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Exploratory Analysis on Blood Flow Models' Data Set

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Abstract

In this paper, we analysed the simulated data generated from our earlier works on the mathematical models built for blood flow in the human arterial system. In those works, four models, the Newtonian model with constant viscosity and variable viscosity, a non-Newtonian model with constant and variable viscosities, have been used to describe the essential constituent of this system, i.e., the blood. In both variable viscosity models, the fluid (blood) viscosity has been described as a function of the Haematocrit, plasma viscosity, and shape factor of the red blood cells. 2-D mathematical models have been developed, and an approximate analytical solution was found. These models are now simulated for blood flow in the human femoral artery using the available clinical and anatomical data. The simulated data generated are considered to build statistical models for estimating the two physical quantities, wall shear stress (WSS) and volumetric flow rate (QF). We developed linear and non – linear (logarithmic) regression models to relate the dependent and the independent variables (model parameters). The results showed a reasonably good R- squared score indicating the reliability of the models. However, the residual analysis has shown non-randomness in the plots indicating a need to explore other predictive models. Basically, this work attempts to bring to the attention of researchers the need for statistical modelling in generating data for a given set of input model parameters instead of performing pretty expensive simulations.

Keywords - Blood flow models, Data Analysis, Wall shear stress, Residual analysis, linear and non-linear statistical models

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Nomenclature:

$\vec{q}$ – Velocity vector

p – Thermodynamic pressure

ρ – Density (kg/m^3)

τ_{ij} – Deviatoric stress tensor

δ_{ij} – Kronecker symbol

μ – Viscosity (Pa. s)

e_{ij} – Rate of deformation tensor

t_{ij} – non-symmetric stress tensor

m_{ij} – Couple stress tensor

$\tau_{ij}^{(s)}$ – Symmetric part of τ_{ij}

$t_{ij}^{(A)}$ – Skew-symmetric part of t_{ij}

d_{ij} – Rate of deformation tensor

c_k – k-component of body moment vector

ε_{ijk} – Levi-Civita symbol

m – trace of m_{ij}

μ_1, λ_1 – Couple stress viscosity coefficients

ϕ – Hematocrit fraction

η_1, η_1' – Couple stress momentum coefficients

$\vec{f}$ – Body force per unit mass

$\vec{c}$ – Body moment per unit mass

σ – Couple stress parameter

a_0 – Average blood pressure (Pa. m^{-1})

a_1 – Pulse difference (Pa. m^{-1})

Ω – Frequency of oscillations $(2\pi \cdot HR)$

HR – Heart rate

F – Taper fraction

L – Length of the artery

R_0 – Inlet radius (at $t = 0$)

β – Contracting/Expanding

I. INTRODUCTION

It is known that cardiovascular diseases (CVDs) have been the leading cause of death globally for the past few decades. While several techniques are available to assess the progression of the disease, there is a need to predict its onset, which would help in taking preventive measures. Consequently, it has been comprehended that understanding local hemodynamic can significantly contribute to related research, surgical planning, and therapy.

Whilst clinical or experimental studies have been the most sought-after methodology for several decades, mathematical modelling has also been a choice for researchers in this field. In recent times, with robust computationally efficient algorithms, numerical modelling is gaining more importance in studying cardiovascular systems. All these studies would finally evaluate some physical quantities for the given input parameters, and the analysis of the generated data plays a crucial role in further investigating and improving our understanding of disease conditions. This understanding would assist in bringing a personalized approach to cardiovascular care.

Regarding data analysis, statistical tools were the most preferred for several decades. Peiffer et al. emphasized the need for using appropriate statistical techniques to improve understanding of the relationship between blood flow and atherogenesis [1]. Statistical analysis during the designing phase of an experiment and analysis phase has been summarised by Merry L. Lindsey et al.[2]. Sylvan Wallenstein et al. detailed statistical techniques for analysing the data on circulation.[3]. Various statistical measures that can extract accurate results from the data are discussed by Hideo Kusuoka et al. [4].

In this work, we propose constructing statistical models for the data generated from the mathematical models developed in our earlier studies to estimate WSS and QF[5-8]. We trust these models would aid in a quick evaluation of the physical quantities associated with the cardiovascular system, which otherwise require tedious procedures such as computing them from the mathematical models each time we have a new set of input parameters.

The current work continues the studies reported by Radhika et al. [5-8], wherein the authors presented their statistical analysis of the data generated by simulating the blood flow in the human femoral artery through various mathematical models developed. However, unlike in these studies, where the objective was only to identify the potential predictors of the two physical quantities, wall shear stress (WSS) and volumetric flow rate (QF), computed through the models, we aim to

1. Find the extent of correlation of the model parameters with WSS and QF.
2. Explore whether the models proposed in the above works are statistically different.
3. Develop expressions or formulae through statistical models for estimating WSS and QF, which avoids the necessity of evaluating them from the mathematical models.
4. Check through residual analysis if the developed statistical models are reliable for estimating WSS and QF.

The paper is organized as follows: For the reader's quick reference, the first section details the features of the Human arterial system, the mathematical models developed in studies [5-8], and the methodology adopted. In the subsequent sections, we present a rigorous exploratory data analysis followed by concluding remarks.

