NUMERICAL SOLUTION FOR THESLOWLY ROTATING POLYTROPIC FLUID SPHERE

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Abstract

We propose a Ramanujan's method for solving a non-linear ordinary differential equation describing the stellar structure of the slowly rotating polytropic fluid sphere. Ramanujan's method is an iterative method which is used to determine the roots of the obtained series through special function as it gives the accurate numerical results to the series solution as any other computational methods. Therefore, it can be used as an application to real stars and the core of degenerate stars. The numerical results are presented for the values of the polytropic index n = 0.0(0.1) to 2.0 and it is found in agreement with the earlier reported results (A.D. Parks 1984).

Key Words: Nonlinear ODE, Slowly Rotating Polytrope, Series Solution and Ramanujan Method.

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1. INTRODUCTION

The structural study of the slowly rotating polytropic fluid problem has engrossed several astrophysicist and mathematicians. In the present study, our purpose is to solve the equation for a rotating polytropic fluid and to find the prospect of solving the rotating stars much more conveniently without any complicated computer programming methods. Previously, the problems of slowly rotating problems were solved by using Sledgehammer technique (Monaghan 1965) which was the only essential technique to solve the problem to the polytropic fluids. In this paper, we present a simple approximation technique extended by the special function (Legendre function and Power series) for solving the rotating polytropic problem in order to find the series that originate at the centre of a polytropic model. The calculation of the boundary values has been done by a Ramanujan's method, a novel method which gives the exact result to the series solution. The applied technique to the polytropic problems can be extended to the case of real stars and the problem of the structure of slowly rotating white dwarf will be dealt with in a subsequent communication. The rotation problems was first investigated by Chandrasekhar(1933) using a first order perturbation, Monaghan J.J & Roxburgh I.W(1965) used an extension of Jean's generalized Roche model, Sackmann I. & Anand S.(1969) constructed ten convective models in the frame work of first order perturbation., Smith B.L(1973) used a simple, analytic & iterative technique, Das M.K. & Tandon J.N(1975) used first order perturbation technique, Seidov Z.F & Kuzakhmedov R (1977) introduced functional series method, Mohan C. & Al-Bayaty A.R (1980) proposed a power series method, Singh & Singh (1984) used Monaghan & Roxburgh method, Parks A.D (1984) used Frobenius method & power series method, Jabbar R.J. (1984) integrated numerically the Chandrasekhar's equation for polytropic gas spheres of zero order and index *n=0(0.1) 4.9* numerically with *CDC7600* automatic computer to 16 decimal places, William P.S(1988) constructed analytical solution for the polytropic index *n=1*, Roxburgh I. & Stockman L.M (1999) used power series method , Hunter C. (2001) used Euler's transformed Series, Daftardar & Jafari H.(2006) proposed an iterative method for solving non-linear Volterra integral equations, Oproui T. & Horedt G.P.(2008) used analytical method, Prince A.M. & Thomas S.(2019) introduced new iterative method to solve second ODE and used Ramanujan's method for series calculation, Kashem B.E & Shihab S.(2020) used modified Hermite operation matrix method to solve the structure of slowly rotating polytropes.

The basic equations governing the structure of polytrope of index n such as,

$$P = k\rho^{1+\frac{1}{n}} \; and \; \rho = \rho_c \sigma^n \tag{1}$$

and where P is the pressure, ρ the density, ρ_c is its central density, σ^n is the dimensionless variable, k is the constant and n is the polytropic index, which can be used to model different polytropic state.

Considering the equation of hydrostatic equilibrium for a rotating spheroid,

$$\nabla P = \rho \nabla (V + V') \tag{2}$$

where V' the rotational potential and V the gravitational potential satisfying the Poisson equation is given by,

$$\nabla^2 V = -4\pi G \rho \; and \; \nabla^2 V' = 2\omega^2 \tag{3}$$

Taking gradient of eqn (2) and substituting the value of eqn (3) in it we get,

$$\vec{\nabla}\left[\frac{1}{\rho}\vec{\nabla}P\right] = -4\pi G\rho + 2\omega^2 \tag{4}$$

Introducing dimensionless variable ξ and α by the relation,

$$r = \alpha\xi, \;\; \alpha^2 = \left[\frac{(n+1)k\rho_c^{\frac{1}{n}-1}}{4\pi G}\right], \;\; \alpha = \frac{\omega^2}{2\pi G \rho_c} \tag{5}$$

Since,

$$\vec{\nabla}\left[\frac{1}{\rho}\vec{\nabla}P\right] = (n+1)K\lambda^{\frac{1}{n}}\nabla^2\sigma \tag{6}$$

Then,

$$\left[\frac{(n+1)k\rho_c^{\frac{1}{n}-1}}{4\pi G}\right]\nabla^2\sigma = -\sigma^n + \frac{\omega^2}{2\pi G\rho_c} \tag{7}$$

The above equation (7) becomes,

$$\nabla^2\sigma = -\sigma^n + \alpha \tag{8}$$

Now, assuming the distance of a point on the surface from the origin r and the θ the angle between the rotation axis (z-axis) and r. We shall denote the cosine of the co-latitudes θ by $\mu = \cos z$.

