# ON BRANCIARI METRIC SPACES THE BEST PROXIMITY POINT RESULTS

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**Abstract**

In order to find the presence of the Best Proximity point, we use contractive maps in Branciari metric space. The results of this study will add to our mathematical quest in terms of intellectual pleasure and contribution to the discipline.

**Keywords:** Best Proximity Point, $p$ −property, Branciari metric space.

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## 1. Introduction and Preliminaries

Many generalizations of metric space have been made, as well as a significant contribution to the subject of fixed point theory. Such a BMS is frequently discovered to be topologically equivalent to a MS known as BMS type.
BMS in terms of BM need not be determined directly from FP theorems on specific MS among the aforementioned BMS. We enhance the best proximity point results in BMS in this paper. Branciari [11] introduced a new set of Branciari metric spaces in the year 2000.

**Define: 1.1**
LetS be a set, let$d: S \times S \to [0, \infty)$be a maps.tfor all s,$t \in S$ and for everyuniqueelementsp,$q \in S$ each are distinctfroms, $t$.
(i) $d(s,t) = 0$ iff s= $t$
(ii) $d(s,t) = d(t,s)$
$(iii)$ $d(s,t) \leq d(s,p) + d(p,q) + d(q,t)$
Then (S,d) is called BMS.

**Define: 1.2**
Let (S,d) be a generalised MS of B type, $\{s_n\}$ be a sequence in S, s∈ $S$.
$(i)$ $\{s_n\}$ converges tosiff$d(s_n, s) \to 0$ as $\to \infty$ . We denote this by$s_n \to s$;
$(ii)$ $\{s_n\}$ is a Cauchy iff$d(s_n, s_m) \to 0$ as $n, m \to \infty$;
(iii) If every Cauchy inS converges to an element in S, then (S,d) is complete.

The Banach FP theorem is one of the necessary tools in functional analysis, and it has been extensively discussed by quite many [1-10, 12-16, 18-25]. Khan et al. at 1982 [22], Branciariat 2000 [12], Jelli and Sametat 2014 [18] gotvery interesting generalisations of FPresults.We analyse various best proximity points theorems from the work of Maryam Eshraghisamani, S. MausourVazpour, and Mehdi Asadi in 2017 [26], which was motivated by research in this approach.

**Define: 1.3**[27]$A_0 = \{s \in A: d(s,t) = d(A,B)$, for t∈ $B$
$B_0 = \{t \in B: d(s,t) = d(A,B)$, for s∈ $A$
where$d(A,B) = inf\{d(s,t): s \in A, t \in B$

**Define: 1.4**[28]
Let $(A,B)$ be a pair of nonempty subsets of MS$(S,d)$with$A_0 \neq 0$. Then the pair $(A,B)$ is said to have $p$property iff for any $s, s_2 \in A_0$ and $t_1, t_2 \in B_0$, $d(s_1, t_1) = d(A,B) = d(s, t_2)$

**Best Proximity Point Theorems**
**Result 2.1**
Let (S,d) be a CMS in B's sense and Let (A,B) be two subsets with the property that$A_0$ is nonempty. Let $T: A \to B$ be a mapsatisfying$T(A_0) \subset B_0$. Suppose that $\psi(d(Ts,Tt)) \leq k\psi(d(s,t) - d(A,B))$ where $k \in (0,1)$ and a function $\psi: [0,\infty) \to [0,\infty)$ for every $\{s_n\} \subset (0,\infty)$ $\lim_{n\to\infty} \psi(s_n) = 0$ iff $\lim_{n\to\infty} s_n = 0$.

T has the BPP.
Proof: select$s_0 \in A$.
Since $Ts_0 \in T(A_0) \subseteq B_0$, there exists $s_1 \in A_0$ such that $d(s_1, Ts_0) = d(A,B)$.
Similarly, when it comes to the assumption, $Ts \in T(A_0) \subseteq B_0$, we determine $s \in A_0$ such that $d(s_2, Ts_1) = d(A,B)$.
we get a sequence recursively $(s_n)$ in $A_0$ satisfying
$d(s_{n+1}, Ts_n) = d(A,B)$for all $n \in N$ (1)

**Claim:**$d(s_n, s_{n+1}) \to 0$
If $s = s_{N+1}$, then $s_N$ is a BPP.

By the $p$ property, we get

$$d(s, s_{n+2}) = d(Ts_n, Ts_{n+1})$$

As a result, we assume$s_n \neq s_{n+1}$ for all $n \in N$.
Since $d(s_{n+1}, Ts_n) = d(A,B)$, from (2), we have for all $n \in N$.
$\psi(d(s_{n+1}, s_{n+2})) = k\psi(d(Ts_n, Ts_{n+1}))$

Since$k \in (0,1)$, we have
$\leq k\psi((d(s_n, s_{n+1}) + d(s, Ts_n) + d(s, Ts_{n+1})) - d(A,B))$(2)

$$\leq \psi((d(s_n, s) - d(A,B))$$

Since$\psi(d(s_n, s)) = d(A = B)$, from (3) we get $d(s_n, s_{n+1}) = 0$ acontradiction.