II. MATHEMATICAL MODEL

A mathematical model built to mimic the blood flow in the human arterial system has to include (through specific mathematical expressions) its salient features,

(i) Blood flow due to the pulsatile pressure gradient induced by the heart

(ii) Blood, a complex fluid that is a suspension of particles and

(iii) the arteries, which are elastic pipes with tapering and branching

for which the following mathematical models have been implemented [5-8].

Flow (Fluid) model:

The fluid (blood) flow is governed by the principles of conservation of mass and momentum, giving rise to a coupled system of partial differential equations as shown below:

$$\nabla \cdot \vec{q} = 0 \text{ in } \Omega, t \in I, \quad (1)$$

$$\rho\left(\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right) = -\nabla p - (\nabla.\tau) + \vec{f} \text{ ιν } \Omega, t \in I, \quad (2)$$

where ρ is the density of the fluid, $\vec{q}$ is the (fluid) velocity vector, τ is the deviatoric stress tensor, $\vec{f}$ is the body force vector and p is the thermodynamic pressure in the flow domain Ω.

Model for the Pulsatile pressure gradient:

Because of the pulsatile nature of the blood flow, the pressure gradient has been assumed to take the form:

$$-\frac{\partial p}{\partial z} = a_0 + a_1 \cos \Omega t, t \geq 0 \quad (3)$$

where a_0 is the average blood pressure, a_1 is the constant amplitude of the pressure gradient (termed pulse difference) (Pa. m^{-1}) and $\Omega = 2\pi \cdot HR/60$, HR being the number of heart beats per minute[5-9].

Models for the Blood:

Newtonian Model:

The constitutive equation for Newtonian fluid is given by

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij}, \quad (4)$$

where p is the thermodynamic pressure, μ is the viscosity and e_{ij} is the rate of deformation tensor.

Couple stress Fluid Model:

The Couple stress fluid model assumes that the fluid medium can sustain couple stresses [5-8]. Thus, the non-symmetric stress tensor τ_{ij} and the couple stress tensor m_{ij} are given by:

$$\tau_{ij} = \tau_{ij}^{(s)} + \tau_{ij}^{(A)}, \qquad \text{where}$$

$$\tau_{ij}^{(s)} = -p\delta_{ij} + \lambda_1 \text{div}(\vec{q})\delta_{ij} + 2\mu_1 d_{ij}, \text{ and}$$

$$\tau_{ij}^{(A)} = -\frac{1}{2}\varepsilon_{ijk}\left(m_{,k} - 4\eta_1\omega_{i,jkk} + \rho c_k\right),$$

$$m_{ij} = \frac{1}{3} m\delta_{ij} + 4\eta_1\omega_{j,i} + 4\eta_1'\omega_{i,j} \quad (5)$$

Here $\tau_{ij}^{(s)}$ is the symmetric part of the stress tensor, $\tau_{ij}^{(A)}$ is the skew-symmetric part of the stress tensor, $\omega_{i,j} = \frac{\partial \omega_i}{\partial x_j}$ is the derivative of the i^{th} component of $\vec{\omega} = \frac{1}{2}\text{curl}\,\vec{q}$, d_{ij} is the rate of deformation tensor, c_k is the k-component of body moment vector, δ_{ij} denotes Kronecker symbol, ε_{ijk} is the Levi-Civita symbol, m is the trace of the couple stress tensor m_{ij}, and $m_{,k} = \text{grad}(m)$, $\omega_{i,jkk}$ represents $\text{grad}(\nabla^2\omega_i)$. The quantities λ_1 and μ_1 are the couple stress viscosity coefficients, while η_1 and η_1' are the couple stress momentum coefficients. These coefficients are material constants and are constrained by the inequalities:

$$\mu_1 \geq 0, 3\lambda_1 + 2\mu_1 \geq 0, \eta_1 \geq 0, |\eta_1'| \leq \eta_1 \quad (6)$$

Variable Viscosity model:

A blood model that is a function of Haematocrit is considered as given in [5]

$$\mu = \mu_0 \left(\frac{1}{1-\kappa\phi}\right) \quad (7)$$

where μ is the suspension viscosity, μ_0 is the plasma viscosity, ϕ is the Haematocrit fraction. Here κ is the shape factor of the particles, which, for RBCs, is given by

$$\kappa = 0.076 \exp\left[2.49\phi + \frac{1107}{T} e^{-1.69\phi}\right] \quad (8)$$

where T is the absolute temperature $(^\circ K)$ taken as $310.15^\circ K$. In this study, we assigned values of 0.3, 0.45, and 0.6 to ϕ to model Anaemia, Healthy, and Polycythaemia conditions, respectively.

A Model for the Artery:

The pipe or the artery has been assumed to be tapered and elastic, with its radius defined in the cylindrical coordinate system as:

$$R(z,t) = R_0 \left(1 - F\frac{z}{L}\right)(1 - \beta\Omega t)^{1/2} \quad (9)$$

Here β is a contraction/expansion coefficient. Positive values of β indicate the contraction state of the pipe, while negative values of this parameter describe the expansion state. It is to be noted that when β assumes positive values, the range of $\Omega \times t$ is fixed, ensuring the validity of the expression (9). However, when β takes on negative values, the expression is valid for any value of Ω and t. R_0 denotes the radius of the pipe at the inlet when the time $t = 0$. F is the fraction of tapering that takes on any value between 0 and 1, L is the length of the pipe (artery) [5-8].

