# COMMON FIXED POINTS THEOREMS FOR D-OPERATOR PAIR ON CONE METRIC TYPE SPACES

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**Abstract**

In this study we prove a common fixed point theorem for D-operator pairs that meet the general contractive condition on cone metric type spaces. The D-operator pair is an extension of mappings that are weakly compatible. The examples are provided to demonstrate the outcome. As an application, Best Approximation is also demonstrated as an application.

**Keywords**: D-Operator Pair, Fixed Point, Point Of Co-Incidence.

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## 1. INTRODUCTION

Huang and Zhang [5] established the explanation of cone metric space by replacing the real numbers R in metric space with an ordered Banach space. Jungck[9] specified compatible maps and after a year or so, he added weak compatibility. Al-Thagafi and shahzad [4] outlined occasionally weakly compatible, which is more general than weakly compatible maps. Later M. Abbas and G. Jungck [1] proposed the idea of a D-operator pair, which is a more widespread concept in space of metric than occasionally weak compatibility. This paper discusses a few common fixed point theorems in cone metric type spaces under some general contractive conditions.

## 2. PRELIMINARIES

**Definition 2.1 [2]**
Let X be a non-empty set and £ be a real Banach space with cone þ. A vector-valued function d: $X \times X \longrightarrow$ þ is said to be a cone metric type function on $X$ with the constant K $\geq 1$ if the following conditions are satisfied:

(1) $\theta \leq$ d$(x,y)$, for all $x,y \in X$, and d$(x,y) = \theta$ iff $x = y$;
(2) d$(x,y)$ =d$(y,x)$ for all $x,y \in X$;
(3) d$(x,z) \leq K\ (d(x,y) + d(y,z))$ for all $x,y,z \in X$.

The pair $(X,d)$ is called the cone metric type space.If $K = 1$ then the ordinary triangle inequality in a cone metric space is satisfied, however it does not hold true if $K > 1$.Thus the class of cone metric type spaces is effectively larger than that of ordinary cone metric spaces.Every cone metric space is a cone metric type space, but the converse need not be true.

**Example 2.2 [2]**
Let X={$-1,0,1$}, E= $\mathbf{R}^2$, þ $= \{(x,y) : x \geq 0, y \geq 0\}$. Define d : $X \times X \longrightarrow$ þ by d$(x,y) = d(y,x)$ for all $x,y \in X, d(x,x) = \theta, x \in X$ and $d(-1,0) =$ (3,3), $d(-1,1) = d(0,1) = (1,1)$. Then $(X,d)$ is a complete cone metric space but the triangle inequality is not satisfied.

We have that, d$(-1,1) + d(1,0)$ =(1,1) + (1,1) = (2,2) $\leq$ (3,3) $= d(-1,0)$.It is clear that $K = \frac{3}{2}$.

**Definition 2.3**

Let (X,d) be a cone metric type space. We say that $\{x_g\}$ is

(a) A cauchy sequence if ∀ ç in £ with ç≫ 0, there is N s.t $\forall\ \vartheta, g > N$, d$(x_\vartheta, x_g)$≪ ç
(b) A convergent sequence if for every c in £ with ç≫ 0,there is N such that for all $g > N$, d$(x_g, x) \ll$ ç for some fixed $x$ in X.

A cone metric type space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_g\}$ convergent to $x$ in X if and only if d$(x_g, x) \to 0\ as\ g \to \infty$.

**Definition 2.4** [9]
Let X be a set and let f, g be a two self mapping of X. A point $x \in$ X is called a coincidence point of f and g iff f$x$ =g$x$. We shall call w=f $x$ =g $x$ a point of coincidence of f and g.

**Definition 2.5**
Two self-maps f and g of a set X are occasionally weakly compatible iff there is a point $x \in$ X which is a coincidence point of f and g at which f and g commute.

