SOME RESULTS ON WEIGHTED MEAN LABELING

R.Christina Mary[1*] , S.Pasunkili Pandian[2], S.Arulraj[3]

Abstract
The labeling of a graph G is said to be weighted Mean labelling (WML), if its vertices are labelled from 0,1,2,...,q where q is the number of edges of G such that the edges of G can be labelled with $\left\lceil\frac{f(\delta)\deg(\delta)+f(\mu)\deg(\mu)}{deg(\delta)+deg(\mu)\}}\right\rceil$ the resulting edge labels are distinct from 1,2,...,q. If a graph G admits WML then, G is said to be Weighted Mean Graph (WMG). In this paper we study the WML of some Cycle related graphs.

Keywords. Weighted Mean Labeling, Weighted Mean Graph.

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DOI: 10.53555/ecb/2022.11.10.97

2010 Mathematics Subject Classification: 05C78

1. Introduction

Rosa introduced the graceful labeling method in 1967 [4] further S.Somasundaram and R.Ponraj developed the concept of mean labeling of graphs and studied the behavior [5][6]. R.Christina et.al defined the Weighted mean labeling of graphs in [1].

The labeling of a graph G is said to be weighted Mean labelling (WML), if its vertices are labelled from 0,1,2,...,q where q is the number of edges of G such that the edges of G can be labelled with $\left\lceil \frac{f(u)deg(u)+f(v)deg(v)}{deg(u)+deg(v)} \right\rceil$ the resulting edge labels are distinct from 1,2,...,q. If a graph G admits WML then, G is said to be Weighted Mean Graph (WMG).

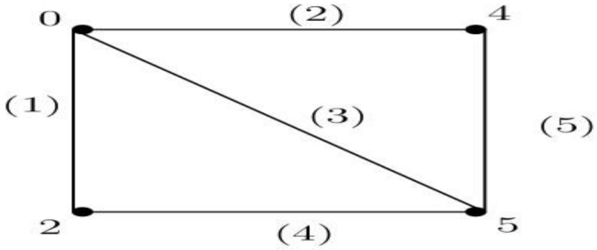

Figure 1

For standard terminology and notation the reader can be refer to [3] and the study the graph labeling the reader can be refer to Gallian(2022) [2].

2. Preliminaries

Definition 2.1[7]

Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of P_n ;a triangular tree is calculated by amalgamating each v_i with a leaf of P_i .We denote this tree by T_n and refer to P_n as the base of T_n .

3. Results

Theorem 3.1. *The graph obtained by identifying a vertex of any two cycles C_m and C_n is WMG.*

Proof. Let $\delta_1, \delta_2, \ldots, \delta_m$ and $\eta_1, \eta_2, \ldots, \eta_n$ be the vertices of the cycle C_m and C_n respectively. Let G be the resultant graph obtained by identifying the vertex δ_m of the cycle C_m to the vertex η_n of the cycle C_n.

We define $\pi : V(G) \to \{0,1,\ldots,m+n\}$ as follows:

$$\pi(\delta_\alpha) = \begin{cases} \alpha - 1 \ \ if \ 1 \le \alpha \le \left\lceil \frac{4m}{6} \right\rceil - 1 \\ \alpha \ \ if \ \left\lceil \frac{4m}{6} \right\rceil \le \alpha \le m \end{cases}$$

$$\pi(\eta_\alpha) = \begin{cases} m + \alpha \ \ if \ 1 \le \alpha \le \left\lceil \frac{3m+n}{3} \right\rceil - m - 2 \\ m + \alpha + 1 \ \ if \ \left\lceil \frac{3m+n}{3} \right\rceil - m - 1 \le \alpha \le n - 1 \end{cases}$$

We attained the following edge labeling as:

$$\pi^*(\delta_\alpha \delta_{\alpha+1}) = \begin{cases} \alpha \ \ if \ 1 \le \alpha \le \left\lceil \frac{4m}{6} \right\rceil - 1 \\ \alpha + 1 \ \ if \ \left\lceil \frac{4m}{6} \right\rceil \le \alpha \le m - 1 \end{cases}$$

$$\pi^*(\delta_1 \delta_n) = \left\lceil \frac{4m}{6} \right\rceil$$

$$\pi^*(\eta_\alpha \eta_{\alpha+1}) = \begin{cases} m + \alpha + 1 \ \ if \ 1 \le \alpha \le \left\lceil \frac{3m+n}{3} \right\rceil - m - 2 \\ m + \alpha + 2 \ \ if \ \left\lceil \frac{3m+n}{3} \right\rceil - m - 1 \le \alpha \le n - 2 \end{cases}$$

$$\pi^*(\eta_{n-1} \eta_n) = \left\lceil \frac{3m+n}{3} \right\rceil$$

$$\pi^*(\eta_n \eta_1) = m + 1$$

Thus the graph labelled with WML and hence we conclude the graph is WMG. □

WML of the graph G obtained by identifying a vertex of the cycles C_8 and C_8 is shown in Figure 2

