Some Results on Deficiency of Meromorphic Functions Related to its Differences and Differential Polynomials

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ABSTRACT

In this article, we focus on investigating the relationship between the characteristic function of meromorphic function with maximal deficiency sum and its difference. And some results on deficiencies of differential polynomials and higher order differences are also considered.

Keywords: Meromorphic function, maximal deficiency sum, differential polynomial, difference operator.

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1. INTRODUCTION

Consider a meromorphic function $f(z)$ in the complex plane C. In Nevanlinna theory the following notations are frequently used [7, 3, 8] such as $T(r,f), m(r,f), N(r,f)$ and $S(r,f)$ any quantity satisfying $S(r,f) = oT(r,f)$, as $r \to \infty$.

As we know, there are many results on the deficiencies of meromorphic functions in the value distribution theory (see [2]). Let $c(\neq 0) \in N$ and $f(z)$ be a meromorphic function then its difference operator is defined as,

$$\Delta_c f(z) = f(z+c) - f(z).$$

The Nevanlinna deficiency of meromorphic function $f(z)$ with respect to a finite complex number a is defined as,

$$\delta(a,f) = \lim\ inf_{r\to\infty} \frac{m(r,\ a)}{T(r,\ f)} = 1 - \lim\ inf_{r\to\infty} \frac{N(r,\ a)}{T(r,\ f)}.$$

If $a = \infty$ then $m(r,a)(N(r,a))$ is replaced by $m(r,f)(N(r,f))$ in the above definition.

Definition 1. Let us define,

$$P(z) = a_m f^m(z) + a_{m-1} f^{m-1}(z) + \cdots + a_0 = \sum_{i=1}^{m} a_i f^i(z), \quad (1.1)$$

$m \in Z^+,\ a_m(\neq 0), a_{m-1}, \ldots, a_0$ are complex constants.

We study the Nevanlinna deficiencies related to a meromorphic function, its difference operator, derivative and polynomial.

$$F_1(z) = \Delta_c^m f(z) f'(z).$$

Theorem 1.1. [9] Let $f(z)$ be a non-constant meromorphic function in $A(R)$, where $1 < R \leq +\infty$. Then,

$$T\left(R, \frac{1}{f-a}\right) = T(R,\ f) + O(1),\ \ 1 < r < R,$$

for every fixed $a \in C$.

In 2013, Zhaojun Wu [4] studied the relationship between the characteristic function of a meromorphic function $f(z)$ with maximal deficiency sum and its exact difference and obtained the following result.

Theorem 1.2. [4] Suppose that $f(z)$ is a transcendental meromorphic function of order less than one with maximal deficiency sum. Then,

$$(i) \lim_{r\to\infty} \frac{T(r, \Delta_c f)}{T(r, f)} = 2 - \delta(\infty, f);$$

$$(ii) \lim_{r\to\infty} \frac{N\left(r, \frac{1}{\Delta_c f}\right)}{T(r, f)} = 0.$$

Consequently, we have that the deficiency of $\delta_c f$ with respect to 0 is 1, i.e.,

$$\delta(0, \Delta_c f) = 1 - \lim_{r\to\infty} \frac{N\left(r, \frac{1}{\Delta_c f}\right)}{T(r,\ f)} = 1.$$

In 2015, Zhaojun Wu et al.,[9] proved the below result by considering the relationship between the deficiency of meromorphic function and their derivatives in the punctured plane.

Theorem 1.3. [9] Suppose that $f(z)$ is an admissible meromorphic function with maximal deficiency sum in the punctured plane $A(+\infty)$ and the order of $f(z)$ is finite. Then,

$(i)\ \lim_{r\to\infty}\dfrac{T_0(r,f')}{T_0(r,f)}=2-\delta_0(\infty,f);$

$(ii)\ \lim_{r\to\infty}\dfrac{N_0\left(r,\frac{1}{f'}\right)}{T_0(r,f')}=0;$

$(iii)\dfrac{1-\delta_0(\infty,f)}{2-\delta_0(\infty,f)}\le K_0(f')\le\dfrac{2\big(1-\delta_0(\infty,f)\big)}{2-\delta_0(\infty,f)}$,

where $K_0(f')=\lim\ sup_{r\to\infty}\frac{N_0(r,\ f')+N_0\left(r,\ \frac{1}{f'}\right)}{T_0(r,\ f')}$.