We find that equation (1)-(3) can be reduced to,

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{2}{\xi}\frac{\partial\psi}{\partial\xi} + \frac{1}{\xi}\frac{\partial^2\psi}{\partial\xi^2} + \frac{2\cot\theta}{\xi^2}\frac{\partial\psi}{\partial z} = [-\sigma^n + \alpha]\psi \tag{9}$$

Boundary Condition : At the centre the polytopes has maximum density by boundary conditions,

$$\text{at} \quad \xi = 0, \theta = 1 \text{ and } \frac{d\theta}{d\xi} = 0 \tag{10}$$

and Assuming that a solution of eqn (9) to the last equation to first order in the small parameter using perturbation technique,

$$\sigma = \theta + \alpha\psi + \alpha^2\phi + \ldots \tag{11}$$

Expanding equation (11) in terms of equation (9) we have,

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left[\xi^2\left(\frac{\partial\theta}{\partial\xi} + \alpha\frac{\partial\psi}{\partial\xi} + \alpha^2\frac{\partial\Phi}{\partial\xi} + \cdots\right)\right] + \frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\left(\frac{\partial\theta}{\partial\mu} + \alpha\frac{\partial\psi}{\partial\mu} + \alpha^2\frac{\partial\Phi}{\partial\mu} + \cdots\right)\right]$$
$$= -\left(\theta^n + n\alpha\theta^{n-1}\psi + n\alpha^2\theta^{n-1}\Phi + \ldots + \frac{n(n-1)}{2}\alpha^2\theta^{n-2}\psi^2 + \cdots\right) + \alpha \tag{12}$$

Since, θ is a spherically-symmetrical function and independent of μ.

Comparing the coefficients of zero, first and second order of α, we obtain the following equation,

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\theta}{\partial\xi}\right) = -\theta^n \tag{13}$$

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Psi}{\partial\xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\Psi}{\partial\mu}\right] = -n\theta^{n-1}\Psi + 1 \tag{14}$$

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Phi}{\partial\xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\right] = -n\theta^{n-1}\Phi - \frac{n(n-1)}{2}\theta^{n-2}\Psi^2 \tag{15}$$

Where, equation (14) is the non linear differential equation of Lane Emden type equation for the slowly rotating polytropic fluid sphere of index n, which is the basic equation in the theory of stellar structure. To obtain the solution of equation (14) consider ψ as a series of Legendre Polynomials $P_m(\mu)$,

$$\Psi = \sum_{m=0}^{\infty} A_m\Psi_m P_m(\mu) \quad \text{and} \quad \sum_{m=0}^{\infty}\Phi_m P_m(\mu) \tag{16}$$

Where $A'_m s$ are the arbitrary constants and $P_m(\mu)$. The Legendre functions of index m satisfies the differential equation

$$\left(\xi^2\frac{\partial^2\Psi_m(\xi)}{\partial\xi^2}\right) + 2\xi\left(\frac{\partial\Psi_m(\xi)}{\partial\xi}\right) + [\xi^2 - m(m+1)]\Psi_m(\xi) = 0, \quad m = 1,2,3 \ldots. \tag{17}$$

$\xi = 0$ is regular singular point for 1. Consider its series solution,

$$\psi_m = a_0\xi^m + a_1\xi^{m+1} + a_2\xi^{m+2} + \ldots \tag{18}$$

The solution of equation (17) is then given by,

$$\psi_m = 1 - \frac{1}{3!}\xi^2 + \frac{n}{5!}\xi^4 - \frac{n(8n-5)}{3\cdot7!}\xi^6 + \frac{n(70-183n+122n^2)}{9\cdot9!}\xi^8 + \ldots\ldots \tag{19}$$

which is the series solution of non-linear second order differential equation for of Lane-Emden equation near origin.