III. CLINICAL AND ANATOMICAL DATA FOR SIMULATION

In what follows, we present the clinical and anatomical data for simulating the blood flow in the human femoral arteries. Table (1) shows different age groups and blood pressure (systole/diastole) in *mmHg* for healthy individuals in that group [9].

TABLE 1. BLOOD PRESSURE IN MMHG

Age Group	Female	Male
19-24 (g1)	120/79	120/79
25-29(g2)	120/80	121/80
30-35(g3)	122/81	123/82
36-39(g4)	123/82	124/83
40-45(g5)	124/83	125/83
46-49(g6)	126/84	127/84
50-55(g7)	129/85	128/85
56-59(g8)	130/86	131/87
60+(g9)	134/84	135/88

Table (2) presents the heart rate range in healthy individuals for different male and female groups [10]. Table (3) contains the values assigned to other model parameters as constrained by the mathematical models described in expressions (1) to (9). The density and viscosity values are from human blood data [11-13].

TABLE 2. HEART RATE PER MINUTE

Age Group	**Heart Rate**	
	Data from the literature (Males)	***Data from the literature (Females)***
19-25	70-73	70-73
26-35	73-76	71-74
36-45	74-78	71-75
46-55	74-77	72-76
56-65	74-77	72-75
65+	73-76	70-73

TABLE 3. OTHER PARAMETERS AND THE VALUES CONSIDERED

Density (ρ) (in $kg.m^{-3}$)	**Viscosity (μ) (in Pa.s)**	**F**	**β**	**ϕ**	**σ**
1055 1058.1 1061.2	0.00552 0.00619 0.00686	0.01 0.03 0.05 0.07 0.09	−0.05 −0.01 0.01 0.05	0.30 0.45 0.60	0.001 0.01 0.3 0.5 0.95 0.99

IV. SOLUTION TO THE MATHEMATICAL MODEL

Mathematical models for the blood flow in the human femoral artery have been developed using the models for the blood, artery, and blood flow mentioned above. These models are solved using a Homotopy Analysis Method (HAM), an appropriate analytical method, and the WSS and QF expressions have been derived. As mentioned earlier, the present work is an extension to that reported in [5-8]; hence the detailed methodology and expressions have not been shown explicitly. The readers may refer to the authors' works in references.

V. STATISTICAL ANALYSIS

The statistical analysis begins with Section A, where we present the correlation analysis of different model parameters with WSS and QF using the simulated dataset provided in reference 15. In section B, we present linear and non–linear (logarithmic) regression models for the data. Section C presents the residual plots to determine the model's reliability in predicting the dependent variables, WSS and QF.

We present our data analysis for the simulated data from four models, namely,

1. Model 1: Newtonian Model with constant viscosity.
2. Model 2: Newtonian Model with variable viscosity.
3. Model 3: Couple stress fluid model with constant viscosity.
4. Model 4: Couple stress fluid model with variable viscosity.

SECTION - A

The summary statistics of model I (female population) are in Table (4). The correlation of different model parameters with WSS and QF using the Kruskal Wallis test is found and shown in Tables (5) and (6). It has been observed that the parameters HR, F, β, and ρ are positively related to WSS whereas μ , a_0 and a_1 are negatively related to it. Further, the parameters that are negatively correlated to QF are HR, F, β, and ρ, whereas those that are positively related are μ, a_0 and a_1.

TABLE 4. SUMMARY STATISTICS – MODEL 1 (FEMALE POPULATION)

Descriptive Statistics	HR	F	β	ρ	μ	WSS	QF	a_0	a_1
Minimum	2.000	2.000	2.0	2.000	2.0	2.000	2.000	2.000	2.000
1st Quartile	2.500	2.222	2.3	2.0	2.0	2.404	2.295	2.250	2.100
Median	2.625	2.556	2.5	2.5	2.5	2.590	2.396	2.500	2.100
Mean	2.620	2.537	2.5	2.5	2.5	2.570	2.408	2.528	2.267
3rd Quartile	2.750	2.889	2.7	3.0	3.0	2.571	2.512	2.875	2.400
Maximum	3.000	3.000	3.0	3.000	3.0	3.000	3.000	3.000	3.000

TABLE 5. CORRELATION OF WSS WITH OTHER MODEL PARAMETERS– MODEL 1 (FEMALE POPULATION)

Model Parameters	Correlation value	P - Value	Inference
HR	0.2060963	2.2×10^{-16}	Significant
F	0.06988329	1.264×10^{-13}	Significant
β	0.08255379	2.2×10^{-16}	Significant
ρ	0.01752659	0.08214	Not Significant
μ	-0.7706699	2.2×10^{-16}	Significant
a_0	-0.08390818	2.2×10^{-16}	Significant
a_1	-0.1489064	2.2×10^{-16}	Significant

TABLE 6. CORRELATION OF QF WITH OTHER MODEL PARAMETERS – MODEL 1 (FEMALE POPULATION)

Model Parameters	Correlation value	P - Value	Inference
HR	-0.2594667	2.2×10^{-16}	Significant

F	-0.3584455	2.2×10^{-16}	Significant
β	-0.4311846	2.2×10^{-16}	Significant
ρ	-0.02254258	0.02536	Significant
μ	0.3107411	2.2×10^{-16}	Significant
a_0	0.1055827	2.2×10^{-16}	Significant
a_1	0.1875756	2.2×10^{-16}	Significant

Kendal's test has been performed on the model parameters to test their significance on the dependent variables. The results are in Tables (7) and (8). All parameters are significant on $WSS\ and\ QF$; however, ρ is not significant in both cases.