**Definition 2.6**
Let X be a non empty set and d be a function, d: X × X →£ ∋

d$(\lambda, \mu) = 0$ if and only if $\lambda = \mu\ \forall\ \lambda, \mu \in X$ → (2.6)

for a space (X,d) satisfying (2.6) and A⊂ X, the diameter of A is defined by

diam(A)=sup{max{ (d$(\lambda,\mu)$, d$(\mu,\lambda,)$ $\lambda,\mu \in$ A}}

**Definition 2.7[1]**
Let λ, $\mu$: X →X be mappings. The pair $(\sigma,\tau)$ is said to be **D-operator pair** if there is a point u in X s.t α ∈ C $(\sigma,\tau)$ and d($\sigma\tau$ u, $\tau\sigma$ u)≤R diam (PC$(\sigma,\tau)$) for some R $> 0$.

**Definition 2.8**
Let M be a nonempty subset of a cone metric space (M,d).The set of best M-approximants to

u∈ X, denoted as $P_M$(u) is defined by

$$P_M(\mathrm{u}) = \{y \in \mathrm{M}: \mathrm{d}(\mathrm{y},\mathrm{u}) = \mathrm{dist}(\mathrm{u},\mathrm{M})\}$$

where $\mathrm{dist}(\mathrm{u},\mathrm{M}) = \inf\{\mathrm{d}(x,\mathrm{u}): x \in \mathrm{M}\}$.

## 3. MAIN RESULTS

**Theorem 3.1**
Let (X,d) be a cone metric type space with constant $K \geq 1$ and P be a cone with a nonempty interior. Suppose that the mappings $\lambda, \mu$: X→X are such that λ(X) ⊆ $\mu$(X) and λ (X) or $\mu$(X) is a complete subspace of X. Suppose $(\lambda, g)$ is a D-operator pair and satisfy the conditio

$$d(\lambda x, \lambda y) \leq \frac{\theta}{K}\left\{max\left\{k\ d(\mu x, \mu y), k\ d(\mu x, \lambda x), k\ d(\mu y, \lambda y), \frac{d(\mu y, \lambda x)+d(\mu x, \lambda y)}{2}\right\}\right\} \quad (3.1)$$

$\forall\ x, y \in$X and for some constant $\theta \in \left(0, \frac{1}{K}\right)$. Then $\lambda\ and\ \mu$ have a unique common fixed point.

**Proof**:
By the definition of D-operator pair there exist u in X and R >0 such that $\lambda u = \mu u\ and$

d($\lambda\mu$u, $\mu\lambda$u)≤Rdiam(PC($\lambda, \mu$))

First, we prove that $PC(\lambda, g)$ is singleton. Suppose w and z be two distinct points in X such that $w = \lambda u = gu$ and $z = \lambda v = gv$ for some $u, v \in C(\lambda, g)$. Then from (3.1) we obtain

d(w,z) = d($\lambda u, \lambda v$)

$$\leq \frac{\theta}{K}\left\{max\left\{k\ d(u, \mu v), k\ d(gu, \lambda u), k\ d(\lambda v, gv), \frac{d(\mu u, \lambda v) + d(\lambda u, \mu v)}{2}\right\}\right\}$$

d(w,z) $\leq \frac{\theta}{K}(d(\lambda u, \lambda v))$

$\leq\ \theta(d(w, z)$

< d(w,z).

Which is a contradiction. Therefore w=z, i.e, w= $\lambda u = \mu u = \lambda v = \mu v = z$. Thus,

PC($\lambda, \mu$) is singleton, i.e., w= $\lambda u = \mu u$ is the unique point of coincidence and diam(PC($\lambda, \mu$)) = 0 from definition of D-operator pair $\lambda\mu u = \mu\lambda u$ for some points u ∈ $C(\lambda, \mu)$.

