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Unique Fixed-Point Theorem for H-contraction mapping in Partially ordered metric space

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ABSTRACT

In the framework of partially ordered metric spaces, a fixed point theorem based on an H-contractive type mapping of rational type is presented in this study.

KEYWORDS

Unique Fixed points, rational type contraction, Partially Ordered metric space

SUBJECT CLASSIFICATION CODE

54H25, 47H10, 47H09

1. INTRODUCTION

The traditional Banach contraction principle [1] has been essential to obtaining a unique solution to the results in fixed point theory and approximation theory. Undoubtedly, it is a crucial and well-liked technique in many areas of mathematics for resolving current nonlinear analysis problems. Since then, the underlying contraction condition has been improved, leading to a wide range of generalizations of the Banach contraction principle in metric fixed point theory. Then, by weakening its hypotheses on a wide range of spaces, including rectangular metric spaces, pseudo-metric spaces, fuzzy metric spaces, quasi-metric spaces, quasi-semi-metric spaces, probabilistic metric spaces, D-metric spaces, G-metric spaces, F-metric spaces, and cone metric spaces, vigorous research work was obtained and is soon to be used to support the findings that have already been made [4-16]. It seems sense that there would be interest in establishing practical fixed point findings given the prominence of work on the existence and uniqueness of a fixed point as well as common fixed point theorems using monotone mappings on cone metric spaces, partly ordered metric spaces, and other spaces.

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The aim of this paper is to give a version of H-contraction [3] theorem in partially ordered metric spaces for a pair of self-mappings satisfying a contractive condition of rational type. These results generalize and extend the results of Harjani et.al.[2] in partially ordered metric space.

2. PRELIMINARIES

Definition 2.1 [3]

A mapping $\Upsilon:\Xi\to\Xi$ be a rational type contraction, where (Ξ,ϱ) is a complete metric space, is called H-rational type Contraction if there exist positive real numbers $\wp,\varphi,\tau$ such that $0\le\wp+\varphi+2\tau<1$ for all $\eta,\kappa\in\Xi$, , the following inequality holds,

$$\varrho(\Upsilon\eta,\Upsilon\kappa)\le\wp\varrho(\eta,\kappa)+\varphi\frac{\varrho(\kappa,\Upsilon\kappa)[1+\varrho(\eta,\Upsilon\eta)]}{1+\varrho(\eta,\kappa)}+\tau[\varrho(\eta,\Upsilon\eta)+\varrho(\kappa,\Upsilon\kappa)]$$

Definition 2.2 [2]

Let $(\Xi,\le)$ be a partially ordered set and $\Upsilon:\Xi\to\Xi$. We say that Υ is a nondecreasing mapping if for $\eta,\kappa\in\Xi,\eta\le\kappa\Rightarrow\Upsilon\eta\le\Upsilon\kappa$.

3. MAIN RESULTS

Theorem 3.1

Let $(\Xi,\le)$ be a partially ordered set and suppose that there exists a metric ϱ in Ξ such that (Ξ,ϱ) is a complete metric space. Let $\Upsilon:\Xi\to\Xi$ be a continuous and nondecreasing mapping such that

$$\varrho(\Upsilon\eta,\Upsilon\kappa)\le\wp\varrho(\eta,\kappa)+\varphi\frac{\varrho(\kappa,\Upsilon\kappa)[1+\varrho(\eta,\Upsilon\eta)]}{1+\varrho(\eta,\kappa)}+\tau[\varrho(\eta,\Upsilon\eta)+\varrho(\kappa,\Upsilon\kappa)] \tag{1}$$

For $\eta,\kappa\in\Xi,\eta\ge\kappa$, with $\wp+\varphi+2\tau<1$. If there exists $\eta_0\le\Upsilon\eta_0$, then Υ has a fixed point.

Proof.