TABLE 7. SIGNIFICANCE TEST OF WSS WITH OTHER MODEL PARAMETERS – MODEL 1 (FEMALE POPULATION)

Model Parameters	**P – Value**	**Inference**
HR	2.2×10^{-16}	Significant
F	1.264×10^{-10}	Significant
β	1.421×10^{-15}	Significant
ρ	0.2233	Not Significant
μ	2.2×10^{-16}	Significant
a_0	2.2×10^{-16}	Significant
a_1	2.2×10^{-16}	Significant

TABLE 8. SIGNIFICANCE TEST OF QF WITH OTHER MODEL PARAMETERS – MODEL 1 (FEMALE POPULATION)

Model Parameters	**P – Value**	**Inference**
HR	2.2×10^{-16}	Significant
F	2.2×10^{-16}	Significant
β	2.2×10^{-16}	Significant
ρ	0.08453	Not Significant
μ	2.2×10^{-16}	Significant
a_0	2.2×10^{-16}	Significant
a_1	2.2×10^{-16}	Significant

A summary of the descriptive statistics and investigations carried out on both the male and female population and all four models are presented in Tables (9) to (19).

TABLE 9. SUMMARY STATISTICS – MODEL 1 (MALE POPULATION)

Descriptive Statistics	$\boldsymbol{HR}$	$\boldsymbol{F}$	$\boldsymbol{\beta}$	$\boldsymbol{\rho}$	$\boldsymbol{\mu}$	**WSS**	**QF**	$\boldsymbol{a_0}$	$\boldsymbol{a_1}$
Minimum	2.000	2.000	2.0	2.000	2.0	2.000	2.000	2.000	2.000

1st Quartile	2.333	2.222	2.3	2.0	2.0	2.506	2.270	2.273	2.000
Median	2.500	2.556	2.5	2.5	2.5	2.636	2.392	2.394	2.167
Mean	2.488	2.537	2.5	2.5	2.5	2.627	2.401	2.451	2.259
3rd Quartile	2.667	2.889	2.7	3.0	3.0	2.770	2.523	2.606	2.333
Maximum	3.000	3.000	3.0	3.000	3.0	3.000	3.000	3.000	3.000

TABLE 10. SUMMARY STATISTICS – MODEL 2 (FEMALE POPULATION)

Descriptive Statistics	HR	F	β	ϕ	μ	**WSS**	**QF**	a_0	a_1
Minimum	2.000	2.000	2.0	2.000	2.000	2.000	2.000	2.000	2.000
1st Quartile	2.500	2.25	2.3	2.0	2.000	2.478	2.127	2.250	2.100
Median	2.625	2.50	2.5	2.5	2.254	2.598	2.426	2.500	2.100
Mean	2.620	2.50	2.5	2.5	2.418	2.584	2.405	2.528	2.267
3rd Quartile	2.750	2.75	2.7	3.0	3.000	2.706	2.621	2.875	2.400
Maximum	3.000	3.000	3.0	3.000	3.000	3.000	3.000	3.000	3.000

TABLE 11. SUMMARY STATISTICS – MODEL 2 (MALE POPULATION)

Descriptive Statistics	HR	F	β	ϕ	**WSS**	**QF**	a_0	a_1
Minimum	2.000	2.000	2.0	2.000	2.000	2.000	2.000	2.000
1st Quartile	2.000	2.25	2.3	2.0	2.523	2.122	2.273	2.000
Median	2.500	2.50	2.5	2.5	2.643	2.416	2.394	2.167
Mean	2.488	2.50	2.5	2.5	2.618	2.397	2.451	2.259
3rd Quartile	2.667	2.75	2.7	3.0	2.739	2.607	2.606	2.333
Maximum	3.000	3.000	3.0	3.000	3.000	3.000	3.000	3.000

TABLE 12. SUMMARY STATISTICS – MODEL 3 (FEMALE POPULATION)

Descriptive Statistics	HR	F	β	σ	μ	**WSS**	**QF**	a_0	a_1
Minimum	2.000	2.000	2.0	2.000	2.0	2.000	2.000	2.000	2.000
1st Quartile	2.500	2.222	2.3	2.014	2.0	2.872	2.158	2.250	2.100
Median	2.625	2.556	2.5	2.309	2.5	2.910	2.229	2.500	2.100
Mean	2.620	2.537	2.5	2.404	2.5	2.899	2.235	2.528	2.267
3rd Quartile	2.750	2.889	2.7	2.699	3.0	2.944	2.304	2.875	2.400
Maximum	3.000	3.000	3.0	3.000	3.0	3.000	3.000	3.000	3.000

TABLE 13. SUMMARY STATISTICS – MODEL 3 (MALE POPULATION)

Descriptive Statistics	HR	F	β	σ	μ	**WSS**	**QF**	a_0	a_1
Minimum	2.000	2.000	2.0	2.000	2.0	2.000	2.000	2.000	2.000

1st Quartile	2.333	2.222	2.3	2.014	2.0	2.466	2.258	2.273	2.000
Median	2.500	2.556	2.5	2.309	2.5	2.611	2.375	2.394	2.167
Mean	2.488	2.537	2.5	2.404	2.5	2.599	2.388	2.451	2.259
3rd Quartile	2.667	2.889	2.7	2.699	3.0	2.750	2.505	2.606	2.333
Maximum	3.000	3.000	3.0	3.000	3.0	3.000	3.000	3.000	3.000