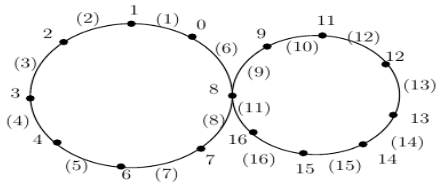

Figure 2

Theorem 3.2. *The graph obtained by identifying an edge of any two cycles C_m and C_n is a WMG.*
Proof. Let $\delta_1,\delta_2,\ldots,\delta_m$ and $\eta_1,\eta_2,\ldots,\eta_n$ be the vertices of the cycle C_m and C_n respectively. Let G be the resultant graph obtained by identifying an edge $\delta_{m-1}\delta_m$ of the cycle C_m with an edge $\eta_{n-1}\eta_n$ of the cycle C_n.
We define $\pi:V(G)\to\{0,1,\ldots,m+n-1\}$ as follows:

$$\pi(\delta_\alpha) = \begin{cases} \alpha - 1 \ \ if \ 1 \le \alpha \le \left\lceil \frac{3m}{5} \right\rceil - 1 \\ \alpha \ \ if \ \left\lceil \frac{3m}{5} \right\rceil \le \alpha \le m \end{cases}$$

$$\pi(\eta_\alpha) = \begin{cases} m + \alpha \ \ if \ 1 \le \alpha \le \left\lceil \frac{5m+2n-3}{5} \right\rceil - m - 2 \\ m + \alpha + 1 \ \ if \ \left\lceil \frac{5m+2n-3}{5} \right\rceil - m - 1 \le \alpha \le n - 1 \end{cases}$$

We attained the following edge labeling as:

$$\pi^*(\delta_\alpha\delta_{\alpha+1}) = \begin{cases} \alpha \ \ if \ 1 \le \alpha \le \left\lceil \frac{3m}{5} \right\rceil - 1 \\ \alpha + 1 \ \ if \ \left\lceil \frac{3m}{5} \right\rceil \le \alpha \le m - 1 \end{cases}$$

$$\pi^*(\delta_1\delta_m) = \left\lceil \frac{3m}{6} \right\rceil$$

$$\pi^*(\eta_\alpha\eta_{\alpha+1}) = \begin{cases} m + \alpha + 1 \ \ if \ 1 \le \alpha \le \left\lceil \frac{5m+2n-3}{5} \right\rceil - m - 2 \\ m + \alpha + 2 \ \ if \ \left\lceil \frac{5m+2n-3}{5} \right\rceil - m - 1 \le \alpha \le n - 3 \end{cases}$$

$$\pi^*(\eta_{n-2}\eta_{n-1}) = \left\lceil \frac{5m+2n-3}{5} \right\rceil$$

Thus the graph labelled with WML and hence we conclude that the graph is WMG.□
WML of the graph G obtained by identifying a edge of the cycles C_{10} and C_7 is shown in Figure 3.

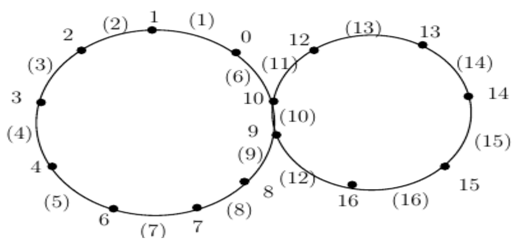

Figure 3

Theorem 3.3 *The graph obtained by joining any two cycles C_m and C_n by a path P_k is a WMG.*

Proof. Let G be a graph obtained by joining any two cycles C_m and C_n by a path P_k. Let $\delta_1,\delta_2,\ldots,\delta_m$ and $\eta_1,\eta_2,\ldots,\eta_n$ be the vertices of the cycle C_m and C_n respectively. Let $\gamma_1,\gamma_2,\ldots,\gamma_k$ be the vertices of the path P_k with $\delta_m=\gamma_1$ and $\gamma_k=\eta_n$.