If $\Upsilon\eta_0=\eta_0$, then the proof is finished. Suppose that $\eta_0<\Upsilon\eta_0$. Since Υ is a nondecreasing mapping, we obtain by induction that

$$\eta_0<\Upsilon\eta_0\le\Upsilon^2\eta_0\le\cdots\le\Upsilon^\gamma\eta_0\le\Upsilon^{\gamma+1}\eta_0\le\cdots$$

Put $\eta_{\gamma+1}=\Upsilon\eta_\gamma$. If there exists $\gamma\ge1$ such that $\eta_{\gamma+1}=\eta_\gamma$, then from $\eta_{\gamma+1}=\Upsilon\eta_\gamma=\eta_\gamma$, η_γ is a fixed point and the proof is finished. Suppose that $\eta_{\gamma+1}\ne\eta_\gamma$ for $\gamma\ge1$.

Then, from equation (1) and as the elements η_γ and $\eta_{\gamma-1}$ are comparable, we get, for $\gamma\ge1$,

$$\begin{aligned}\varrho(\eta_\gamma,\eta_{\gamma+1})&=\varrho(\Upsilon\eta_{\gamma-1},\Upsilon\eta_\gamma)\\&\le\wp\varrho(\eta_{\gamma-1},\eta_\gamma)+\varphi\frac{\varrho(\eta_\gamma,\Upsilon\eta_\gamma)[1+\varrho(\eta_{\gamma-1},\Upsilon\eta_{\gamma-1})]}{1+\varrho(\eta_{\gamma-1},\eta_\gamma)}\\&\quad+\tau[\varrho(\eta_{\gamma-1},\Upsilon\eta_{\gamma-1})+\varrho(\eta_\gamma,\Upsilon\eta_\gamma)]\\&=\wp\varrho(\eta_{\gamma-1},\eta_\gamma)+\varphi\frac{\varrho(\eta_\gamma,\eta_{\gamma+1})[1+\varrho(\eta_{\gamma-1},\eta_\gamma)]}{1+\varrho(\eta_{\gamma-1},\eta_\gamma)}\\&\quad+\tau[\varrho(\eta_{\gamma-1},\eta_\gamma)+\varrho(\eta_\gamma,\eta_{\gamma+1})]\end{aligned}$$

$$\varrho(\eta_\gamma,\eta_{\gamma+1})\le\frac{\wp+\tau}{[1-\varphi-\tau]}\varrho(\eta_{\gamma-1},\eta_\gamma)$$

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Again, using induction,

$$\varrho(\eta_\gamma, \eta_{\gamma+1}) \le \left[\frac{\wp + \tau}{1 - \varphi - \tau}\right]^\gamma \varrho(\eta_0, \eta_1)$$

Put $\varpi = \frac{\wp+\tau}{1-\varphi-\tau} < 1$

Moreover, by the triangular inequality, we have, for $\delta \ge \gamma$

$$\begin{aligned}\varrho(\eta_\delta, \eta_\gamma) &\le \varrho(\eta_\delta, \eta_{\delta-1}) + \varrho(\eta_{\delta-1}, \eta_{\delta-2}) + \cdots + \varrho(\eta_{\gamma+1}, \eta_\gamma)\\ &\le (\varpi^{\delta-1} + \varpi^{\delta-2} + \cdots + \varpi^\gamma)\varrho(\eta_0, \eta_1)\\ &\le \left(\frac{\varpi^\gamma}{1-\varpi}\right)\varrho(\eta_0, \eta_1)\end{aligned}$$

and this proves that $\varrho(\eta_\delta, \eta_\gamma) \to 0$ as $\delta, \gamma \to \infty$.

So, $\{\eta_\gamma\}$ is a Cauchy sequence and, since Ξ is a complete metric space, there exists $\sigma \in \Xi$ such that $\lim_{\gamma\to\infty} \eta_\gamma = \sigma$.

Further the continuity of Υ implies

$$\Upsilon\sigma = \Upsilon\left(\lim_{\gamma\to\infty} \eta_\gamma\right) = \lim_{\gamma\to\infty} \Upsilon\eta_\gamma = \lim_{\gamma\to\infty} \eta_{\gamma+1} = \sigma$$

And this proves that σ is a fixed point.

This finishes the proof.