We extend the above theorem to the exact difference of meromorphic function with maximal deficiency sum in the complex plane and prove the following theorem.

Theorem 1.4. Suppose that $f(z)$ is a transcendental meromorphic function with maximal deficiency sum in C and the order of $f(z)$ is finite. Then,

$(i)\ \lim_{r\to\infty}\dfrac{T(r,\ \Delta_c f)}{T(r,f)}=2-\delta(\infty,f);$

$(ii)\ \lim_{r\to\infty}\dfrac{N\left(r,\frac{1}{\Delta_c f}\right)}{T(r,\ \Delta_c f)}=0;$

$(iii)\ \dfrac{1-\delta(\infty,f)}{2-\delta(\infty,f)}\le K(\ \Delta_c f)\le\dfrac{2\big(1-\delta(\infty,f)\big)}{2-\delta(\infty,f)}$,

where $K(\Delta_c f)=\lim\ sup_{r\to\infty}\frac{N(r,\ \Delta_c f)+N\left(r,\frac{1}{\Delta_c f}\right)}{T(r,\ \Delta_c f)}$.

In 2018, S. S. Bhoosnurmath et al.,[1] proved the following result for n^{th} difference operator of meromorphic functions.

Theorem 1.5. [1] Suppose that $f(z)$ is a transcendental meromorphic function of finite order. Then we have,

$\sum_{a\epsilon C}\delta(a,f)\le\ \lim\inf_{r\to\infty}\frac{T(r,\ \Delta_c^n f)}{T(r,\ f)}\le\lim\sup_{r\to\infty}\frac{T(r,\ \Delta_c^n f)}{T(r,\ f)}\le(n+1)-n\delta(\infty,f).$

We extend the above result for the difference polynomial of a meromorphic function to get the following result.

Theorem 1.6. Suppose that $f(z)$ is a transcendental meromorphic function of finite order and let $A(z) = f^n(z)f(z + c)P(f(z))$. Then we have,

$$2 - \delta(\infty, f) \leq \lim\, inf_{r\to\infty} \frac{T(r,\ A)}{T(r,\ f)} \leq \lim\, sup_{r\to\infty} \frac{T(r,\ A)}{T(r,\ f)} \leq (n + m + 1) - (n + m\,\delta(\infty, f).$$

In 2016, X.M. Zheng and H.Y. Xu [10] studied the Nevanlinna deficiencies of a meromorphic function and its derivative to prove the below result.

Theorem 1.7. [10] Let $f(z)$ be a meromorphic function of finite order satisfying,

$$\lim\, sup_{r\to\infty} \frac{T(r,\ f)}{T(r,\ f')} < +\infty.$$

If $F_1^*(z) = f^m(z)f(z + c)f'(z)$ and c be a non-zero complex constant. Then,

$$\delta(\infty,\ F_1^*) \geq \delta(\infty, f').$$

In this paper we prove the results on Nevanlinna deficiencies of meromorphic functions and the higher derivative of its difference operator.

Theorem 1.8. Let $f(z)$ be a meromorphic function of finite order satisfying,

$$\limsup_{r\to\infty} \frac{T(r,\ \Delta_c^m f)}{T(r,\ f)} < +\infty, \tag{1.2}$$

where c be a non-zero complex constant. Then,

$$\delta(\infty, f') \leq \delta(\infty, F_1).$$

In 2016, X.M. Zheng and H.Y. Xu [10] proved the following theorem.

Theorem 1.9. [10] Let $f(z)$ be a meromorphic function of finite order, c be a non-zero complex constant and F_1^* is as defined in theorem 1.7. If $\delta = \delta(\infty, f) > \frac{8}{m+5}$, then

$$\delta(\infty,\ F_1^*) > 0.$$

At the end of this paper, we prove the Nevanlinna deficiencies related to a meromorphic function, its difference operator, derivative and polynomial as follows.

