APPROXIMATION OF FRACTIONAL ORDER SYSTEM AND COMPARISON OF VARIOUS METHODS-

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ABSTRACT-

Most of the systems used in day to day life are of fractional order. So analysis of fractional system is most important. But as it is difficult to design and realize a fractional order system in hardware by using available electrical parameters like resistance, capacitance etc; integral approximation of fractional plant is required. Several methods of approximation of fractional plant to integral order are discussed. Every method is analyzed in time and frequency domain. Also stability analysis is done using pole-zero realization.

KEYWORDS-Crone, Carlson, Matsuda, Oustaloup, Modified Oustaloup

1. INTRODUCTION:

The part of universe which is under consideration (separated by suitable boundary that may exist physically or imaginary) is called a system. Thereexists a particular relationship between inputs and outputs of a system which are usually described by differential equations. In our day today life we come across such systems in which these differential equations are not of integer orders. Such systems are called fractional order systems. This thesis mainly deals with the systems which are of fractional order.

By utilization of fractional order system memory and inherited properties of different materials and methods can be portrayed magnificently. This is the fundamental point of interest of fractional order system over established integral model where such element practices are not considered. In recent time applications of fractional integrals and derivatives in theory of control dynamics is quite obvious and description of controllers with the help of fractional differential equation is not uncommon. Some other fields where utilization of fractional order difference/differential equation is necessary are-transmission line , chemical analysis of aqueous solutions, heat-flux meter, viscoelasticity[1] , dielectric polarization[2] , propagation of electromagnetic waves, rheology of soil, quantum mechanical calculations etc.

Fractional P-I-D(proportional integral derivative) controller is also an excellent tool for controlling parameters of a system. Because of more degrees of flexibility when contrasted with conventional integral P-I-D controller adaptable control of system process is acquired with the assistance of fractional PID controller.

2. STATEMENT OF PROBLEM-:

Designing of infinite transmission line is one of the prime applications of fractional order system. Considering such transmission line having series impedanceZ_a and shunt admittanceZ_b:

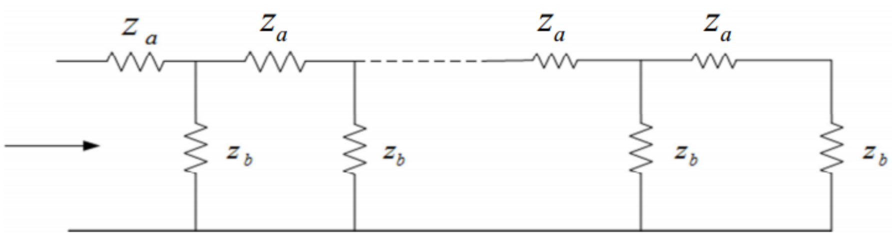

Figure 1.1: Infinite Line Transmission System

Referring to the diagram characteristic impedance of the line is given as=$\sqrt{Z_a * Z_b}$

Z_a=series impedance of line=R+SL

Z_b=shunt admittance of the line=1/SC

So characteristics impedance is (Z_{char}) = (R+SL)*($\frac{1}{SC}$)

But as frequency terms in series inductance and shunt capacitance is cancelled out so considering only series resistance term we get=$(\frac{R}{SC})^{0.5}$

So here characteristics impedance of practical line is of -0.5 th order .

So as real life systems are of fractional order, in our discussion we will be confined to fractional order system.

3. OBJECTIVE OF THE STUDY-

- To approximate fractional order system to integral order.
- To compare various approximations.
- To determine stability using pole-zero analysis.

4. LITERATURE REVIEW

This section provides a brief idea on how to deal with a system which is of fractional order. In our day to day life we come across many such things which are perfect examples of fractional order system. Starting from infinite transmission line to viscos elasticity [1], polarization characteristics of dielectric materials [2] are some mostly observed fractional order systems.

According to Cauchy[3] , Grunwald-Letinkov [4] , Riemann-Liouville [5] fractional derivative of of a function holds all the memories from past(negative infinite time) to present time. They also defined state space of fractional order system is of infinite dimensions and thus fractional order system suffers from memory effects.

A .Oustaloup , P . Melchior , O.Colis , F.Dancla in their work “The Crone Tool box for Matlab ”[8] continuous(s-domain) and discrete(z-domain) highlighted approximation of fractional order systems,its necessity,advantage of using approximated integer transfer functions . The work done by A .Oustaloup , P . Melchior in “Crone tool box for Matlab ” [8] suggests FOMCON is a superset of already existing toolbox FOTF. This also suggests that this toolbox serves as a missing link between CRONE and NINTEGER toolboxes. D.Valerio in his work showed NINTEGER tool box contains various folders among which some are accessible to clients, some are used for graphical user interface, some contains functions that deals with continued fractions [9]. Using this toolbox Crone suggested to approximate a system where fractional power is also a complex number[10].Matsuda approximated a system by continuous fraction approximations where value of a function at different frequencies are known beforehand [11].Carlson did an excellent work [12] for approximation of fractional capacitor $\frac{1}{s^{1/n}}$ by regular Newton method which is a great contribution to fractional theory. Oustaloup designed a BPF[15] to approximate fractional power of ‘s’ which shows its best performance when fractional power lies between 0 to 1. Next some methods are suggested known as modified Oustaloup method which provided opportunity to follow step command without steady stateerror when the plant contains open loop poles at origin.