In what follows, we prove that **theorem 3.1** is still valid for Υ, not necessarily continuous, assuming the following hypothesis in Ξ:

If $\{\eta_\gamma\}$ is a nondecreasing sequence in Ξ such that $\eta_\gamma \to \eta$, then $\eta = \sup\{\eta_\gamma\}$ (2)

Theorem 3.2

Let $(\Xi, \le)$ be a partially ordered set and suppose that there exists a metric ϱ in Ξ such that (Ξ, ϱ) is a complete metric space. Assume that Ξ satisfies equation (2). Let $\Upsilon: \Xi \to \Xi$ be a nondecreasing mapping such that

$$\varrho(\Upsilon\eta, \Upsilon\kappa) \le \wp\varrho(\eta, \kappa) + \varphi\frac{\varrho(\kappa, \Upsilon\kappa)[1 + \varrho(\eta, \Upsilon\eta)]}{1 + \varrho(\eta, \kappa)} + \tau[\varrho(\eta, \Upsilon\eta) + \varrho(\kappa, \Upsilon\kappa)]$$

With $\wp + \varphi + 2\tau < 1$. If there exists $\eta_0 \in \Xi$ with $\eta_0 \le \Upsilon\eta_0$, then Υ has a fixed point.

Proof.

Following the proof of **Theorem 3.1**, we only have to check that $\Upsilon\sigma = \sigma$.

As $\{\eta_\gamma\}$ is a nondecreasing sequence in Ξ and $\eta_\gamma \to \sigma$, then, by **(2)**, $\sigma = \sup\{\eta_\gamma\}$. Particularly, $\eta_\gamma \le \sigma$ for all $\gamma \in \mathbb{N}$.

Since, Υ is a nondecreasing mapping, then $\Upsilon\eta_\gamma \le \Upsilon\sigma$, for all $\gamma \in \mathbb{N}$ or, equivalently, $\eta_{\gamma+1} \le \Upsilon\sigma$ for all $\gamma \in \mathbb{N}$. Moreover, as $\eta_0 < \eta_1 \le \Upsilon\sigma$ and $\sigma = \sup\{\eta_\gamma\}$, we get $\sigma \le \Upsilon\sigma$.

Suppose that $\sigma < \Upsilon\sigma$. Using a similar argument that in proof of Theorem 2 for $\eta_\gamma \le \Upsilon\eta_\gamma$, we obtain that $\{\Upsilon^\gamma\eta\}$ is a nondecreasing sequence and $\lim_{\gamma\to\infty} \Upsilon^\gamma\eta = \kappa$ for certain $\kappa \in \Xi$.

Again, using **equation (2)**, we have that $\kappa = \sup\{\Upsilon^\gamma\sigma\}$.

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Moreover, from $\eta_0 \leq \sigma$, we get $\eta_\gamma = \Upsilon^\gamma \eta_0 \leq \Upsilon^\gamma \sigma$ for $\gamma \geq 1$ and $\eta_\gamma < \Upsilon^\gamma \sigma$ for $\gamma \geq 1$ because $\eta_\gamma \leq \sigma < \Upsilon\sigma < \Upsilon^\gamma \sigma$ for $\gamma \geq 1$.

As η_γ and $\Upsilon^\gamma \sigma$ for comparable and distinct for $\gamma \geq 1$, applying the contractive condition we get

$$\begin{aligned}\varrho\left(\eta_{\gamma+1}, \Upsilon^{\gamma+1}\sigma\right) &= \varrho\left(\Upsilon\eta_\gamma, \Upsilon(\Upsilon^\gamma\sigma)\right) \\ &\leq \wp\varrho\left(\eta_\gamma, \Upsilon^\gamma\sigma\right) + \varphi\frac{\varrho(\Upsilon^\gamma\sigma, \Upsilon^{\gamma+1}\sigma)\left[1 + \varrho\left(\eta_\gamma, \Upsilon\eta_\gamma\right)\right]}{1 + \varrho\left(\eta_\gamma, \Upsilon^\gamma\sigma\right)} \\ &\quad + \tau\left[\varrho\left(\eta_\gamma, \Upsilon\eta_\gamma\right) + \varrho(\Upsilon^\gamma\sigma, \Upsilon^{\gamma+1}\sigma)\right] \\ &= \wp\varrho\left(\eta_\gamma, \Upsilon^\gamma\sigma\right) + \varphi\frac{\varrho(\Upsilon^\gamma\sigma, \Upsilon^{\gamma+1}\sigma)\left[1 + \varrho\left(\eta_\gamma, \eta_{\gamma+1}\right)\right]}{1 + \varrho\left(\eta_\gamma, \Upsilon^\gamma\sigma\right)} \\ &\quad + \tau\left[\varrho\left(\eta_\gamma, \eta_{\gamma+1}\right) + \varrho(\Upsilon^\gamma\sigma, \Upsilon^{\gamma+1}\sigma)\right]\end{aligned}$$