We define $\pi : V(G) \to \{0,1,\ldots,m+k+n\}$ as follows:

$$\pi(\delta_\alpha) = \begin{cases} \alpha - 1 \ if \ 1 \le \alpha \le \left\lceil \frac{3m}{5} \right\rceil - 1 \\ \alpha \ if \ \left\lceil \frac{3m}{5} \right\rceil \le \alpha \le m \end{cases}$$

$$\pi(\gamma_a) = m + \alpha - 1 \ if \ 2 \le \alpha \le k$$

$$\pi(\eta_\alpha) = \begin{cases} m + k + \alpha \ if \ 1 \le \alpha \le \left\lceil \frac{5m+5k+2n-2}{5} \right\rceil \\ m + k + \alpha + 1 \ if \ \left\lceil \frac{5m+5k+2n-2}{5} \right\rceil + 1 \le \alpha \le n-2 \end{cases}$$

We attained the following edge labeling as:

$$\pi^*(\delta_\alpha\delta_{\alpha+1}) = \begin{cases} \alpha - 1 \ if \ 1 \le \alpha \le \left\lceil \frac{3m}{5} \right\rceil - 1 \\ \alpha + 1 \ if \ \left\lceil \frac{3m}{5} \right\rceil \le \alpha \le m-1 \end{cases}$$

$$\pi^*(\delta_1\delta_m) = \left\lceil \frac{3m}{5} \right\rceil$$

$$\pi^*(\gamma_a\gamma_{a+1}) = m + \alpha \ if 1 \le \alpha \le k-1$$

$$\pi^*(\delta_m\gamma_2) = m + 1 \ if \ 2 \le \alpha \le k$$

$$\pi^*(\eta_\mu\eta_{\alpha+1}) = \begin{cases} m + k + \alpha + 1 \ if \ 1 \le \alpha \le \left\lceil \frac{5m+5k+2n-2}{5} \right\rceil \\ m + k + \alpha + 2 \ if \ \left\lceil \frac{5m+5k+2n-2}{5} \right\rceil + 1 \le \mu \le n-2 \end{cases}$$

$$\pi^*(\eta_{\mu-1}\eta_\mu) = \left\lceil \frac{5m+5k+2n}{5} \right\rceil$$

$$\pi^*(\eta_\mu\eta_1) = m + k + 1$$

Thus the graph is labelled with WML and hence we conclude that the graph isWMG. □

WML of the graph G obtained by joining two cycles C_9 and C_{11} by a path P_4 is shown in Figure 4.

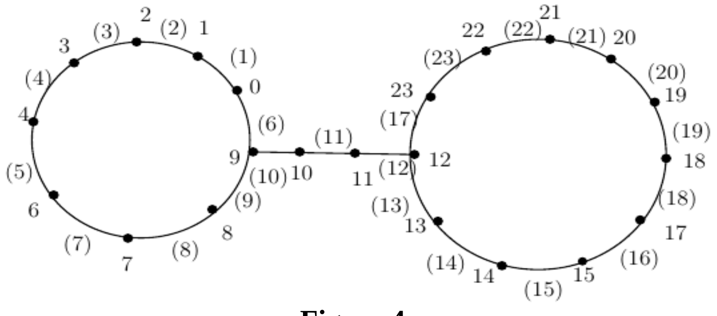

Figure 4

Theorem 3.4. $A(D(T_n))$ is a WMG.

Proof. Let $V(A(D(T_n))) = \{\delta_1,\delta_2,\ldots,\delta_n,\eta_1,\eta_2,\ldots,\eta_{\frac{n}{2}}, \mu_1,\mu_2,\ldots,\mu_{\frac{n}{2}}\}$ and the $E\left(A\left(D(T_n)\right)\right) = \{\delta_\alpha\delta_{\alpha+1} : 1 \le \alpha \le n-1\} \cup \left\{\delta_{2\alpha-1}\eta_\alpha : 1 \le \alpha \le \frac{n}{2}\right\} \cup \left\{\delta_{2\alpha-1}\mu_\alpha : 1 \le \alpha \le \frac{n}{2}\right\} \cup \left\{\delta_{2\alpha}\eta_\alpha : 1 \le \alpha \le \frac{n}{2}\right\} \cup \left\{\delta_{2\alpha}\mu_\alpha : 1 \le \alpha \le \frac{n}{2}\right\}$.