Next to controlled performance of plant; design of PID controller is highlighted by ArjitBiswas ,Sambrat Dasgupta in their excellent work [16] which also shows advantage of using fractional PID controller over traditional PID controller. I.Padlubny gave a clear idea how fractional order controller gives more degrees of freedom over integral controllers [17].The research work of J.G Ziegler and N.B Nichols “Optimum setting for Automatic controller” [19] showed how to approximate a plant into its equivalent FOPDT model and to calculate gains of proportional, integral, derivative controller to design IOPID controller. Nelder-Meld proposed an algorithm for tuning of IOPID controller and calculates fractional powers by choosing suitable number of iterations for designing FOPID controller. Monje-Vingre proposed frequency domain tuning to design tilt controller(TID) which provides better tuning methods, low sensitivity, to parameter variations and less dominance of integral terms in low frequency region[20].

5. RESEARCH METHODOLOGY-

5.1 . CRONE APPROXIMATION:

This function is used to design a controller given by frequency domain transfer function:

C(s)=ks^{v} ;v is a fractional number.

If 'v' is a complex number given by a+ib then the approximation is given by[10] =

$$C(s)=K'\frac{\prod_{n=1}^{M} 1+\frac{s}{\omega_{zn}}}{\prod_{n=1}^{N} 1+\frac{s}{\omega_{pn}}}$$

Here from these formulae it is clear that no of poles and no of zeros are not equal in the approximation. So K′ is so adjusted is so adjusted that if K is equal to 1 then gain is 0 dB for 1 rad/sec frequency. Zeros and poles depends on frequencies ω_h and ω_l given by[10]

$$\alpha =\left(\frac{\omega_h}{\omega_l}\right)^{\alpha\left(\frac{N+M}{NM}\right)} \quad ,¶= 10^{\frac{\eta\log(\alpha)}{\eta-2b\log(\alpha)\mathrm{tgh}\frac{b\eta}{2}}}$$

$$\omega_{z1=}\omega_l\sqrt{¶}\ ,\omega_{pl}=\omega_l\sqrt{\alpha}$$

$\omega_{pn}=\omega_{p,n-1}\alpha$,   n=2……N

$\omega_{zn}=\omega_{z,n-1}¶$,   n=2…..M

If v is a negativethen role of poles and zeros are interchanged.

One of the major advantages of this method is that it is suitable for complex number.

5.2. MATSUDA APPROXIMATION:

This approximation method is used to approximate a function by continuous fraction approximation method. Here we know value of function 'f' at various frequenciesx_0,x_1….x_n. The number of frequencies where we want to find gains depends on number of poles and zeros. For M no of zeros and N number of poles number of frequencies used are 2M+1. It is always preferable to use odd no of frequencies terms because if we useeven number of frequencies number of zeros exceeds number of poles by one. So the model will be improper.

Then we can defineas[11]:

d_0(x)=f(x)

d_{k+1}(x)=$\frac{x-x_k}{d_k(x)-d_k(x_k)}$

Then the approximation of f(x) is given by:

f(x)= $d_0(x)+\frac{x-x_0}{d_1(x_1)+\frac{x-x_1}{d_2(x_2)+\cdots}}$

For frequency terms we have to replace x terms by s terms. So theequations will be =

$|d_0(\omega)|=|C(j\omega)|$

$d_{k+1}(\omega)=\frac{\omega-\omega_k}{d_k(\omega)-d_k(\omega_k)}$

C(ω)= $d_0(\omega)+\frac{\omega-\omega_0}{d_1(\omega_1)+\frac{\omega-\omega_1}{d_2(\omega_2)+\cdots}}$

5.3 . CARLSON APPROXIMATION:

This is one of the continuous approximation method used to write system as :

$F^{1/v}=s$

After this we use Newton's iteration method to solve theequation. But it is applicable to those cases only where 1/v ε Z that is v can only take values +1/2,-1/2,+1/3,-1/3,+1/4,-1/4…This leads to following approximation[12] :

$F_i(s)=F_{i-1}(s)\frac{\left(\frac{1}{v}-1\right)F^{\frac{1}{v}}{}_{i-1}+\ \left(\frac{1}{v}+1\right)s}{\left(\frac{1}{v}+1\right)F^{\frac{1}{v}}{}_{i-1}+\left(\frac{1}{v}-1\right)s}$, given $F_0(s)=1$

It is required that (1/v) must be an integer. If this condition is not satisfied fractional powers of's' cannot be removed from theexpression given in $F_i(s)$.

As no of iterations increase; ripples in the gain and phase of approximation decreases and frequency range increases. Increase in number of iteration continues till these characteristics are within particular limit. The center of frequency interval of this approximation is set to 1.

5.4. CONTINIOUS CONTINUED FRACTION APPROXIMATION METHOD:

It is of 2 types. They are

1. Continuous fraction approximation at high frequency(CFE HIGH)
2. Continuous fraction approximation at low frequency(CFE LOW)

The approximation is given as [13,14] : $s^v=(1+s)^v$; if ω>>>1

$$= \quad (1+\frac{1}{s})^{v} \text{ ; if } \omega<<<1$$

The broad range of frequencies where that approximation is valid depends on number of terms of continued fraction that is kept. Different validity of frequency range is obtained by writing as:

$(\frac{s}{\lambda})^{v}$ $=(1+\frac{s}{\lambda})^{v}$ if $\frac{\omega}{\lambda}$>>1

$=(1+\frac{\lambda}{s})^{-v}$ if $\frac{\omega}{\lambda}$<<1

So we can also state that

$(s)^{v}$ $=\lambda^{v}(1+\frac{s}{\lambda})^{v}$ if ω >>λ

$=(1+\frac{\lambda}{s})^{-v}$ if ω<<λ

So theexpansion is given as:

F(s)= $\lambda^{v}[0 \;;1/1;(\frac{-sv}{\lambda})/1;\frac{\left\{\frac{i(i+v)s}{(2i-1)2i\lambda}\right\}}{1};\left\{\frac{i(i-v)s}{(2i-1)2i\lambda}\right\}/1]$ for ω >>λ , here I varies from 0 to ∞……………………………1

$= \lambda^{v}[0 \;;1/1;(\frac{sv}{\lambda})/1;\frac{\left\{\frac{i(i-v)s}{(2i-1)2i\lambda}\right\}}{1};\left\{\frac{i(i+v)s}{(2i+1)2i\lambda}\right\}/1]$ for ω <<λ , here also I varies from 0 to ∞…………………………2

For higher frequency continued fraction approximation truncating after N terms of equation 1 result in N poles and N-1 zeros. For lower frequency continued fraction approximation truncating after N terms of continued fraction approximation of equation 2 results in N poles and N zeros.

