{x(\varsigma)\}$ is increasing, there exists a constant $F>0$ and an integer $\varsigma_2\geq\varsigma_1$ and $x(\varsigma)\geq F$ for all $\varsigma\geq\varsigma_2$. Then we have the following

$$\Delta(u(\varsigma)(\Delta x(\varsigma))^{\gamma})\leq\Delta(u(\varsigma_2)(\Delta x(\varsigma_2))^{\gamma})-F^{\frac{\mu}{\kappa}}\sum_{i=\varsigma_2}^{\varsigma-1} p(i)\eta^{\frac{\mu}{\kappa}}(i-d_2),$$

which tends to $-\infty$ as $\varsigma\to\infty$. This inconsistency rules out Case (C1).

Case(C2): Since $\{x(\varsigma)\}$ is decreasing and there is a constant e such that $\lim\limits_{k\to\infty} x(\varsigma)=e\geq 0$. If $e>0$, then $x(\varsigma)\geq e\ \forall\ \varsigma\geq\varsigma_1\geq\varsigma_0$. Applying this in (7), we have

$$\Delta^2(u(\varsigma)(\Delta x(\varsigma))^{\gamma})\leq -e^{\frac{\mu}{\kappa}}p(\varsigma)\chi^{\frac{\mu}{\kappa}}(\varsigma-d_2),\varsigma\geq\varsigma_1.$$

Summing up the last inequality from ς_1 to $\varsigma-1$ gives,

$$\Delta(u(\varsigma)(\Delta x(\varsigma))^{\gamma})\leq\Delta(u(\varsigma_1)(\Delta x(\varsigma_1))^{\gamma})-e^{\frac{\mu}{\kappa}}\sum_{s=\varsigma_1}^{\varsigma-1} p(s)\chi^{\frac{\mu}{\kappa}}(\varsigma-d_2),$$

which approaches to $-\infty$ as $\varsigma\to\infty$, which is contradiction.

If $e=0$, then $y(\varsigma)\leq x(\varsigma)\Rightarrow y(\varsigma)\to 0$ as $\varsigma\to\infty$. Hence the proof.

Remark Let us assume that (S7) $\lim\limits_{k\to\infty}\frac{A^{\frac{1}{\kappa}}(\varsigma+2d_1)}{A(\varsigma+d_1)\delta^{\frac{1}{\kappa}}(\varsigma+2d_1)}=0.$

Since $\{A(\varsigma)\}$ is positive and increasing and $\frac{1}{\kappa}>1$, thus $(S7)$ implies

$$\lim_{k\to\infty}\frac{1}{\delta^{\frac{1}{\kappa}}(\varsigma+2d_1)}=0. \tag{12}$$

We can apply these conditions to simplify $\eta(\varsigma)$ and $\chi(\varsigma)$ as follows. Inview of (S7), there exist $b_1\in(0,1)$ and an integer $\varsigma_1\geq\varsigma_0$ such that

$$\frac{A^{\frac{1}{\kappa}}(\varsigma+2d_1)}{A(\varsigma+d_1)\delta^{\frac{1}{\kappa}}(\varsigma+2d_1)}\leq\epsilon^{1-\frac{1}{\kappa}}(1-b_1).$$

Applying the above inequality with $\eta(\varsigma)$, we get

$$\eta(\varsigma)\geq\frac{b_1}{\delta(\varsigma+d_1)},\varsigma\geq\varsigma_1. \tag{13}$$

Again from (12), there exists $b_2\in(0,1)$ and an integer $\varsigma_2\geq\varsigma_1$ and

$$\frac{1}{\delta^{\frac{1}{\kappa}}(\varsigma+2d_1)} \leq \epsilon^{1-\frac{1}{\kappa}}(1-b_2).$$

Using above with $\chi(\varsigma)$, we get

$$\chi(\varsigma) \geq \frac{b_2}{\delta(\varsigma+d_1)}, \varsigma \geq \varsigma_2. \tag{14}$$