Define $\pi: \mathrm{V}(\mathrm{A}(\mathrm{D}(T_n))) \rightarrow \{0,1,2,..,3n-1\}$ as follows:
$\pi(\delta_1)$=0,

$$\pi(\delta_\alpha) = \begin{cases} 3\alpha - 1 & \text{if } \alpha = even\ and 2 \leq \alpha \leq n \\ 3\alpha - 3 & \text{if } \alpha = odd\ and\ 3 \leq \alpha \leq n' \end{cases}$$

$\pi(\eta_1)$=1,
$\pi(\eta_\alpha) = 6\alpha - 4$, if $2 \leq \alpha \leq \frac{n}{2}$,
$\pi(\mu_1)$=3 and
$\pi(\mu_\alpha) = 6\alpha - 2$, if $2 \leq \alpha \leq \frac{n}{2}$.
we attain the following edge labeling as:
$\pi^*(\delta_\alpha \delta_{\alpha+1}) = 3\alpha$, if $1 \leq \alpha \leq n - 1$.
$\pi^*(\delta_1 \eta_1) = 1$,
$\pi^*(\delta_{2\alpha-1} \eta_\alpha) = 3\alpha - 2$, if $1 \leq \alpha \leq \frac{n}{2}$,
$\pi^*(\delta_{2\alpha} \eta_\alpha) = 3\alpha - 2$, if $1 \leq \alpha \leq \frac{n}{2}$,
$\pi^*(\delta_1 \mu_1) = 2$,
$\pi^*(\delta_{2\alpha-1} \mu_\alpha) = 3\alpha - 1$, if $2 \leq \alpha \leq \frac{n}{2}$ and
$\pi^*(\mu_\alpha \delta_{2\alpha}) = 3\alpha - 1$, if $1 \leq \alpha \leq \frac{n}{2}$.
Thus A(D(T_n)) is labelled with WML and hence we conclude that A(D(T_n)) is a WMG.

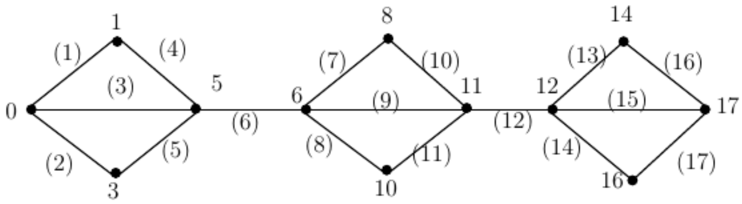

Figure 5:WML of A(D(T_3))

Theorem 3.5.$P_m \times P_n$ is a WMG for m = 3 and n ≥ 2.
Proof. Let $\mathrm{V}(P_m \times P_n) = \{\delta_{\alpha\beta}: 1 \leq \alpha \leq m, 1 \leq \beta \leq n\}$ be the vertex set of $P_m \times P_n\ and E(P_m \times P_n) = \{\delta_{\alpha\beta}\delta_{(\alpha+1)\beta}: 1 \leq \alpha \leq m-1, 1 \leq \beta \leq n\} \cup \{\delta_{\alpha\beta}\delta_{\alpha(\beta+1)}: 1 \leq \alpha \leq m, 1 \leq \beta \leq n - 1\}$be the edge set of $P_m \times P_n$.
If $m = 3$ and n ≥ 2.
Define $\pi: V(P_m \times P_n) \rightarrow \{0,1,2,..,5n-3\}$ as follows:

$$\pi(\delta_{1\beta}) = \begin{cases} 0, & \text{if } \beta = 1 \\ 5\beta - 7, & \text{if } 2 \leq \beta \leq n \end{cases}$$

$\pi(\delta_{2\beta}) = 5\beta - 4,$ if $1 \leq \beta \leq n,$
$\pi(\delta_{3\beta}) = 5\beta - 1,$ if $1 \leq \beta \leq n.$
The induced edge labeling is as follows:
$\pi^*(\delta_{\alpha\beta}\delta_{(\alpha+1)\beta}) = \alpha + 5(\beta - 1)$,if $1 \leq \alpha \leq 2$ and $2 \leq \beta \leq n$
$\pi^*(\delta_{\alpha\beta}\delta_{\alpha(\beta+1)}) = \alpha + 5\beta - 3$,if $1 \leq \alpha \leq 3$ and $2 \leq \beta \leq n - 1$,
$\pi^*(\delta_{\alpha 1}\delta_{(\alpha+1)1}) = 2\alpha - 1$, if $1 \leq \alpha \leq 2$,

$$\pi^*(\delta_{\alpha 1}\delta_{(\alpha+1)1}) = \begin{cases} 2\alpha, & \text{if } 1 \leq \alpha \leq 2 \\ 7, & \text{if } \alpha = 3. \end{cases}$$

Thus $P_m \times P_n$ is labelled with WML and hence we conclude that $P_m \times P_n$is a WMG.