Making $\gamma \to \infty$ in the last inequality, we obtain

$$\varrho(\sigma, \kappa) \leq \wp\varrho(\sigma, \kappa)$$

As $\wp < 1$ $\varrho(\sigma, \kappa) = 0$ thus, $\sigma = \kappa$

Particularly, $\sigma = \kappa = \sup\{\Upsilon^\gamma\sigma\}$ and, consequently, $\Upsilon\sigma \leq \sigma$ and this is a contradiction.

Hence, we conclude that $\sigma = \Upsilon\sigma$.

Now, we present an example where it can be appreciated that hypotheses in **Theorem 3.1** do not guarantee uniqueness of the fixed point. This example appears in [10].

Let $\Xi = \{(1,0), (0,1)\} \subset \mathbb{R}^2$ and consider the usual order

$$(\eta, \kappa) \leq (\sigma, t) \Leftrightarrow \eta \leq \gamma, \kappa \leq t.$$

Thus, $(\Xi, \leq)$ is a partially ordered set whose different elements are not comparable.

Besides, (Ξ, d_2) is a complete matric space considering, d_2, the Euclidean distance. The identity map $\Upsilon(\eta, \kappa) = (\eta, \kappa)$ is trivially continuous and nondecreasing and assumption **equation (1)** of **Theorem 3.1** is satisfied since elements in Ξ are only comparable to themselves. Moreover, $(1,0) \leq \Upsilon(0,1)$ and Υ has two fixed points in Ξ.

In what follows, we give a sufficient condition for the uniqueness of the fixed point in **Theorem 3.1** and **3.2**. This condition appears in [17] and

For $\eta, \kappa \in \Xi$, there exists a lower bound or an upper bound. (3)

In [10], it is proved that the above-mentioned condition is equivalent,

For $\eta, \kappa \in \Xi$, there exists $\sigma \in \Xi$ which is comparable to η and κ. (4)

Theorem 3.3.

Adding condition (equation (4)) to the hypothesis of Theorem 1 (or Theorem 2) one obtains uniqueness of the fixed point of Υ.

Proof.

Suppose that there exists $\sigma, \kappa \in \Xi$ which are fixed point.

We distinguish two cases.

Case 1. If κ and σ are comparable and $\kappa \neq \sigma$, then using the contractive condition we have

$$\varrho(\kappa, \sigma) = \varrho(\Upsilon\kappa, \Upsilon\sigma)$$

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$$\leq \wp\varrho(\kappa,\sigma) + \varphi\frac{\varrho(\sigma,\Upsilon\sigma)[1+\varrho(\kappa,\Upsilon\kappa)]}{1+\varrho(\kappa,\sigma)} + \tau[\varrho(\kappa,\Upsilon\kappa)+\varrho(\sigma,\Upsilon\sigma)]$$
$$= \wp\varrho(\kappa,\sigma) + \varphi\frac{\varrho(\sigma,\sigma)[1+\varrho(\kappa,\kappa)]}{1+\varrho(\kappa,\sigma)} + \tau[\varrho(\kappa,\kappa)+\varrho(\sigma,\sigma)]$$
$$= \wp\varrho(\kappa,\sigma)$$

As $\wp < 1$ is the last inequality, it is a contradiction. Thus, $\kappa = \sigma$

Case 2. If κ is not comparable to σ, then by equation (4) there exists $\eta \in \Xi$ comparable to κ and σ.