2. METHODOLOGY

2.1RAMANUJAN'S METHOD

An iterative method to determine the smallest root of the equation, $f(x) = 0$ where $f(x)$ is the form of $f(x) = 1 - (a_1x + a_2x^2 + a_3x^3 + \ldots)$ described by Srinivasa Ramanujan [10]. For the smallest root of $f(x)$,we can write

$$[f(x) = 1 - (a_1x + a_2x^2 + a_3x^3 + \cdots)]^{-1} = bx + b_2x^2 + b_3x^3 + \ldots \tag{20}$$

Expanding L.H.S by Binomial theorem and equating the coefficients of like powers of x on both the sides, we obtain,

$$b_1 = 1, b_2 = a_1 = a_1b_1, b_3 = a_2 + a_1^2 = a_2b_1 + a_1b_2,\ b_{n=}a_1b_{n-1} + a_2b_{n-2} + \cdots + a_{n-1}b_1 \tag{21}$$

The ratio $b_n/b_{n-1,}$,is called the convergent of the above equation (21).

THE SERIES SOLUTION FOR ROTATING LANE EMDEN EQUATION

The series solution of non-linear second order differential equation (19) of Lane-Emden equation near origin is given by,

$$\psi_m = 1 - \frac{1}{3!}\xi^2 + \frac{n}{5!}\xi^4 - \frac{n(8n-5)}{3\cdot7!}\xi^6 + \frac{n(70-183n+122n^2)}{9\cdot9!}\xi^8 + \ldots\ldots$$

$$\text{for } n = 0,\ \ \psi_0 = 1 - \frac{1}{3!}\xi^2 \text{ and for } n = 1, \psi_1 = \frac{\sin\xi}{\xi} \tag{22}$$

Substituting equation (19) in (20) we obtain,

$$[f(\xi) = 1\text{-}(\frac{1}{6}\xi^2 - \frac{1}{120}\xi^4 + \frac{1}{5040}\xi^6 - \frac{1}{362880}\xi^8 + \ldots\ldots)]^{-1} = b_1\xi^2 + b_2\xi^4 + b_3\xi^6 + b_4\xi^8 + \ldots\ldots\ldots$$

$$a_1 = \frac{1}{6}, a_2 = -\frac{1}{120}, a_3 = \frac{1}{5040}, a_4 = -\frac{1}{362880}$$

$$b_1 = 1, b_2 = a_1 = \frac{1}{6} = 0.016667, b_3 = a_2 + a_1^2 = a_2b_1 + a_1b_2 = 0.01944444\ldots\ldots \tag{24}$$

The ratio of $b_n/b_{n-1,}$, of equation (24) determines the smallest root (ξ) of the equation.

$$\frac{b_1}{b_2} = 2.449489743, \frac{b_2}{b_3} = 2.927700219, \frac{b_3}{b_4} = 2.8419928\ldots\ldots\ldots \frac{b_9}{b_{10}} = 3.14145478$$

Hence, 3.1414578 is the root of equation (24) for the polytropic index $n = 1$.

3. RESULT & DISCUSSION

In the present paper our purpose, is to solve the equation for a polytropic fluid and it is found that the solution of the rotating polytropic fluid sphere can be obtained much more conveniently without complicated computer programming methods.

Case 1: The series solution for the slowly rotating polytropic fluid sphere equation, $\left(\xi^2 \frac{\partial^2 \Psi_m(\xi)}{\partial \xi^2}\right) + 2\xi \left(\frac{\partial \Psi_m(\xi)}{\partial \xi}\right) + [\xi^2 - m(m+1)]\Psi_m(\xi) = 0$ is evaluated for different polytropic index i.e.

n= 0.0(0.1) to 2.0 by using special function (Legendre function and power series).

Case 2: The boundary values $\xi = 0,\ \theta = 1$ *and* $\theta' = 0$ for small values of ξ has been obtained using Ramanujan's Method and it is compared with the A.D. Parks's boundary value (reference), as presented in the table. The % error shows the efficiency of solving such equations by Ramanujan's Method. As, for slowly rotating polytropic fluid this method is highly accurate for the values of the polytropic index n = 0.0(0.1) *to* 1.7 and a simple calculator or MS-excel sheet can be used for the calculation.

Case 3: The polytropic model gives good results practically in the whole star, except for the nuclear region, suggesting that in this region a slightly higher polytropic index would be more appropriate. It also helps us to estimate the main properties of stellar interiors such as pressure P, density ρ, temperature T which can be compared with real stars because the physical laws derive from the laboratory also holds in the whole universe.