Theorem 1.10. Let $f(z)$ be a meromorphic function of finite order satisfying,

$$\lim_{r\to\infty} sup \frac{N(r,\ F_1)}{N(r,\ F_1)} \leq \frac{3+(1-\delta)(m+2)}{m-3+2\delta} < 1$$

and $c(\neq 0)$ be a complex constant. Then,

$$\delta(\infty, F_1) > 0.$$

2. LEMMAS

Lemma 2.1. [9] Let $f(z)$ be an admissible meromorphic function of finite order in the punctured plane $A(+\infty)$. Then,

$$m\left(r, \frac{f\prime}{f}\right) = S(r,f) = o(T(r,f)).$$

Lemma 2.2. [1] Let $f(z)$ be a non-constant meromorphic function of fnite order, $c \epsilon C$ and $\delta < 1$. Then,

$$m\left(r, \frac{\Delta_c^n f(z)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^\delta}\right) = S(r,f),$$

where $S(r,f) = o(T(r,f))$.

Lemma 2.3. [1] Let $f(z)$ be a non-constant meromorphic function of fnite order, $c \epsilon C$. Then,

$$N(r,\ \Delta_c^n f(z)) \leq (n+1)\ N(r,f) + S(r,f).$$

Lemma 2.4. [6] Let $f(z)$ is a non-constant meromorphic function of finite order and c be a non-zero complex constant. Then,

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r,f).$$

Lemma 2.5. [10] Let $f(z)$ be a meromorphic function of finite order and c be a non-zero complex constant. Then,

$$T\big(r, f(z+c)\big) = T(r,f) + S(r,f)$$

$$N\big(r, f(z+c)\big) = N(r,f) + S(r,f)$$

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r,f).$$

3. PROOFS OF THEOREMS

Proof of Theorem 1.4. It follows from [5] that

$$T(r,\Delta_c f) \leq T(r,f) + N(r,f) + S(r,f). \quad (3.1)$$

Hence (3.1) and Lemma 2.1 imply that

$$\limsup_{r\to\infty} \frac{T(r,\Delta_c f)}{T(r,f)} \leq 2 - \delta(\infty.f).$$

(3.2)

$\Delta_c f(z)$ has at most countably many finite deficient values and we denote by $a^{[j]}$. For any positive integer p, Wu and Chen [5] proved the following inequality.

$$\sum_{j=1}^{p} m\left(r,\frac{1}{f-a^{[j]}}\right) \leq T(r,\Delta_c f) - N\left(r,\frac{1}{\Delta_c f}\right) + S(r,f). \quad (3.3)$$

Thus from (3.3) and theorem 1.1 we deduce

$$pT(r,f) \leq T(r,\Delta_c f) + \sum_{j=1}^{p} N\left(r,\frac{1}{f-a^{[j]}}\right) - N\left(r,\frac{1}{\Delta_c f}\right) + S(r,f). \quad (3.4)$$

And hence

$$p \leq \liminf_{r\to\infty} \frac{T(r,\Delta_c f)}{T(r,f)} + \sum_{j=1}^{p}\left(1-\delta\left(a^{[j]},f\right)\right) + \liminf_{r\to\infty}\frac{S(r,f)}{T(r,f)},$$

$$= p + \liminf_{r\to\infty} \frac{T(r,\Delta_c f)}{T(r,f)} - \sum_{j=1}^{p} \delta\left(a^{[j]},f\right).$$

(3.5)

Therefore, we have

$$\sum_{j=1}^{p} \delta\left(a^{[j]},f\right) \leq \liminf_{r\to\infty} \frac{T(r,\Delta_c f)}{T(r,f)}.$$

(3.6)

As p is arbitrary, combining (3.2) and (3.6), we get

$$2 - \delta(\infty,f) \leq \liminf_{r\to\infty} \frac{T(r,\Delta_c f)}{T(r,f)} \leq \limsup_{r\to\infty} \frac{T(r,\Delta_c f)}{T(r,f)} \leq 2 - \delta(\infty,f).$$

That is

$$\liminf_{r\to\infty} \frac{T(r,\ \Delta_c f)}{T(r,\ f)} = 2 - \delta(\infty, f).$$