5 .5. OUSTALOUP APPROXIMATION METHOD-

This method is proposed by scientist Oustaloup. He designed a band pass filter for approximation. This filter has a lower cutoff frequency, a higher cut off frequency and order of filter. He approximated thefractional power of **s** as[15]

s^{r} = K * $\prod_{k=-N}^{k=N}\frac{(s+\omega'_{k})}{(s+\omega_{k})}$

Where ω'_{k}=$\omega_{b}(\frac{\omega_{h}}{\omega_{b}})^{\{k+N+0.5(1-r)\}/(2N+1)}$

ω_k=$\omega_b(\frac{\omega_h}{\omega_b})^{\{k+N+0.5(1-r)\}/(2N+1)}$

K=$(\omega_h)^r$

r=fractional number between 0 and 1(such assumption gives better result)

ω_b=lower cutoff frequency of filter
ω_h=higher cutoff frequency of filter
N=order of filter

5.6 MODIFIED OUSTALOUP APPROXIMATION METHOD:

The problem with ORA is that currently there is no method available for determining the suitable value ofω_b ,ω_h, and N. In practice, the value of these parameters is determined by trial and error which may be inefficient in somecases. Also, applying ORA to the fractional order integrator of a FOPID prompts an approximate feedback system without the capacity of following the step command without steady stateerror (take note of that giving a fractional order integrator in any stable feedback system ensures the following of step command without steady stateerror). To overcome such bad marks we utilizeanother filter technique called "Modified oustaloup approximation method". The estimate is given as:

s^r=K$\prod_{k=-N}^{k=N}\frac{s+\omega'_k}{s+\omega_k}$

Whereω'_k=$\omega_b(\frac{\omega_h}{\omega_b})^{\frac{k+N+0.5(1-r)}{2N+1}}$

ω_k=$\omega_b(\frac{\omega_h}{\omega_b})^{\frac{k+N+0.5(1+r)}{2N+1}}$

K=${\omega_h}^r$

s^r=[G_p[d $*$ $\frac{\omega_h}{b}$]^{r}r* (d$s^2 + b\omega_h s$)]/ [d (1-r)$s^2 + b\omega_h s + d_r$]

G_p=K$\prod_{k=-N}^{k=N}\frac{s+\omega'_k}{s+\omega_k}$

ω_k=(bω_h/d)$^{(r+2k)/(2N+1)}$, $\omega'_k = (d\omega_b$/b)$^{(r-2k)/(2N+1)}$

Default value of a=10 and b=9

6. RESULTS AND DISCUSSION-

6.1 COMPARISON OF RESULTS OF VARIOUS APPROXIMATION

Taking frequency range[0.1,10]rad/sec and number of poles and zeros to be 4.

Original TF	Crone	Carlson	Matsuda	CFE high	CFE low	Oustaloup	Mod Oustaloup
$s^{0.5}$	3.162 s^4 + 19.31 s^3 + 27.7 s^3 + 10.86 s + 1 -------- s^4 + 10.86 s^3 + 27.7 s^2 + 19.31 s + 3.162	9 s^6 + 138 s^5 + 711 s^4+ 1548 s^3 + 1351 s^2 + 330 s + 9 ………… s^6 + 42 s^5 + 351 s^4 + 1164 s^3 + 1647 s^2 + 810 s + 81	6.819 s^2 + 15.96 s + 1 ----------- s^2 + 15.96 s + 6.819	59.29 s^3 - 35.58 s^2 - 6.099 s - 0.3162 -------- 11.16 s^7 - 35.71 s^6 - 53.57 s^5 - 14.29 s^4- s^3	3.162 s^4 + 47.43 s^3+ 197.6 s^2 + 197.6 s ----- s^4+ 20 s^3+ 125 s^2 + 250 s + 78.12	3.162 s^9 + 53.25 s^8 + 333.9 s^7 + 1013 s^6 + 1628 s^5 + 1432 s^4+ 690.1 s^3 + 176.1 s^2+ 21.75 s + 1 ------------- s^9+ 21.75 s^8 + 176.1 s^7 + 690.1 s^6 + 1432 s^5 + 1628 s^4 + 1013 s^3 + 333.9 s^2 + 53.25 s + 3.162	6 s (s+0.1136)(s+0.1896)(s+0.3162)(s+0.5275)(s+0.8799)(s+1.468)(s+2.448)(s+4.084)(s+6.813)(s+11.11) ------------- (s+0.1468)(s+0.2448)(s+0.4084)(s+0.6813)(s+1.136)(s+1.896)(s+3.162)(s+5.275)(s+8.799)(s+22.18)(s+0.04509)

TABLE . 1 (Transfer functions of various approximation methods)