TABLE 14. SUMMARY STATISTICS – MODEL 4 (FEMALE POPULATION)

Descriptive Statistics	HR	F	β	σ	ϕ	WSS	QF	a_0	a_1
Minimum	2.000	2.000	2.0	2.000	2.0	2.000	2.000	2.000	2.000
1st Quartile	2.500	2.222	2.3	2.009	2.0	2.188	2.232	2.250	2.100
Median	2.625	2.556	2.5	2.403	2.5	2.304	2.472	2.500	2.100
Mean	2.620	2.537	2.5	2.463	2.5	2.288	2.457	2.528	2.267
3rd Quartile	2.750	2.889	2.7	2.960	3.0	2.367	2.635	2.875	2.400
Maximum	3.000	3.000	3.0	3.000	3.0	3.000	3.000	3.000	3.000

TABLE 15. SUMMARY STATISTICS – MODEL 4 (MALE POPULATION)

Descriptive Statistics	HR	F	β	σ	ϕ	WSS	QF	a_0	a_1
Minimum	2.000	2.000	2.0	2.000	2.0	2.000	2.000	2.000	2.000
1st Quartile	2.333	2.222	2.4	2.009	2.0	2.190	2.230	2.273	2.000
Median	2.500	2.667	2.6	2.505	2.5	2.306	2.470	2.394	2.167
Mean	2.488	2.537	2.5	2.463	2.5	2.291	2.455	2.451	2.259
3rd Quartile	2.667	2.889	2.9	2.960	3.0	2.370	2.635	2.606	2.333
Maximum	3.000	3.000	3.0	3.000	3.000	3.000	3.000	3.000	3.000

TABLE 16. RESULTS OF KRUSKAL WALLIS TEST

Model		WSS								
		HR	F	β	ϕ	μ	a_0	a_1	ρ	σ
I	Female	+	+	+	NA	-	-	-	+	NA
	Male	+	+	+	NA	-	-	-	+	NA
II	Female	+	+	+	+	+	-	-	No	NA
	Male	+	+	+	+	No	-	-	No	NA
III	Female	-	+	-	NA	-	-	-	No	-
	Male	-	+	+	NA	+	-	-	No	-
IV	Female	-	+	+	+	No	-	-	No	-
	Male	-	+	+	+	No	-	-	No	-

Note: +: positive correlation, - : Negative correlation, No: No correlation, NA: Not applicable

TABLE 17. RESULTS OF KENDAL'S TEST

Model		WSS								
		HR	F	β	ϕ	μ	a_0	a_1	ρ	σ
I	Female	✓	✓	✓	NA	✓	✓	✓	×	NA
	Male	✓	✓	✓	NA	✓	✓	✓	×	NA
II	Female	✓	✓	✓	✓	✓	✓	✓	×	NA
	Male	✓	✓	✓	✓	NA	✓	✓	×	NA
III	Female	✓	✓	✓	NA	✓	✓	✓	NA	✓
	Male	✓	✓	✓	NA	✓	✓	✓	NA	✓
IV	Female	✓	✓	✓	✓	NA	✓	✓	NA	✓
	Male	✓	✓	✓	✓	NA	✓	✓	NA	✓

Note: ✓: Significant, ×: Not significant, NA: Not applicable

TABLE 18. RESULTS OF KRUSKAL WALLIS TEST

Model		QF								
		HR	F	β	ϕ	μ	a_0	a_1	ρ	σ
I	Female	-	-	-	NA	+	+	+	-	NA
	Male	-	-	-	NA	+	+	+	-	NA
II	Female	-	-	-	-	-	+	+	No	NA
	Male	-	-	-	-	No	+	+	No	NA
III	Female	+	-	-	NA	-	+	+	No	+
	Male	+	-	-	NA	-	+	+	No	+
IV	Female	+	-	-	-	No	+	+	No	+
	Male	+	-	-	-	No	+	+	No	+

Note: +: positive correlation, -: Negative correlation, No: No correlation, NA: Not applicable

TABLE 19. RESULTS OF KENDAL'S TEST

Model		QF								
		HR	F	β	ϕ	μ	a_0	a_1	ρ	σ
I	Female	✓	✓	✓	NA	✓	✓	✓	✓	NA
	Male	✓	✓	✓	NA	✓	✓	✓	×	NA
II	Female	✓	✓	✓	✓	✓	✓	✓	×	NA

	Male	✓	✓	✓	✓	NA	✓	✓	×	NA
III	Female	✓	✓	✓	NA	✓	✓	✓	NA	✓
	Male	×	✓	✓	NA	✓	✓	✓	NA	✓
IV	Female	✓	✓	✓	✓	NA	✓	✓	NA	✓
	Male	×	✓	✓	✓	NA	✓	✓	NA	✓

Note: ✓: Significant, ×: Not significant, NA: Not applicable

To test our second objective mentioned earlier, we performed Kendal's test on models 1 and 3, 2 and 4, and the results presented in Table (20) show that they are significantly different.