Remark 2.6 *Let $\varsigma \geq N \geq \varsigma_0$ be fixed. Applying (14) in (7), we have*

$$\Delta^2(u(\varsigma)(\Delta x(\varsigma))^\gamma) + p(\varsigma) b_2^{\frac{\mu}{\kappa}} \delta^{\frac{-\mu}{\kappa}}(\varsigma+d_1-d_2) x^{\frac{\mu}{\kappa}}(\varsigma+d_1-d_2) \leq 0, \varsigma \geq N. \tag{15}$$

Theorem 2.7 *Let $(S1)-(S4)$ and $(S7)$ be hold. Moreover, suppose there exists an integer a_1 such that $\varsigma+d_1-d_2 \leq \varsigma-a_1$ for any $\varsigma \geq \varsigma_0$. If the second order difference inequality*

$$\Delta^2(E(\varsigma))^\gamma + \frac{p(\varsigma) b_1^{\frac{\mu}{\kappa}}}{\delta^{\frac{\mu}{\kappa}}(\varsigma+d_1-d_2)} A^{\frac{\mu}{\kappa}}(\varsigma+d_1-d_2) E^{\frac{\mu}{\kappa}}(\varsigma+d_1-d_2) \leq 0. \tag{16}$$

has no positive increasing solution and the second order difference inequality

$$\Delta^2(U(\varsigma))^\gamma - p(\varsigma) b_2^{\frac{\mu}{\kappa}} \delta^{\frac{-\mu}{\kappa}}(\varsigma+d_1-d_2)\left(\sum_{i=\varsigma+d_1-d_2}^{\varsigma-a_1-1} \frac{1}{u(i)}\right) U^{\frac{\mu}{\kappa}}(\varsigma-a_1) \geq 0. \tag{17}$$

has no positive decreasing solution, then (1) is oscillatory.

Proof. Let $\{y(\varsigma)\}$ be a non oscillatory solution of (1), say $y(\varsigma)>0$, $y(\varsigma-d_1)>0$ and $y(\varsigma-d_2)>0$ $\forall$ $\varsigma \geq \varsigma_1$ for some integer $\varsigma_1 \geq \varsigma_0$. Then from Lemma 2.1, $\{x(\varsigma)\}$ fulfills either case $(C1)$ or case $(C2)$ for $\varsigma \geq \varsigma_1$.

Case(C1) Similar to how the Lemma 2.3 was proved, we get here at (4). Applying (13) in (4), we obtain

$$\Delta^2(u(\varsigma)(\Delta x(\varsigma)^\gamma) + p(\varsigma)\left(\frac{b_1}{\delta(\varsigma+d_1-d_2)}\right)^{\frac{\mu}{\kappa}} x^{\frac{\mu}{\kappa}}(\varsigma+d_1-d_2) \leq 0, \varsigma \geq \varsigma_1. \tag{18}$$

.

Applying (2) in (7) and by above (18), we have

$$\begin{aligned}&\Delta^2(u(\varsigma)(\Delta x(\varsigma))^\gamma)\\ &+\frac{p(\varsigma) b_1^{\frac{\mu}{\kappa}}}{\delta^{\frac{\mu}{\kappa}}(\varsigma+d_1-d_2)} A^{\frac{\mu}{\kappa}}(\varsigma+d_1-d_2)(u(\varsigma+d_1-d_2)(\Delta x(\varsigma+d_1-d_2))^{\frac{\mu}{\kappa}}) \leq 0.\end{aligned} \tag{19}$$

Let $E(\varsigma)=u(\varsigma)(\Delta x(\varsigma))>0$ for $\varsigma \geq \varsigma_1$. Then from (19), we can see that

$$\Delta^2(E(\varsigma))^\gamma + \frac{p(\varsigma) b_1^{\frac{\mu}{\kappa}}}{\delta^{\frac{\mu}{\kappa}}(\varsigma+d_1-d_2)} A^{\frac{\mu}{\kappa}}(\varsigma+d_1-d_2) E^{\frac{\mu}{\kappa}}(\varsigma+d_1-d_2) \leq 0. \tag{20}$$

has a positive increasing solution, which is contradiction.