6.2 ROOTS OF THE TRANSFER FUNCTION FOR VARIOUS APPROXIMATIONS:

APPROXIMATION	ZEROS	POLES	COMMENT
Crone	-4.2211 , -1.2365 -0.7583 0.0545 + 0.2774i 0.0545 - 0.2774i	-7.5041 -2.3681 -0.7508 -0.2370	Alternative stable real poles and zeros. Complex zeros are unstable. Method is good in the frequency range mentioned. But performance decreases sharply at its limits.
Carlson	-7.5486 -3.0000 -3.0000 -1.4203 -0.3333 -0.0311	-32.1634 -3.0000 + 0.0000i -3.0000 - 0.0000i -3.0000 -0.7041 -0.1325	All poles and zeros are stable. Some are complex and some are real. Seems to be best approximation method .Approximation is valid in wide frequency range.
Matsuda	-2.2761 -0.0644	-15.5206 -0.4394	Alternative stable zeros and poles. Good approximation method. But ripples in phase behavior are significant.
CFE high	0.7473 -0.0736 + 0.0415i -0.0736 - 0.0415i	0 0 0 4.3672 , -0.8428 -0.2069 ,-0.1176	Presence of unstable poles and zeros.
CFE low	0 -8.5379 -4.9975 -1.4646	-9.6194 -6.9133 -3.0867 -0.3806	Stable poles and zeros are present. But approximation is valid below the chosen frequency λ only for very short range of frequency

Oustloup	-6.7986 , -4.1149 -2.4135 , -1.5132 -0.8266 ,-0.5627 -0.3060 , -0.1916 -0.1135	-8.8115 -5.2180 -3.2685 -1.7771 -1.2097 -0.6608 -0.4143 -0.2430 -0.1471	All the poles and zeros are stable. Approximation is equivalent to crone approximation.
Modified Oustaloup	0,-0.1136,-0.1896,-0.3162,-0.5275,-0.8799,-1.468,-2.448,-4.084,-6.813,-11.11	-0.1468,-0.2448,-0.4084,-0.6813,-1.136,-1.896,-3.162,-5.275,-8.799,-22.18,-0.04509	Stable system. Better performance than oustaloup for low frequency region.

TABLE. 2 (Roots of various approximations)

6.3 FREQUENCY DOMAIN COMPARISON OF VARIOUS APPROXIMATIONS:

[Frequency range [0.1, 10] , no of poles and zeros=4 in all approximations]

6.3.1 OBSERVATIONS FROM FREQUENCY RESPONSEOF $s^{0.5}$:

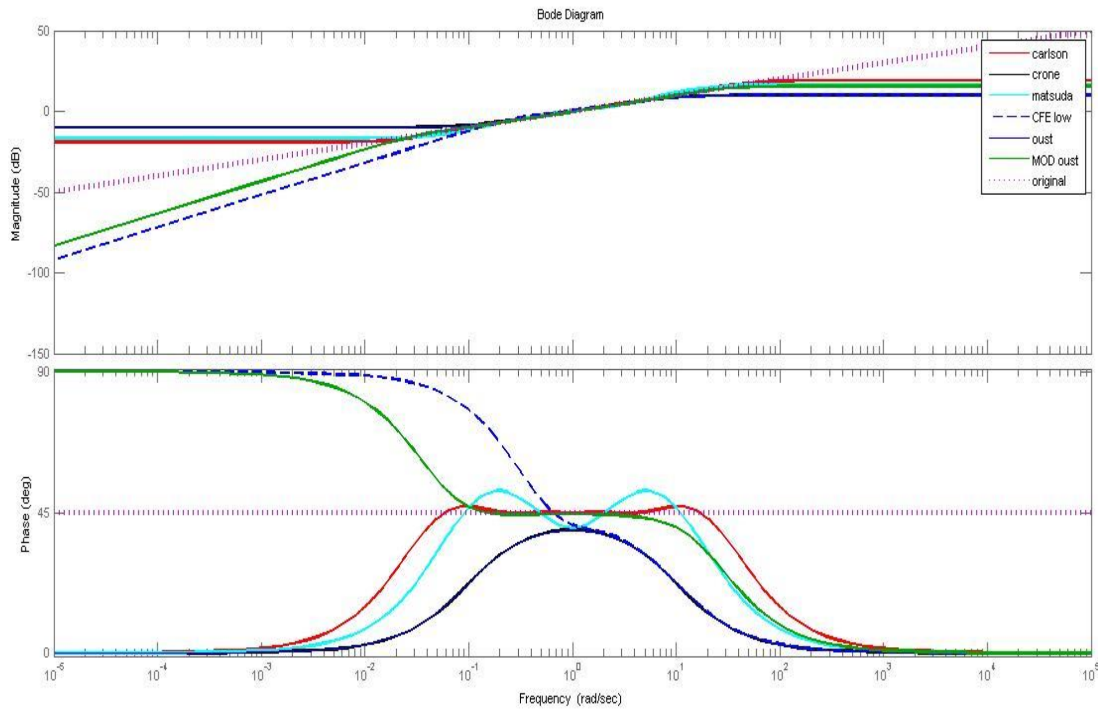

FIG 2:- Frequency response of open loop transfer function$s^{0.5}$

Here we take Modified Oustaloup method is the best method in approximating the system. After this is Carlson and Matsuda both method shows equal success in approximation. Crone and Oustaloup methods completely match and are not good enough in approximation comparable to other approximations. But all the approximations are good in wide range of frequencies in this case.