Now from (3.1) we have

d($\lambda\lambda u, \lambda v) = d(\lambda\lambda u, \lambda u)$

$$\leq \frac{\theta}{K}\left\{max\left\{k\ d(\mu\lambda u, \mu u), k\ d(\mu\lambda u, \lambda\lambda u), k\ d(\mu u, \mu u), \frac{d(\mu u, \lambda\lambda u) + d(\mu\lambda u, \lambda u)}{2}\right\}\right\}$$

$$\leq \frac{\theta}{K}\left\{max\left\{k\ d(\lambda\lambda u, \lambda u), k\ d(\lambda\lambda u, \lambda\lambda u), k\ d(\lambda u, \lambda u), \frac{d(\lambda u, \lambda\lambda u) + d(\lambda\lambda u, \lambda u)}{2}\right\}\right\}$$

$\leq \frac{\theta}{K}\{d(\lambda\lambda u, \lambda u\}$

$\leq \theta\big(d(\lambda\lambda u, \lambda u)\big)$

$\leq d(\lambda\lambda u, \lambda u)$

This is a contradiction. Hence $\lambda\lambda u = \mu\lambda u = \lambda u$ and therefore $\lambda, \mu have$ a common fixed point. For uniqueness, suppose that $u, v \in X, such\ that\ \lambda u = \mu u = u\ and\ \lambda v = \mu v = v\ and\ u \neq v$.Then (3.1) gives,

$d(u,v) = (d(\lambda u, \lambda v))$

$$\leq \frac{\theta}{K}\left\{max\left\{k\, d(\mu u, \mu v), k\, d(\mu u, \mu u), k\, d(\lambda v, \mu v), \frac{d(\mu u, \lambda v) + d(\lambda u, \mu v)}{2}\right\}\right\} \leq \frac{\theta}{K}\ (d(u,v)$$

$$\leq \big(d(u,v)\big)$$

Which is a contradiction. Therefore $u = v$ and hence the common fixed point of $\lambda\ and\ \mu$ is unique.

**Example 3.2**
Let E=$R^2$,P=$\{(x,y) \in E: x,y \geq 0, \subset R\}$and define d:R×R→ £ by d$(x,y) = (|x-y|, \alpha|x-y|), where\ \alpha > 0\ is$ constant.Define $\lambda, \mu: X \to X$ by, $\lambda(x) = \left(\frac{\theta}{2K}\right)^{\frac{1}{2}} x$ and $\mu(x) = \left[\left(\frac{\theta}{2K}\right)^{\frac{1}{2}} + \theta^{-\frac{1}{2}}\right] x$, for $x \in X$ where $\theta \in (0, \frac{1}{K})$ and $K \geq 1$. (X, d) is a cone metric type space. $\lambda\ and\ \mu$ are D-operator pair satisfy condition (3.1). $\lambda\ and\ \mu$ have a coincidence point 0 and unique point of coincidence which is 0. Since $\lambda\ and\ \mu$ commute at 0, it is the unique common fixed point.

**Corollary 3.3**
Let (X, d) be a cone metric type space with the constant $K \geq 1\ and$ þ a cone having a nonempty interior. Suppose that the mappings $\lambda, \mu$ : X→X are such that $\lambda$(X) ⊆ $\mu$(X) and $\lambda$(X) or $\mu$(X) is a complete subspace of X, and that for some constant $\theta \in (0,1)$ and for every $,y \in X$, we have d$(\lambda x, \lambda y) \leq \theta\, d(\mu x, \mu y)$. Then $\lambda\ and\ \mu$ have a unique point of coincidence in X. Moreover if $\lambda\ and\ \mu\ are$ D-operator p, $\lambda\ and\ \mu$ have a unique common fixed point.

**Corollary 3.4**
Let (X, d) be a cone metric type space with the constant $K \geq 1\ and$ þ a cone having a nonempty interior. Suppose that the mappings $\lambda, \mu$ : X → X are such that $\lambda$(X) ⊆ $\mu$(X) and $\lambda$(X) or $\mu$(X) is a complete subspace of X, and that for some constant $\theta \in (0,1)$ and for every $x, y \in X$, we have

d$(\lambda x, \lambda y) \leq \theta \left\{\frac{d(\mu x, \lambda x) + d(\mu y, \lambda y)}{2}\right\}$

Then $\lambda\ and\ \mu$ have a unique point of coincidence in X. Moreover if $\lambda\ and\ \mu\ are$ D-operator pair, $\lambda\ and\ \mu$ have a unique common fixed point.