Case(C2) For some $s \geq n \geq \varsigma_2 \geq \varsigma_1$, we can write

$$x(n) = x(s) + \sum_{i=n}^{s-1} \frac{u(i)\Delta x(i)}{u(i)},$$

$$\geq -u(s)\Delta x(s) \sum_{i=n}^{s-1} \frac{1}{u(i)}. \qquad (21)$$

Fix $n = \varsigma + d_1 - d_2$ and $s = \varsigma - a_1$ in (21), we get

$$x(\varsigma + d_1 - d_2) \geq \left(\sum_{i=\varsigma+d_1-d_2}^{\varsigma-a_1-1} \frac{1}{u(i)}\right)(-u(\varsigma - a_1)(\Delta x(\varsigma - a_1))). \qquad (22)$$

By the above remark , we can say the inequality (15) holds. By (15) and (22), we obtain the ensuing inequality

$$\Delta^2(u(\varsigma)(\Delta x(\varsigma))^\gamma) + p(\varsigma) b_2^{\frac{\mu}{\kappa}} \delta^{\frac{-\mu}{\kappa}}(\varsigma + d_1 - d_2)\left(\sum_{i=\varsigma+d_1-d_2}^{\varsigma-a_1-1} \frac{1}{u(i)}\right)(-u(\varsigma - a_1)(\Delta x(\varsigma - a_1))^\gamma) \leq 0.$$

Now, we define $U(\varsigma) = -u(\varsigma)(\Delta x(\varsigma)) > 0$ for $\varsigma \geq \varsigma_1 \geq N \geq \varsigma_0$, then we have

$$\Delta^2(U(\varsigma))^\gamma - p(\varsigma) b_2^{\frac{\mu}{\kappa}} \delta^{\frac{-\mu}{\kappa}}(\varsigma + d_1 - d_2)\left(\sum_{i=\varsigma+d_1-d_2}^{\varsigma-a_1-1} \frac{1}{u(i)}\right) U^{\frac{\mu}{\kappa}}(\varsigma - a_1) \geq 0.$$

has a positive decreasing solution which comprises a contradiction. Hence the theorem.

3 Example

Example 3.1 *Examine the third order neutral delay difference equation*

$$\Delta^2\left(\frac{1}{\varsigma^4}(\Delta x(\varsigma))^4\right) + \varsigma^3 y^{\frac{1}{4}}(\varsigma - 5) = 0, \ \varsigma \geq 5 \qquad (23)$$

where $x(\varsigma) = y(\varsigma) + \varsigma^2 y^{\frac{1}{2}}(\varsigma - 2)$. Here $p(\varsigma) = \varsigma^3$, $\delta(\varsigma) = \varsigma^2$, $\kappa = \frac{1}{4}$, $d_2 = 5$, $d_1 = 2$, $\mu(\varsigma) = \frac{1}{4}$, $\gamma = 4$, $u(\varsigma) = \frac{1}{\varsigma^4}$. We can easily say $(S1)$- $(S4)$ hold. From the direct calculation shows that $A(\varsigma) = \frac{\varsigma-1}{2}$, $\eta(\varsigma) \geq \frac{1}{(\varsigma+3)^2}\left(1 - \frac{(\varsigma+3)^4 \epsilon^3}{8(\varsigma+1)(\varsigma+4)^8}\right) > 0$ and $\chi(\varsigma) \geq \frac{1}{(\varsigma+2)^2}\left(1 - \frac{\Gamma^3}{(\varsigma+4)^3}\right) > 0$, so $(S5)$ and $(S6)$ are satisfied. This follows from (10) and (11)

$$\sum_5^\infty \frac{\varsigma^3}{(\varsigma-3)^2}\left(1 - \frac{(\varsigma-2)^4 \quad \epsilon^3}{8(\varsigma-4)(\varsigma-1)}\right) = \infty$$

and

$$\sum_5^\infty \frac{\varsigma^3}{(\varsigma-1)^3}\left(1 - \frac{\Gamma^3}{(\varsigma-1)^3}\right) = \infty.$$

Hence by theorem 2.5, every solution of (23) is either oscillatory or tends to zero as $\varsigma \to \infty$.

4 Conclusion

The oscillation conditions of the third order sublinear neutral delay difference equations are obtained in this study. In order to improve and reduced third order difference equation to second order difference equation with known oscillation characteristics. Furthermore one example is provided to dwell upon the importance of our main results.

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