6.3.2 OBSERVATIONS FROM FREQUENCY RESPONSE OF $s^{1.5}$

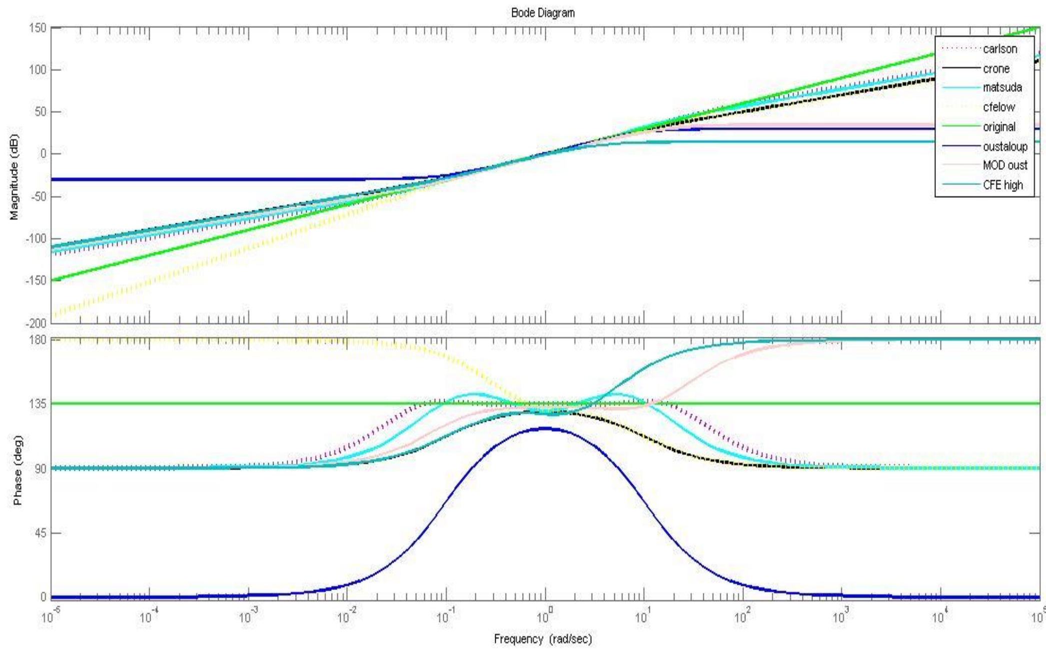

FIG 3:-Frequency response of open loop transfer function$s^{1.5}$

Matsuda and Carlson approximation is more close to original system for longer range of frequencies as compared to any other approximation. But ripple content in phase response of Matsuda approximation is higher. For CFE low approximation is only valid below the frequency which is chosen and that is also for short range only. For CFE high approximation is valid above the chosen frequency. Oustaloup approximation method is not so accurate and approximates the system accurately to someextent around the center frequency only. Modified oustaloupmethod is superior to Oustaloup method and satisfies the approximation for longer range of frequencies than Oustaloup approximation method.

6.3.3 OBSERVATIONS FROM FREQUENCY RESPONSE OF $s^{-1.5}$

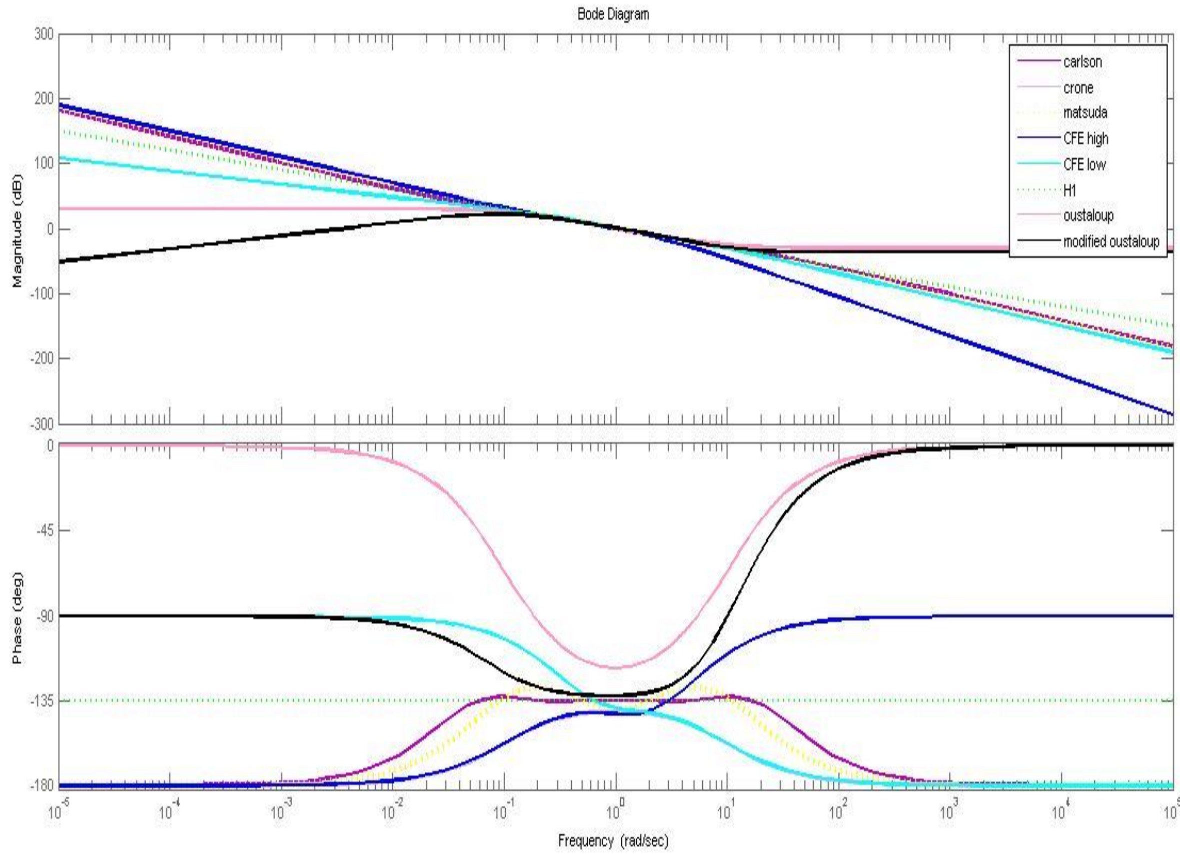

FIG 4:- Frequency response of open loop transfer function$s^{-1.5}$

Again Carlson method is proved to be the best approximation method followed by Matsuda. Modified Oustaloup method satisfies the approximation for quite long range of frequencies. Oustaloup method shows a large variation from original system especially in case of phase behavior and proved to be the worst method.