(3.7)

Given $\epsilon > 0$, we choose q sufficiently large. So that

$$\sum_{j=1}^{p} \delta\left(a^{[j]}, f\right) > 2 - \delta(\infty, f) - \epsilon. \quad (3.8)$$

For these q, (3.3) implies

$$\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{\Delta_c f}\right)}{T(r,\ \Delta_c f)} + \liminf_{r\to\infty} \frac{T(r, f)}{T(r,\ \Delta_c f)} \sum_{j=1}^{p} \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f - a^{[j]}}\right)}{T(r, f)}$$

$$\leq 1 + \limsup_{r\to\infty} \frac{S(r,\ f)}{T(r, \Delta_c f)}.$$

(3.9)

Thus, from (3.7) - (3.9) we deduce

$$\limsup_{r\to\infty} \frac{N\left(r,\ \frac{1}{\Delta_c f}\right)}{T(r,\ \Delta_c f)} \leq \frac{\epsilon}{2 - \delta(\infty, f)}.$$

(3.10)

As $\epsilon > 0$ is arbitrary, we have

$$\limsup_{r\to\infty} \frac{N\left(r,\ \frac{1}{\Delta_c f}\right)}{T(r,\ \Delta_c f)} = 0.$$

(3.11)

As $N(r,\ \Delta_c f) \leq 2N(r, f)$ then

$$\frac{N(r,\ \Delta_c f)}{T(r,\ \Delta_c f)} \frac{T(r,\ \Delta_c f)}{T(r,\ f)} \leq \frac{2N(r,\ f)}{T(r,\ f)}. \quad (3.12)$$

Combining (3.7) and (3.12)

$$(2 - \delta(\infty, f)) \limsup_{r\to\infty} \frac{N(r,\ \Delta_c f)}{T(r,\ \Delta_c f)} \leq 2(1 - \delta(\infty, f)).$$

(3.13)

(3.11) and (3.13) yields

$$\limsup_{r\to\infty} \frac{N(r,\ \Delta_c f) + N\left(r, \frac{1}{\Delta_c f}\right)}{T(r,\ \Delta_c f)} \leq \frac{2(1 - \delta(\infty,\ f))}{2 - \delta(\infty,\ f)}.$$

Therefore, we obtain

$$K(\Delta_c f) \leq \frac{2(1-\delta(\infty, f))}{2-\delta(\infty, f)}. \tag{3.14}$$

Furthermore, we have $N(r,f) \leq N(r, \Delta_c f)$.

By (3.7) we get

$$\frac{N(r, f)}{T(r, f)} \leq \frac{T(r, \Delta_c f)}{T(r, f)} \frac{N(r, \Delta_c f)}{T(r, \Delta_c f)} \leq (2 - \delta(\infty, f)) \frac{N(r, \Delta_c f)}{T(r, \Delta_c f)}.$$

Thus

$$\limsup_{r\to\infty} \frac{N(r, \Delta_c f)}{T(r, \Delta_c f)} \geq \frac{1}{2 - \delta(\infty, f)} \limsup_{r\to\infty} \frac{N(r, f)}{T(r, f)}$$

$$= \frac{1-\delta(\infty, f)}{2-\delta(\infty, f)}.$$

Hence

$$K(\Delta_c f) \geq \frac{1-\delta(\infty, f)}{2-\delta(\infty, f)}. \tag{3.15}$$

Combining (3.14) and (3.15), we have

$$\frac{1-\delta(\infty, f)}{2-\delta(\infty, f)} \leq K(\Delta_c f) \leq \frac{2(1-\delta(\infty, f))}{2-\delta(\infty, f)}.$$