**Corollary 3.5**
Let (X, d) be a cone metric type space with the constant $K \geq 1\ and\ P$ a cone having a nonempty interior. Suppose that the mappings $\lambda, \mu$ : X→ X are such that $\lambda$(X) ⊆ $\mu$(X) and $\lambda$(X) or $\mu$(X) is a complete subspace of X, and that for some constant $\theta \in (0,1)$ and for every $,y \in X$, we have

d$(\lambda x, \lambda y) \leq \frac{\theta}{K}\left\{\frac{d(gy, \lambda x) + d(gx, \lambda y)}{2}\right\}$

Then $\lambda\ and\ \mu$ have a unique point of coincidence in X.Moreover if $\lambda\ and\ \mu\ are$ D-operator pair, $\lambda\ and\ \mu$ have a unique common fixed point.

**Corollary 3.6**
Let (X, d) be a cone metric type space with the constant $K \geq 1\ and\ P$ a cone having a nonempty interior. Suppose that the mappings $\lambda, \mu$ : X→X are such that $\lambda$(X) ⊆ $\mu$(X) and $\lambda$(X) or $\mu$(X) is a complete subspace of X, and that for some constant $\theta \in (0,1)$ and for every $,y \in X$, we have

$$d(\lambda x, \lambda y) \leq ad(\mu x, \mu y) + b(max\{(d(\lambda x, \mu x), d(\lambda y, \mu y)\}) + c(max\{(d(\mu x, \mu y), d(\mu x, \lambda x), d(\mu y, \lambda y)\}) \to (3.6)$$

for all $x, y$ ∈X, where a,b,c $> 0$, a+b+c=1. Then $\lambda\ and\ \mu$ have a unique point of coincidence in X.Moreover if $\lambda\ and\ g\ are$ D-operator pair, $\lambda\ and\ \mu$ have a unique common fixed point.

Proof: In the theorem (3.1) replacing the condition (3.1) by (3.4) we get the result that $\lambda\ and\ \mu$ have a unique common fixed point.

## 4. APPLICATION TO BEST APPROXIMATION

**Theorem 4.1:**
Let (X,) be a metric space of the cone type.. Suppose that u ∈ X, $\lambda\ and\ \mu$ satisfy inequality (3.1) in theorem 3.1. $\lambda$ leaves $\mu$- invariant compact subset M of a closed subspace $\mu X$ as invariant. For each b∈ $P_M$(u), let d$(x, \lambda b) <$ d$(x, \mu b)$ and $fb \in P_M$(u).If $\lambda\ and\ \mu$ are D-operator pair, then u has a best approximation in M which is also a common fixed point of $\lambda\ and\ \mu$

Proof**:**

Let u$\in F(\lambda) \cap F(\mu)$. Since M is a compact subset of $\mu X$, $P_M$(u)$\neq \emptyset$. To prove that $\lambda(P_M(\text{u})) \subset \mu(P_M(\text{u})$, assume the contrary.Then there exist b$\in P_M$(u) with $\lambda b \nsubseteq \mu(P_M(\text{u}).)$

Now, d(u, $\mu b) = dist$(u,M)

$$< d(\text{u}, \lambda b)$$

$$< d(\text{u}, \lambda b)$$

As a result of Contradiction, we now have $\lambda(P_M(\text{u})) \subset \mu(P_M(\text{u})$.Now $f(P_M(\text{u}))$ being a closed subset of a complete cone metric space, it is complete. Hence $P_M(\text{u}) \cap F(\lambda) \cap F(\mu)$ is singleton.

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