6.3.4 OBSERVATION FROM FREQUENCY RESPONSE OF $s^{-0.5}$

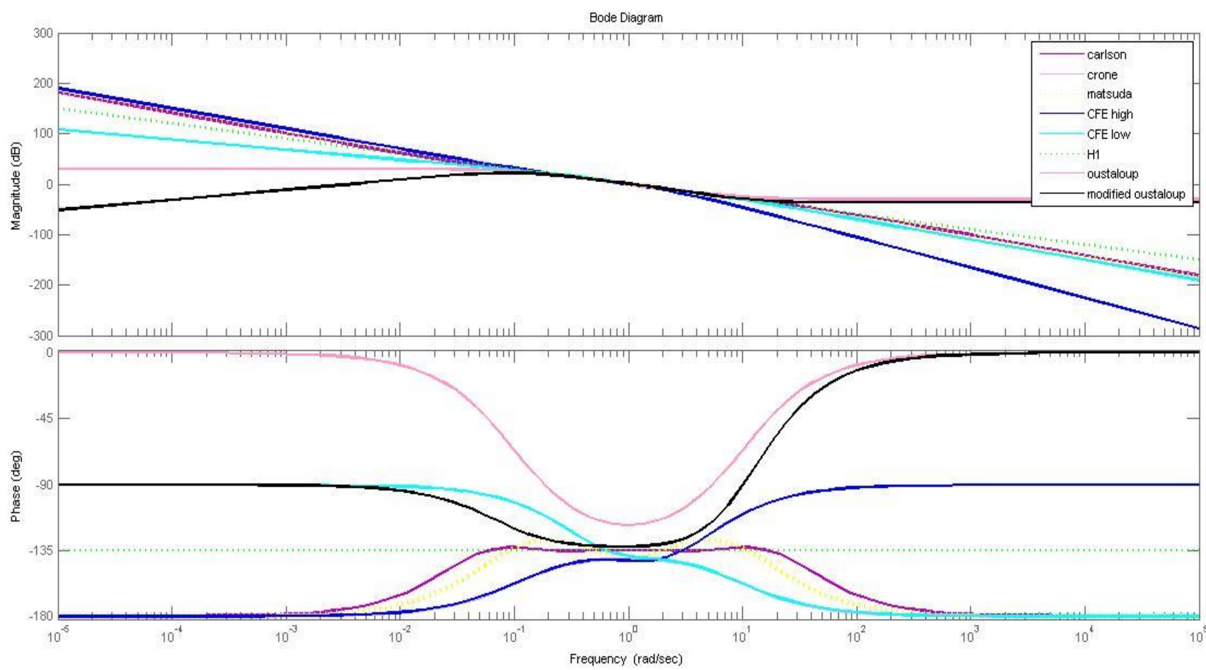

FIG 5:- Frequency response of open loop transfer function$s^{-0.5}$

Again Carlson method is proved to be the best approximation method followed by Matsuda. Modified Oustaloup method satisfies the approximation for quite long range of frequencies. Oustaloup method shows a large variation from original system especially in case of phase behavior and proved to be the worst method.

6.8TIME DOMAIN COMPARISON OF VARIOUS APPROXIMATIONS:

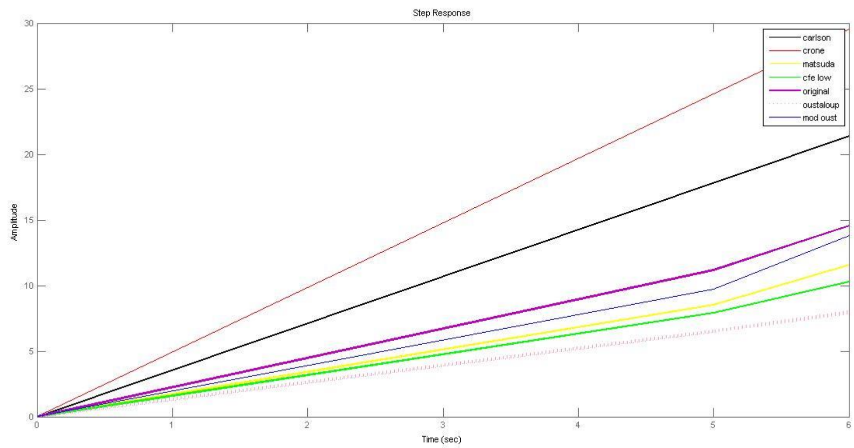

FIG 6 :-Step response of open loop transfer functions$s^{-1.5}$

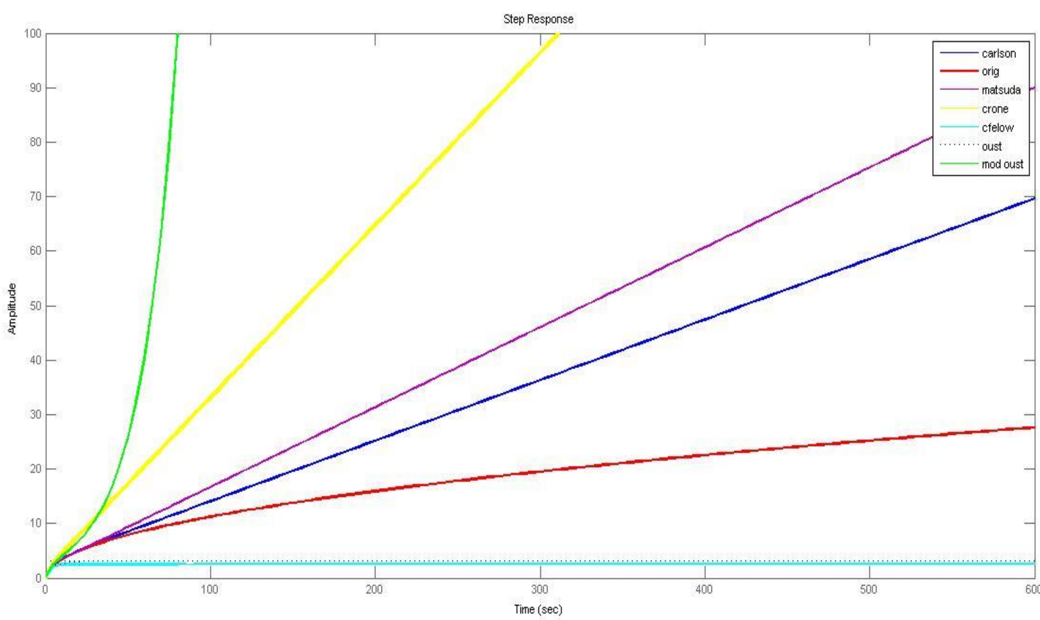

FIG 7 :- Step response of open loop transfer functions $s^{-0.5}$

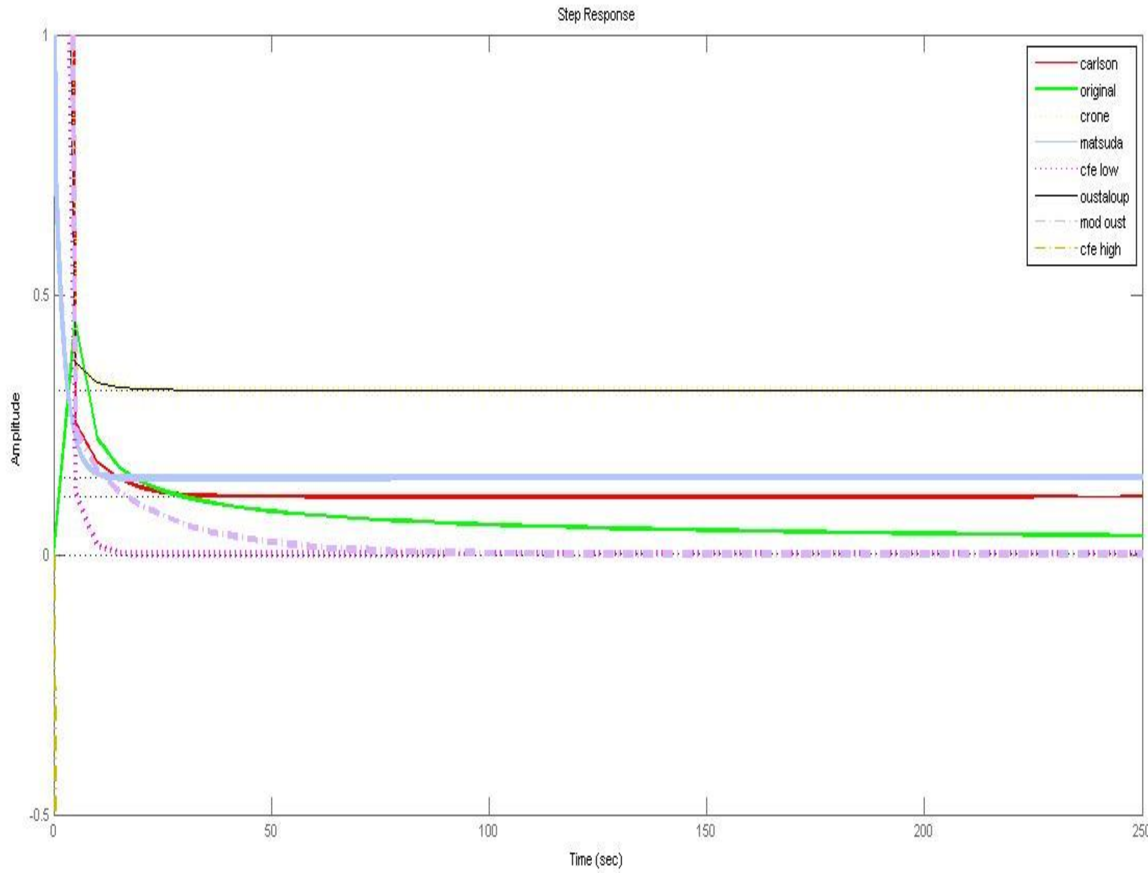

FIG 8 :-Step response of open loop transfer functions $s^{0.5}$

TRANSFER FUNCTION	OBSERVATION FROM STEP RESPONSE	COMMENT
$s^{-1.5}$	Modified oustaloup method gives best approximation followed by Matsuda and then CFE low. Carlson and Crone methods are not so appropriate. Oustaloup method is most inferior.	• For -1<v<1 Carlson approximation is a good approximation method. • Modified Oustaloup method is also a very good approximation method for 0<v<1. • Oustaloup method is always inferior to modified oustaloup whatever the value of v. • For 0<v<1 Crone and