Proof of Theorem 1.6

Given $A(z) = f^n(z)f(z+c)P(f(z))$. Using the 1st fundamental theorem of Nevanlinna and Lemmas 2.2 and 2.3, we have,

$$T(r,A) = m(r,A) + N(r,A)$$

$$= m\left(r, \frac{fA}{f}\right) + N(r,A)$$

$$\leq m(r,f) + m\left(r, \frac{A}{f}\right) + N(r,f^n) + N(r,f(z+c)) + N(r,P(f)) + O(1)$$

$$\leq m(r,f) + nN(r,f) + N(r,f) + mN(r,f) + O(1)$$

$$\leq T(r,f) + (n+m)N(r,f) + O(1).$$

Hence

$$\lim\ \sup_{r\to\infty}\frac{T(r,\ A)}{T(r,f)}\leq\ 1+(n+m)\ \lim\ \sup_{r\to\infty}\frac{N(r,\ f)}{T(r,\ f)}$$

$$\leq\ 1+(n+m)(1-\delta(\infty,f))$$

$$\leq\ (1+n+m)-(n+m)\delta(\infty,f). \qquad (3.16)$$

Let $\{b_i(i\ =\ 1,2,\ldots,q)\}$ be distinct complex numbers containing all the deficient values of $f(z)$. Consider

$$F_2=\sum_{i=1}^{q}\frac{1}{f-b_i}.$$

We know that, $T(r,f-b_i)=T(r,f)\ +\ O(1)$. Since $A(z)-b_i=A(z)$, we deduce from Lemma 2.2 that

$$m(r,F_2(z)A(z))\ \leq\sum_{i=1}^{q}m\left(r,\frac{A-b_i}{f-b_i}\right)+logq$$

$$=\ S(r,f).$$

Now the above relation yields

$$m\big(r,F_2(z)\big)=m\left(\ r,F_2(z)A(z)\frac{1}{A(z)}\right)$$

$$\leq\ m(r,F_2(z)A(z))+m\left(\ r,\frac{1}{A(z)}\right)+S(r,f)$$

$$\leq m\left(\ r,\frac{1}{A(z)}\right)+S(r,f). \qquad (3.17)$$

In view of Valiron-Mo'honko identity, we have

$$qT(r,f)=T\big(r,A(z)\big)+N\left(\ r,\frac{1}{A(z)}\right)+S(r,f)$$

$$=\ m(r.A)+\ N(r,A)+N\left(\ r,\frac{1}{A(z)}\right)+S(r,f)$$

$$\leq\ m\left(\ r,\frac{1}{A(z)}\right)+N\left(\ r,\frac{1}{A(z)}\right)+\ \sum_{i=1}^{q}N(r,b_i)+S(r,f)$$

$$\leq\ T(r,A)+\sum_{i=1}^{q}N(r,b_i)+S(r,f).$$

$$q\ \leq\ \liminf_{r\to\infty}\frac{T(r,\ A)}{T(r,\ f)}+\sum_{i=1}^{q}\ \limsup_{r\to\infty}\frac{N(r,b_i)}{T(r,f)}\ +\ \liminf_{r\to\infty}\frac{S(r,\ f)}{T(r,\ f)}$$

$$\leq \liminf_{r\to\infty} \frac{T(r,\ A)}{T(r,f)} + \sum_{i=1}^{q}[1 - \delta(b_i\,, f)]$$

$$= q + \liminf_{r\to\infty} \frac{T(r,\ A)}{T(r,f)} - \sum_{i=1}^{q} \delta(b_i\,, f).$$

Hence

$$\liminf_{r\to\infty} \frac{T(r,\ A)}{T(r,\ f)} \geq \sum_{i=1}^{q} \delta(b_i\,, f).$$

Since q is arbitrary, we have

$$\liminf_{r\to\infty} \frac{T(r,\ A)}{T(r,\ f)} \geq \sum_{i=1}^{q} \delta(b_i\,, f) = 2 - \delta(\infty, f).$$

(3.18)

From (3.16) and (3.18) we get

$2 - \delta(\infty, f) \leq \liminf_{r\to\infty} \frac{T(r,\ A)}{T(r,\ f)} \leq \limsup_{r\to\infty} \frac{T(r,\ A)}{T(r,\ f)} \leq (1 + n + m) - (n + m\delta(\infty, f).$

Hence the proof.