Monotonicity implies that $\Upsilon^\gamma\eta$ is comparable to $\Upsilon^\gamma\kappa = \kappa$ and $\Upsilon^\gamma\sigma = \sigma$ for $\gamma = 0,1,2,\ldots$.

If there exists $\gamma_0 \geq 1$ such that $\Upsilon^{\gamma_0}\eta = \kappa$ then as κ is a fixed point, the sequence $\{\Upsilon^\gamma\eta : \gamma \geq \gamma_0\}$ is constant, and, consequently, $\lim_{\gamma\to\infty}\Upsilon^\gamma\eta = \kappa$.

On the other hand, if $\Upsilon^\gamma\eta \neq \kappa$ for $\gamma \geq 1$, using the contractive condition, we obtain, for $\gamma \geq 2$,

$$\varrho(\Upsilon^\gamma\eta,\kappa) = \varrho(\Upsilon^\gamma\eta,\Upsilon^\gamma\kappa)$$
$$= \varrho\left(\Upsilon(\Upsilon^{\gamma-1}\eta),\Upsilon(\Upsilon^{\gamma-1}\kappa)\right)$$
$$\leq \wp\varrho(\Upsilon^{\gamma-1}\eta,\Upsilon^{\gamma-1}\kappa) + \varphi\frac{\varrho(\Upsilon^{\gamma-1}\kappa,\Upsilon^\gamma\kappa)[1+\varrho(\Upsilon^{\gamma-1}\eta,\Upsilon^\gamma\eta)]}{1+\varrho(\Upsilon^{\gamma-1}\eta,\Upsilon^{\gamma-1}\kappa)}$$
$$+\tau[\varrho(\Upsilon^{\gamma-1}\eta,\Upsilon^\gamma\eta)+\varrho(\Upsilon^{\gamma-1}\kappa,\Upsilon^\gamma\kappa)]$$
$$\leq \wp\varrho(\Upsilon^{\gamma-1}\eta,\kappa) + \varphi\frac{\varrho(\kappa,\kappa)[1+\varrho(\Upsilon^{\gamma-1}\eta,\Upsilon^\gamma\eta)]}{1+\varrho(\Upsilon^{\gamma-1}\eta,\kappa)}$$
$$+\tau[\varrho(\Upsilon^{\gamma-1}\eta,\Upsilon^\gamma\eta)+\varrho(\kappa,\kappa)]$$
$$\varrho(\Upsilon^\gamma\eta,\kappa) \leq \frac{\wp+\tau}{1-\tau}(\Upsilon^{\gamma-1}\eta,\kappa)$$

Using induction,

$$\varrho(\Upsilon^\gamma\eta,\kappa) \leq \left(\frac{\wp+\tau}{1-\tau}\right)^\gamma(\Upsilon^{\gamma-1}\eta,\kappa) \qquad \text{for } \gamma \geq 2$$

And as $\frac{\wp+\tau}{1-\tau} < 1$, the last inequality gives us $\lim_{\gamma\to\infty}\Upsilon^\gamma\eta = \kappa$.

Hence, we conclude that $\lim_{\gamma\to\infty}\Upsilon^\gamma\eta = \kappa$.

Using a similar argument, we can prove that $\lim_{\gamma\to\infty}\Upsilon^\gamma\eta = \sigma$

Now, the uniqueness of the limit gives us $\kappa = \sigma$.

This finishes the proof.

Remark 3.4. It is easily proved that the space $C[0,1] = \{\eta : [0,1] \to \mathbb{R}, continuous\}$ with the partial order given by

$$\eta \leq \kappa \Leftrightarrow \eta(t) \leq \kappa(t), \quad \text{for } t \in [0,1]$$

And the metric given by

$$\varrho(\eta,\kappa) = \sup\{|\eta(t)-\kappa(t)| : t \in [0,1]\}$$

Satisfies condition (equation (2)). Moreover, as for $\eta,\kappa \in C[0,1]$, the function $\max(\eta,\kappa)(t) = \max\{\eta(t),\kappa(t)\}$ is continuous, $(C[0,1],\leq)$ satisfies also condition (equation (4)).

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