		Oustaloup method completely matches. • Matsuda method also shows good approximation for-1<v<1. • CFE low shows medium range of success in approximation. • CFE high is the worst method.
$s^{-0.5}$	Carlson is much more close to original one. Next to Carlson it is Matsuda and next it is Crone. CFE low and Oustaloup method completely matches. Modified Oustaloup method shows much more variation then original. During low time period all approximation holds good but as time increases it becomes poor.	
$s^{0.5}$	Carlson is very good approximation. It nearly matches to original response for medium and higher time periods. Next to Carlson, Matsuda and Modified oustaloup shows equal success in approximation for medium and higher time. But for lower timeOustaloup is better than Matsuda. Next to modified Oustaloup it is CFE low. It shows pretty good approximation during low time period. Crone and Oustaloup method match completely but these are not good approximation methods. CFE high is not at all good approximation method.	

TABLE.3 :-Comparison of step responses for various approximations

6.5 SUMMING UP COMPARISONS OF FREQUENCY AND TIME RESPONSES:

It cannot be said which method is the best method because:

1. It depends on values of fractional order ‘v’ whether $v<-1$ or $-1<v<0$ or

$0<v<1$ or $v>1$.

2.It depends on instant of time in time response and instant of frequency in frequency response at which we are considering the approximations.

3. It also depends on whether we are interested in gain or phase behavior of bode plot in case of frequency response.

But one thing is concluded that almost all the approximations shows good performances when $0<v<1$. So this range of values of v is always preferred.

7. CONCLUSION:

In this dissertation approximation of fractional order system into suitable integral order system and design of integral and fractional order PID controller for fractional order plant is discussed. As in our day to day life we come across many systems whose order is fractional ; importance should be given to such systems. As software developed by us deals with only integral order and fractional order system suffers from memory effect due to infinite dimensions of its state space it is necessary to approximate such system into integral order system. Here in this dissertation I have discussed various approximation methods such as Crone , Matsuda , CFE high , CFE low , Oustaloup and modified Oustaloup methods are discussed. Step response and impulse responses for various methods are obtained and compared with responses of original system for various values of fractional order. Similarly frequency domain comparisons (Bode plot) for various approximation is done. At last for what range of frequency and what values of fractional order which method is best is suggested

8. REFERENCE

1. http://en.wikipedia.org/wiki/Viscoelasticity

2. http://en.wikipedia.org/wiki/Dielectric

3. K.S. Miller, B. Ross "An Introduction to the Fractional Calculus and Fractional Differential Equations", NewYork: Wiley, 1993.

4. D. Xue, Y.Q. Chen MATLAB Solutions to Advanced Applied Mathematical Problems. Beijing: Tsinghua University Press, 2004.

5. R. Hilfer "Applications of Fractional Calculus in Physics", Singapore: World Scientific, 2000.

6. I. Podlubny Fractional Differential Equations. San Diego: Academic Press, 1999.

7. I. Petras, I. Podlubny. P. O'Leary. "Analogue Realization of Fractional Order Controllers" Fakulta BERG,TU Kosice,2002

8.A. Oustaloup, P. Melchior, P. Lanusse, O. Cois, and F. Dancla, "The CRONE toolbox for Matlab," in*Proc. IEEE Int. Symp. Computer-Aided Control System Design CACSD 2000*, 2000, pp. 190–195

9.D. Valério. (2005) Toolbox ninteger for MatLab, v. 2.3.

10. Oustaloup, A. (1991). La commandeCRONE :CommandeRobusted'Ordre Non Entier. Herm`es, Paris.

11. Matsuda et al., Vinagre, p. 153-154,2001.

12. Approximation of Fractional Capacitors $1/s^{1/n}$ by a Regular Newton Process G. E. CARLSON, MEMBER, IEEE, AND C. A. HALIJAH, MEMBER, IEEE-1964

13. VINAGRE B.M., PODLUBNY I., HERNANDEZ A., FELIU V.: 'Some approximations of fractional order operators used in control theory and applications', Fractional Calculus Appl. Anal., 2000, 3, (3), pp. 231– 248

14. PODLUBNY I., PETRAS I., VINAGRE B.M., O'LEARY P., DORCAK L.: 'Analogue realizations of fractional-order controllers', Nonlinear Dyn., 2002, 29, (1– 4), pp. 281– 296

15. A. Oustaloup, F. Levron, B. Mathieu, and F.M. Nanot, "Frequencyband complex noninteger differentiator: characterization and synthesis," IEEE Trans. Circuits Syst. I, vol. 47, pp. 25-39, 2000.

16.ArijitBiswas, Swagatam Das, Ajith Abraham and SambartaDasgupta, "Design of fractional-order controllers with an improved differential evolution", Engineering Applications of Artificial Intelligence, Volume 22, Issue 2, pp. 343-350, March 2009.

17. I. Podlubny "Fractional-order systems and controllers", IEEE Transactions on Automatic Control, 1999, 44(1):208–214.

18.R. Oldenhuis, *Optimize*, Math Works File Exchange, [online]http://www.mathworks.com/matlabcentral/ fileexchange/24298-optimize, 2009.

19. J. G. Ziegler, N. B. Nichols, Optimum Setting for Automatic Controller, Transactions ASME, vol. 64, pp. 759-768, 1942

20. C.A. Monje, B.M. Vinagre, Y.Q. Chen, V. Feliu, P. Lanusse and J. Sabatier, "Proposals for Fractional PID Tuning", The First IFAC Symposium on Fractional Differentiation and its Applications 2004, Bordeaux, France, July 19-20, 2004.