TABLE 20: SIGNIFICANCE TEST BETWEEN MODELS

S. No	**Model I**	**Model II**	**Model III**	**Model IV**
Model I	NA	NA	✓	NA
Model II	NA	NA	NA	✓
Model III	✓	NA	NA	NA
Model IV	NA	✓	NA	NA

Note: ✓: Significant, NA: Not applicable

SECTION – B

This section represents the linear and non–linear regression expressions developed for both populations in all four models.

Model-1: (Newtonian fluid with constant viscosity)

Female population:

The linear regression models of WSS and QF for normalized data are given by

$$\mathrm{WSS} = 2.97183996 + 0.33531422\mathrm{HR} + 0.06194844\mathrm{F} + 0.06955973\beta + 0.01275438\rho - 0.46490933\mu - 0.09233235a_0 - 0.10917868a_1 \quad (10)$$

$$\mathrm{QF} = 3.57625605 - 0.30212227\mathrm{HR} - 0.22017732\mathrm{F} - 0.24661427\beta - 0.01147001\rho + 0.15690168\mu + 0.08365908a_0 + 0.09857376a_1 \quad (11)$$

The logarithmic regression models of WSS and QF for normalized data are

$$\mathrm{WSS} = 2.89714370 + 0.84977613\mathrm{HR} + 0.15322529\mathrm{F} + 0.17078328\beta + 0.03135541\rho - 1.14002901\mu - 0.21270789a_0 - 0.29830550a_1 \quad (12)$$

$$\mathrm{QF} = 3.47702819 - 0.76563663\mathrm{HR} - 0.51511934\mathrm{F} - 0.60649113\beta - 0.02819788\rho + 0.38459460\mu + 0.19272543a_0 + 0.26938761a_1 \quad (13)$$

Male population:

The linear regression models of WSS and QF for normalized data are given by

$$\mathrm{WSS} = 2.9322505 + 0.2799881\mathrm{HR} + 0.07309223\mathrm{F} + 0.07926105\beta + 0.01137225\rho - 0.36518952\mu - 0.37432237a_0 + 0.18449581a_1 \quad (14)$$

$$\mathrm{QF} = 4.17853418 - 0.24168607\mathrm{HR} - 0.29752368\mathrm{F} - 0.32904658\beta - 0.01016268\rho + 0.02323550\mu + 0.37606337a_0 - 0.24488902a_1 \quad (15)$$

The logarithmic regression models of WSS and QF for normalized data are

$$WSS = 2.90510556 + 0.68457936HR + 0.1808080117F + 0.19459612\beta + 0.02795604\rho - 0.89722863\mu - 0.83000085a_0 + 0.34985530a_1 \quad (16)$$

$$QF = 3.96948133 - 0.58434682HR - 0.73711283F - 0.80930490\beta - 0.02498258\rho + 0.05695662\mu + 0.82755875a_0 - 0.48640010a_1 \quad (17)$$

Model 2: (Newtonian Model with variable viscosity)

Female population:

The linear regression models of WSS and QF for normalized data are given by

$$WSS = 1.61875355 + 0.40708322HR + 0.16380323F + 0.157005638\beta + 0.04819460\phi - 0.2533818a_0 - 0.19965154a_1 + 0.02843231\mu \quad (18)$$

$$QF = 4.52996240 - 0.07796552HR - 0.12150059F - 0.14074776\beta - 0.45232664\phi + 0.04830322a_0 + 0.04013512a_1 - 0.14346402\mu \quad (19)$$

The logarithmic regression models of WSS and QF for normalized data are

$$WSS = 1.69535218 + 1.04324970HR + 0.40334404F + 0.38533339\beta + 0.08425473\phi - 0.59069866a_0 - 0.55749074a_1 + 0.10473050\mu \quad (20)$$

$$QF = 4.2929225 - 0.1996678HR - 0.2996258F - 0.3460480\beta - 0.8393895\phi + 0.1122443a_0 + 0.1117414a_1 - 0.6300111\mu \quad (21)$$

Male population:

The linear regression models of WSS and QF for normalized data are given by

$$WSS = 2.40848020 + 0.09856032HR + 0.15317097F + 0.14868052\beta + 0.06991393\phi - 0.32649318a_0 - 0.11566743a_1 \quad (22)$$

$$QF = 4.35167177 - 0.01974618HR - 0.11896029F - 0.13827410\beta - 0.58326471\phi + 0.06513078a_0 + 0.02427359a_1 \quad (23)$$

The logarithmic regression models of WSS and QF for normalized data are

$$WSS = 2.3155936 + 0.3315618HR + 0.3771634F + 0.3649072\beta + 0.1708208\varphi - 0.7090875a_0 - 0.3671999a_1 \quad (24)$$

$$QF = 4.15271038 - 0.06645224HR - 0.29336107F - 0.33996924\beta - 1.42795552\phi + 0.14129149a_0 + 0.07627396a_1 \quad (25)$$

Model 3: (Couple Stress Model with constant viscosity)

Female population:

The linear regression models of WSS and QF for normalized data are given by

$$WSS = 3.296147612 + 0.024929876HR + 0.037177634F - 0.008005173\beta - 0.100754410\sigma - 0.031833089\mu - 0.052672410a_0 - 0.036002720a_1 \quad (26)$$

$$QF = 2.97446475 - 0.02166091HR - 0.14379398F - 0.11697513\beta + 0.05151478\sigma - 0.14694691\mu + 0.0530548a_0 + 0.03727736a_1 \quad (27)$$