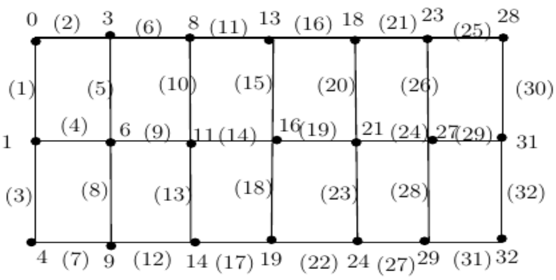

Figure 6 : WML of P_3X P_7

Theorem 3.6. Each triangular tree T_n is a WMG.

Proof. Assume $\delta_1, \delta_2, \dots, \delta_n$ be the vertices of the path P_n and each vertex adjoining the path that is represented by $\delta_{\alpha\beta}$, for $\alpha = 1,2, \dots n$ and $\beta < \alpha$.

Define $\pi: V(T_n) \to \left\{0,1,2,\dots,\frac{n(n+1)}{2}\right\}$ as follows:

Case 1. n is odd

$$\pi(\delta_\alpha) = \begin{cases} \left\lfloor\frac{\alpha(\alpha+1)}{2}\right\rfloor - 1, & \text{if } \alpha \text{ is odd} \\ \left\lfloor\frac{\alpha^2-\alpha+2}{2}\right\rfloor - 1, & \text{if } \alpha \text{ is even} \end{cases}.$$

$$\pi(\delta_{\alpha\beta}) = \begin{cases} \left\lfloor\frac{\alpha^2-\alpha+2}{2}\right\rfloor - (\beta-1) - 1, & \text{if } \alpha \text{ is odd and } \beta < \alpha \\ \left\lfloor\frac{\alpha^2-\alpha+4}{2}\right\rfloor + (\beta-1) - 1, & \text{if } \alpha \text{ is even and } \beta < \alpha. \end{cases}$$

we attained the following edge labeling as:

$\pi^*(\delta_1\delta_2) = 1,$

$\pi^*(\delta_\alpha\delta_{\alpha+1}) = \left\lceil\frac{\alpha(\alpha+1)}{2}\right\rceil$, if $2 \le \alpha \le n,$

$$\pi^*(\delta_\alpha\delta_{\alpha\beta}) = \begin{cases} \left\lfloor\frac{\alpha(\alpha+1)}{2}\right\rfloor - \beta - 1, & \text{if } \alpha \text{ is odd and } \beta < \alpha \\ \left\lfloor\frac{\alpha^2-\alpha}{2}\right\rfloor + \beta + 1, & \text{if } \alpha \text{ is even and } \beta < \alpha. \end{cases}$$

$$\pi^*(\delta_\alpha\delta_{\alpha\beta+1}) = \begin{cases} \left\lfloor\frac{\alpha(\alpha+1)}{2}\right\rfloor - \beta - 1, & \text{if } \alpha \text{ is odd and } \beta < \alpha \\ \left\lfloor\frac{\alpha^2-\alpha}{2}\right\rfloor + \beta + 1, & \text{if } \alpha \text{ is even and } \beta < \alpha. \end{cases}$$

Thus T_n is labelled with WML and hence we conclude that T_n is a WMG.