Proof of Theorem 1.8

Let

$$[f'(z)]^{m+1} = F_1 f(z) \frac{[f'(z)]^m}{f(z)} \frac{f(z)}{\Delta_c^m f(z)}$$

it follows that

$$(m + 1)m(r, f') \leq m(r, F_1) + mm\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f}{\Delta_c^m f}\right)$$

then by Lemma 2.4, we have

$$m(r, F_1) \geq (m + 1)m(r, f') + S(r, f). \qquad (3.19)$$

Since $N(r, f') = N(r, f) + \overline{N}(r, f)$, it follows by Lemma 2.5 that

$$N(r, F_1) \leq (m + 1)N(r, f) + \overline{N}(r, f) \leq (m + 1)N(r, f'). \qquad (3.20)$$

From (1.2) we have

$$\limsup_{r\to\infty} \frac{S(r,\ f)}{T(r,\ f)} = \limsup_{r\to\infty} \frac{S(r,\ f)}{T(r,\ f)} \frac{T(r,\ f)}{T(r,\ f')} = 0.$$

(3.21)

Then from (3.19) - (3.21), we have

$$\frac{N(r,F_1)}{T(r,F_1)} \leq \frac{(m+1)N(r,f')}{(m+1)N(r,f')+(m+1)m(r,f')+S(r,f)}$$

$$\leq \frac{N(r,f')}{T(r,f')+S(r,f)}$$

$$= \frac{N(r,\ f')}{(1+o(1))\,T(r,\ f')}.$$

It follows that $\delta(\infty,f') \leq \delta(\infty,F_1)$. Hence the proof.

Proof of Theorem 1.10.

Let $F_3 = \Delta^{m+1}f(z)$, then

$$N(r,F_2) = (m+1)N(r,f) \text{ and } T(r,F_3) = (m+1)T(r,f)$$

It follows that $\delta(\infty,F_3) = \delta(\infty,f) = \delta$.

Since $F_3 = \Delta^{m+1}f(z)$, we have

$$\overline{N}(r,f) \leq N(r,f) \leq \frac{1}{m+1} N(r,F_3) \leq \frac{1-\delta}{m+1} T(r,F_3)+S(r,F_3). \tag{3.22}$$

$$\overline{N}\left(r,\frac{1}{f}\right) \leq N\left(r,\frac{1}{f}\right) \leq \frac{1}{m+1} N\left(r,\frac{1}{F_3}\right) \leq \frac{1}{m+1}T(r,F_3) + O(1). \tag{3.23}$$

From (3.22), (3.23) and Lemmas 2.4 and 2.5 we have

$$T(r,F_3) = T\left(r,\ F_1\frac{\Delta f(z)}{f(z)}\frac{f(z)}{f'(z)}\right)$$

$$\leq T(r,\ F_1)+N\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+N(r,f)+\overline{N}(r,f)+S(r,f)$$

$$\leq T(r,\ F_1)+\frac{4-2\delta}{m+1}\ T(r,F_3)+S(r,f).$$

Which is

$$T(r,F_1) \geq \left[\frac{m-3+2\delta}{m+1}+o(1)\right]T(r,F_3). \tag{3.24}$$

Again from (3.22), (3.23),

$$N(r,F_1) \le N(r,F_3) \le N(r,F_3) + N\left(r,\frac{f(z)}{\Delta f(z)}\right) + N\left(r,\frac{f'}{f}\right)$$

$$\le N(r,F_3) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f}\right) + N(r,f) + \overline{N}(r,f) + S(r,f)$$

$$\le \left[\frac{(m+2)(1-\delta)+3}{m+1}\right] T(r,F_3) + S(r,f).$$

That is

$$N(r,F_1) \le \left[\frac{3+(1-\delta)(m+2)}{m+1} + O(1)\right] T(r,F_3). \qquad (3.25)$$

Hence from (3.24), (3.25) and $\delta = \delta(\infty,f) = \delta(\infty,F_3)$

$$\limsup_{r\to\infty} \frac{N(r,F_1)}{T(r,F_1)} \le \frac{3+(1-\delta)(m+2)}{m-3+2\delta} < 1$$

that is

$$\delta(\infty,F_1) = 1 - \limsup_{r\to\infty} \frac{N(r,F_1)}{T(r,F_1)} > 0.$$

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