The logarithmic regression models of WSS and QF for normalized data are

$$WSS = 3.24377465 + 0.06581677HR + 0.09234815F - 0.02301210\beta - 0.24160877\sigma - 0.07644953\mu - 0.12187089a_0 - 0.10210723a_1 \quad (28)$$

$$QF = 2.90209846 - 0.05735493HR - 0.35659964F - 0.28445625\beta + 0.12231908\sigma - 0.36430751\mu + 0.12236977a_0 + 0.10555333a_1 \quad (29)$$

Male population:

The linear regression models of WSS and QF for normalized data are given by

$$WSS = 3.489845060 + 0.004858846HR + 0.165810687F + 0.025240261\beta - 0.329635394\sigma + 0.012814236\mu - 0.280558335a_0 - 0.105613985a_1 \quad (30)$$

$$QF = 3.76995148 - 0.24128906F - 0.21523820\beta + 0.05201724\sigma - 0.29323200\mu + 0.11338320a_0 + 0.04364628a_1 \quad (31)$$

The logarithmic regression models of WSS and QF for normalized data are

$$WSS = 3.61690437 + 0.02548080HR + 0.41003612F + 0.06150832\beta - 0.80105332\sigma + 0.03211521\mu - 0.59909031a_0 - 0.34755268a_1 \quad (32)$$

$$QF = 3.637795719 - 0.006909651HR - 0.597698805F - 0.528994520\beta + 0.126343673\sigma - 0.723609734\mu + 0.24304047a_0 + 0.14174613a_1 \quad (33)$$

Model 4: (Couple Stress Model with variable viscosity)

Female population:

The linear regression models of WSS and QF for normalized data are given by

$$WSS = 2.85499829 + 0.05198128F + 0.01257197\beta - 0.20458855\sigma + 0.01932719\phi - 0.06152483a_0 - 0.05257346a_1 \quad (34)$$

$$QF = 3.92916812 - 0.11379411F - 0.10278122\beta + 0.04284790\sigma - 0.48169291\phi + 0.03842387a_0 + 0.03306265a_1 \quad (35)$$

The logarithmic regression models of WSS and QF for normalized data are

$$\mathrm{WSS} = 2.778214815 + 0.007660109\mathrm{HR} + 0.128546041\mathrm{F} + 0.035760582\beta - 0.503780404\sigma + 0.052258205\phi - 0.143101529\mathrm{a}_0 - 0.144078924\mathrm{a}_1 \tag{36}$$

$$\mathrm{QF} = 3.769357042 - 0.003783119\mathrm{HR} - 0.281883106\mathrm{F} - 0.255623188\beta + 0.106220750\sigma - 1.182352489\phi + 0.08877948\mathrm{a}_0 + 0.091044310\mathrm{a}_1 \tag{37}$$

Male population:

The linear regression models of WSS and QF for normalized data are given by

$$\mathrm{WSS} = 2.85984207 + 0.05185607\mathrm{F} + 0.01397957\beta - 0.20426511\sigma + 0.01943804\phi - 0.08515374\mathrm{a}_0 - 0.03215044\mathrm{a}_1 \tag{38}$$

$$\mathrm{QF} = 3.93362329 - 0.11215147\mathrm{F} - 0.10266653\beta + 0.04270734\sigma - 0.48465694\mathrm{phi} + 0.04560317\mathrm{a}_0 + 0.02527213\mathrm{a}_1 \tag{39}$$

The logarithmic regression models of WSS and QF for normalized data are

$$\mathrm{WSS} = 2.775661303 + 0.008731785\mathrm{HR} + 0.128239074\mathrm{F} + .039192280\beta - 0.503013075\sigma + 0.052530944\phi - 0.185453238\mathrm{a}_0 - 0.102497074\mathrm{a}_1 \tag{40}$$

$$\mathrm{QF} = 3.777791003 - 0.004057671\mathrm{HR} - 0.277816989\mathrm{F} - 0.255328899\beta + 0.106020578\sigma - 1.189751304\phi + 0.096202946\mathrm{a}_0 + 0.077106412\mathrm{a}_1 \tag{41}$$

SECTION – C

This section explores further the statistical models presented in the previous section. We computed the coefficient of determination for both linear and non–linear models and presented the results in Table (21).

TABLE 21: COEFFICIENT OF DETERMINATION VALUES FOR LINEAR AND NON-LINEAR REGRESSION MODELS

Model	**Gender**	**Variable under study**	**Linear**	**Logarithmic**
Model 1	***Female***	WSS	0.9979468	0.9955042
		QF	0.9876963	0.9773179
	Male	WSS	0.9983073	0.9983073
		QF	0.987456	0.987456
Model 2	***Female***	WSS	0.9820751	0.9739404
		QF	0.9757273	0.971062
	Male	WSS	0.9456723	0.9392179
		QF	0.9521916	0.9466861
	Female	WSS	0.484211	0.4832149

Model 3		QF	0.9177822	0.913292
	Male	WSS	0.4864623	0.4853612
		QF	0.9186411	0.9142756
Model 4	***Female***	WSS	0.5624823	0.550387
		QF	0.8300894	0.8289864
	Male	WSS	0.9795731	0.9756096
		QF	0.9896296	0.9907651

Also, the residual plots are shown in figures (1 – 4).