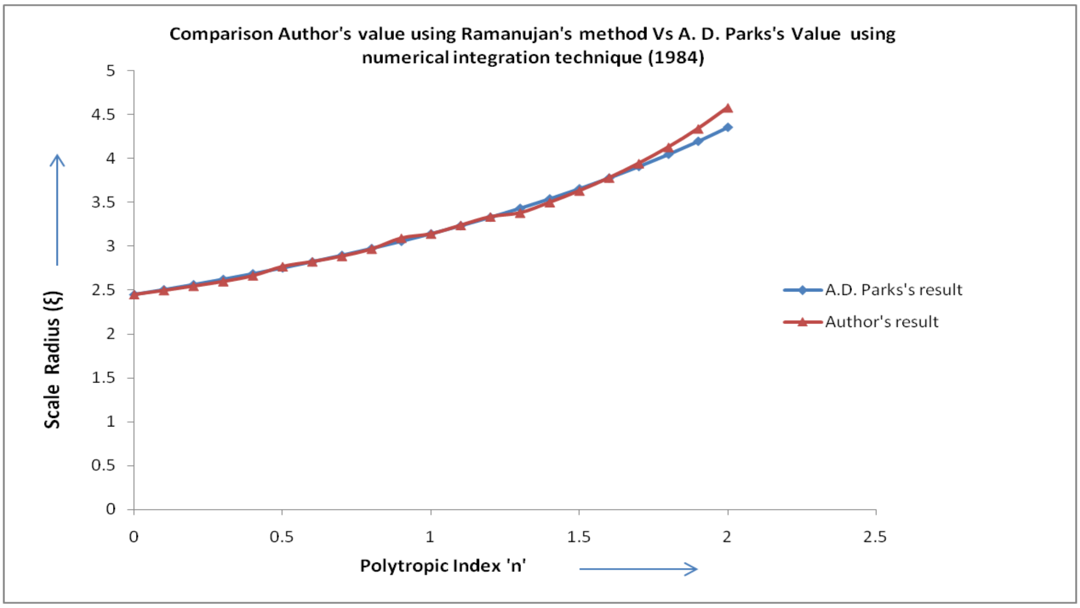
Case 4: The study of gaseous filaments or of spiral arms, where it would provide less idealized models than some which have been considered in the past. The solution to the polytropic model of index n can be used for the description of celestial objects such as:

- A polytropic index $n=0$ has constant density; it has in-compressible interior and used to model rocky planets.
- A polytropic index $n=0.5$ *to* 1.0 models the neutron stars.
- A polytropic index $n=1.5$ is a good model for white dwarf with low mass, fully convective star cores, brown dwarfs.

which can further lead to an application to the astronomy and astrophysics.

Table: Comparison table for the boundary values (ξ_1) using Ramanujan's Method with A.D. Parks's Value.

n	A.D. Parks's Result	Author's Result (ξ1)	Error (%)
0.0	2.44949	2.44949	0.00000
0.1	2.50454	2.49624	0.33100
0.2	2.56222	2.54570	0.64443
0.3	2.62268	2.59812	0.93617
0.4	2.68610	2.66402	0.82197
0.5	2.75270	2.76911	0.59623
0.6	2.82268	2.82421	0.05442
0.7	2.89628	2.88681	0.32663
0.8	2.97376	2.96913	0.15538
0.9	3.05543	3.09307	1.23191
1.0	3.14159	3.14145	0.00432
1.1	3.23261	3.23955	0.21471
1.2	3.32887	3.33646	0.22812
1.3	3.43081	3.38042	1.46848
1.4	3.53893	3.50050	1.08583
1.5	3.65375	3.63316	0.56344
1.6	3.77590	3.78063	0.12537
1.7	3.90606	3.94554	1.01086
1.8	4.04501	4.13129	2.13299
1.9	4.19361	4.34186	3.53521
2.0	4.35287	4.58257	5.27711



4. SUMMARY & CONCLUSION

Several iterative methods has been studied previously to determine the solution to initial boundary problems of the non linear differential equation but it was not very efficient. In the present study, we have calculated the roots (boundary values) for the slowly rotating polytropic fluid spheres using Ramanujan's method which can be seen as a useful tool for solving the in-homogeneous second order differential equation as, it is a novel method used to determine the accurate numerical results to the series solution as any other computational method. The numerical comparison of the results discussed here justified the relevance and the efficiency of the present study as the numerical values obtained are also close to the exact results. Hence, it can be used as an application in the field of astronomy such as real stars, white dwarfs and core of the degenerate star.

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