Case 2. n is even

$$\pi(\delta_\alpha) = \begin{cases} 5 - \left\lfloor\frac{\alpha^2-\alpha}{2}\right\rfloor, & \text{if } 1 \le \alpha \le 3 \\ \left\lfloor\frac{\alpha^2-\alpha+1}{2}\right\rfloor - 1, & \text{if } \alpha \text{ is odd and } \alpha \ge 5 \\ \left\lfloor\frac{\alpha(\alpha+1)+2}{2}\right\rfloor - 2, & \text{if } \alpha \text{ is odd and } \alpha \ge 4. \end{cases}$$

.

$$\pi(\delta_{\alpha\beta}) = \begin{cases} 5 - \left\lfloor\frac{\alpha^2-\alpha+2\beta}{2}\right\rfloor, & \text{if } 1 \le \alpha \le 3 \text{ and } \beta < \alpha \\ \left\lfloor\frac{\alpha^2-1}{2}\right\rfloor + \beta, & \text{if } \alpha \text{ is odd and } \alpha \ge 4 \text{ and } \beta < \alpha \\ \left\lfloor\frac{\alpha(\alpha+1)}{2}\right\rfloor - \beta - 1, & \text{if } \alpha \text{ is even and } \alpha \ge 4 \text{ and and } \beta < \alpha. \end{cases}$$

we attained the following edge labeling as:

$$\pi^*(\delta_\alpha\delta_{\alpha+1}) = \begin{cases} 5 - \left\lfloor\frac{\alpha^2-\alpha+1}{2}\right\rfloor, & \text{if } 1 \le \alpha \le 2 \\ 6, & \text{if } \alpha = 3 \\ \left\lfloor\frac{\alpha(\alpha+1)}{2}\right\rfloor, & \text{if } 4 \le \alpha \le n \end{cases}$$

$$\pi^*(\delta_\alpha \delta_{\alpha\beta}) = \begin{cases} 5 - \left\lfloor \dfrac{\alpha^2 - \alpha + \beta}{2} \right\rfloor, & \text{if } 1 \le \alpha \le 3 \text{ and } \beta < \alpha \\ 6, & \text{if } \alpha \text{ is odd and } \alpha \ge 4 \text{ and } \beta < \alpha \\ \left\lfloor \dfrac{\alpha(\alpha+1)}{2} \right\rfloor - \beta, & \text{if } \alpha \text{ is even and } \alpha \ge 4 \text{ and and } \beta < \alpha \end{cases}$$

$$\pi^*(\delta_\alpha \delta_{\alpha(\beta+1)}) = \begin{cases} 1, & \text{if } \alpha = 3 \text{ and } \beta = 1 \\ \left\lfloor \dfrac{\alpha(\alpha+1)}{2} \right\rfloor - \beta - 1, & \text{if } \alpha \text{ is odd and} \beta < \alpha \\ \left\lfloor \dfrac{\alpha^2 - \alpha}{2} \right\rfloor + \beta + 1, & \text{if } \alpha \text{ is even and } \beta < \alpha. \end{cases}$$

Thus T_n is labelled with WML and hence we conclude that $T_n$is a WMG.

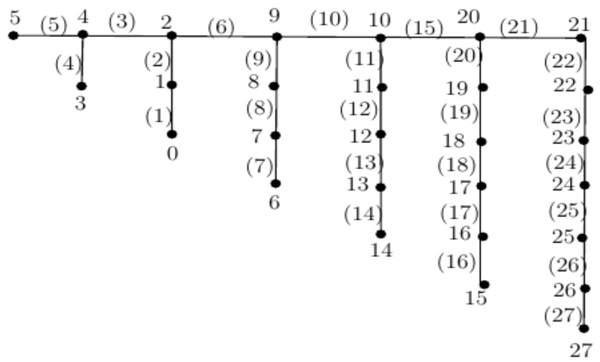

Figure 7:WML of T$_7$

References

1. Christina Mary, S.Pasunkili Pandian and S.Arulraj, Weighted Mean Labeling of Standard Graphs , European Chemical Bulletin 2022,11(Regular issue 8,)390-395.
2. J.A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, **25**(2022).
3. F. Harary, *Graph theory*, Addison Wesely, Reading Mass., 1972.
4. A. Rosa, On certain valuation of the vertices of graph, *Theory of Graphs*,(International Symposium, Rome , July 1966, Gordon and Breach, N.Y. and Dunod Paris) (1967), 349-355.
5. S. Somasundaram and R. Ponraj, Mean labeling of graphs, *National Academy Science Letter*,**26**(2003), 210-213.
6. S. Somasundaram and R. Ponraj, Some results on mean graphs, *Pure and Applied Mathematika Sciences*, **58**(2003), 29-35.
7. Thamaraiselvi Baskaran and Rajasekaran Ganapathy, A Study on C- Exponential Mean Labeling of Graphs ,Hindawi Journal of Mathematics, (2022),