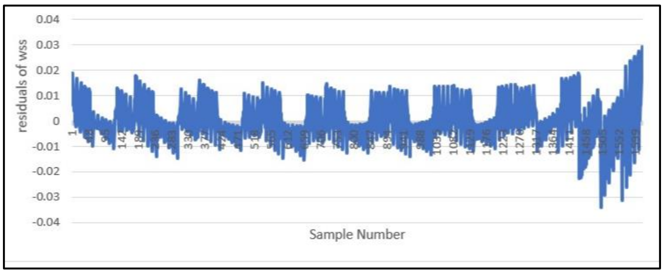

Fig. 1: Residual Plot of WSS for the Normalized data of Linear Model

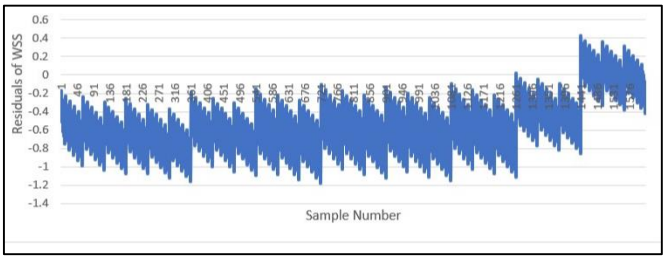

Fig. 2: Residual Plot of WSS for the normalized data of non - linear model

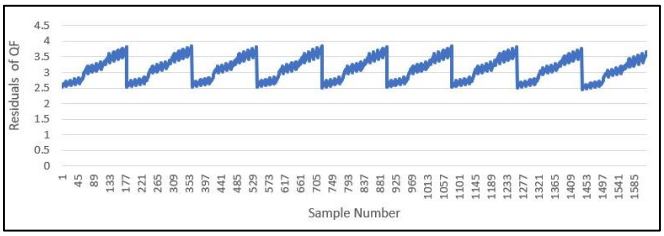

Fig. 3: Residual Plot of QF for the Normalized data of Linear Model

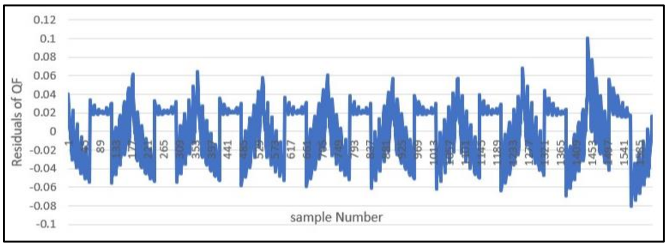

Fig. 4: Residual Plot of QF for the normalized data of non - linear model

It is observed that even though the coefficient of determination shows that the models are a good fit, the residual plots showed patterns indicating that these models may not be accurate for predicting the WSS and QF.

VI. CONCLUSION

This work is focused on building statistical models for data simulated using the mathematical models developed in our earlier works. We worked on fitting both linear and non-linear regression models for the data and obtained expressions for computing WSS and QF, given the values of the model parameters. To explore the suitability of these models, we computed the coefficient of determination, which showed an excellent dependency of the independent model parameters on the dependent parameters. We calculated the estimates for WSS using these models and presented them in Tables (22) and (23). These estimated values for different age groups are compared with the reported values of 2.77 pas – 5.31 pas by Kornet et al. [14]. Though the estimated values are in close correspondence with the clinical values, it is desirable to have greater accuracy.

TABLE 22: ESTIMATES OF WSS FROM THE LINEAR MODEL.

Model	**Gender**	**20 - 29**	**30 - 39**	**40 - 49**	**50 - 59**
I	***Male***	2.742566642	2.685418042	2.645421745	2.560340624
	Female	2.659340165	2.689888613	2.63641682	2.489873015
2	***Male***	2.683522581	2.627060753	2.667342715	2.59005427
	Female	2.565485559	2.603406421	2.634720122	2.545955328
3	***Male***	2.333467816	2.315020173	2.290738628	2.787754769
	Female	2.926852355	2.917928545	2.290895567	2.395962147
4	***Male***	2.926852355	2.917928545	2.903122262	2.884010457
	Female	2.739483708	2.665988152	2.599374014	2.475777723

TABLE 23: ESTIMATES OF WSS FROM THE NON-LINEAR MODEL.

Model	**Gender**	**20 - 29**	**30 - 39**	**40 - 49**	**50 - 59**
I	***Male***	4.680025272	4.351090017	4.959948175	4.92326775
	Female	5.697382313	5.971355136	6.032814625	5.992157088
2	***Male***	3.068328572	3.115927766	3.239938516	3.231111481

	Female	2.706267702	2.803248684	3.116389184	3.039471635
3	***Male***	1.918269847	1.878923774	1.849308226	1.807707999
	Female	1.910518248	1.871781484	3.108499517	1.791899831
4	***Male***	2.55950952	2.655885465	2.65083886	2.626263316
	Female	2.344195949	2.56640607	2.170463483	2.064325171

Continuing our exploration, we plotted the residual plots to check the predictive efficiency of the developed models. However, we observed patterns in these plots indicating non – randomness. Hence, we built linear and non–linear regression models for group-wise data but still observed non-random patterns in the residual plots. All these efforts made us conclude that these regression models could not capture the predictive information completely. Thus, it is indicative of performing data analysis using advanced predictive algorithms, which is the scope of our future study.

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