$\{\psi(d(s_{n+1}, s_{n+2}))\}$converges since it is in a decreasing sequence and $\lim_{n\to\infty} \psi(d(s_{n+1}, s_{n+2})) = r > 0$.
We can now demonstrate this $r = 0$ from (3), we get

$$0 < \psi(d(s_{n+1}, s_{n+2})) \leq k\psi(d(s_{n+1}, s_n)) \leq k^n\psi(d(s_1, s)) \quad (3)$$

Since $0 < k < 1$, therefore
$\lim_{n\to\infty} \psi(d(s_{n+1}, s_n)) = 0$. So
$\lim_{n\to\infty} d(s_{n+1}, s_{n+2}) = 0$ (4)
by the condition of $\psi$.
We then demonstrate that $\{s_n\}$ is Cauchy.
If there exits$\varepsilon > 0$, two positive integer sequences can be derived for$(m_k)$ and $(n_k)$ such that for all positive integers
$m_k > n_k > k$, $d(s_{m_k}, s_{n_k}) > \varepsilon$and $d(s_{m_k}, s_{n_{k-1}}) < \varepsilon$.
Now$\varepsilon \leq d(s_{m_k}, s_{n_k}) \leq d(s_{m_k}, s_{n_{k-1}}) + d(s_{n_{k-1}}, s_{n_k})$,
that is $\varepsilon \leq d(s_{m_k}, s_{n_k}) < \varepsilon + d(s_{n_{k-1}}, s_{n_k})$

Taking the limit as $k \to \infty$in the preceding inequity and using (5) we get

$$\lim_{k\to\infty} d(s_{m_k}, s_{n_k}) = \varepsilon \qquad (5)$$

Again $d(s_{m_k}, s_{n_k}) \leq d(s_{m_k}, s_{m_{k+1}}) + d(s_{m_{k+1}}, s_{n_{k+1}}) + d(s_{n_{k+1}}, s_{n_k})$.

Taking the limit as $k \to \infty$in the preceding inequity and using (5) and (6) we get

$$\lim_{k\to\infty} d(s_{m_{k+1}}, s_{n_{k+!}}) = \varepsilon \qquad (6)$$

Again $d(s_{m_k}, s_{n_k}) \leq d(s_{m_k}, s_{n_{k+1}}) + d(s_{n_{k+1}}, s_{n_k})$
$\leq d(s_{m_k}, s_{n_k}) + d(s_{n_k}, s_{n_{k+1}})$

Letting $k \to \infty$in the preceding inequity and using (5) and (6) we get

$$\lim_{k\to\infty} d(s_{m_k}, s_{n_{k+1}}) = \varepsilon \qquad (7)$$

$$\lim_{k\to\infty} d(s_{n_k}, s_{m_{k+1}}) = \varepsilon \qquad (8)$$

For s= $s_{m_k}$, t= $t_{m_k}$ we get $d(s_{m_k}, Ts_{m_k}) - d(A,B) \leq d(s_{m_k}, s_{m_{k+1}}) + d(s_{m_{k+1}}, Ts_{m_{k+1}}) - d(A,B) = d(s_{m_k}, s_{m_{k+1}})$
Similarly, $d(s_{n_k}, Ts_{n_k}) - d(A,B) = d(s_{m_k}, s_{n_{k+1}})$.
Also $d(s_{m_k}, Ts_{n_k}) - d(A,B) = d(s_{m_k}, s_{n_{k+1}})$ and $d(s_{n_k}, Ts_{m_k}) - d(A,B) = d(s_{n_k}, s_{m_{k+1}})$.
From (1) we get$\psi(d(s_{m_{k+1}}, s_{n_{k+1}})) = \psi(d(Ts_{m_k}, Ts_{n_k}))$

$$\leq k\psi((d(s_{m_k}, s_{n_k}) + d(s_{m_k}, Ts_{m_k}) + d(s_{n_k}, Ts_{n_k})) - d(A,B))$$
$$\leq \psi((d(s_{m_k}, s_{n_k}) + d(s_{m_k}, Ts_{m_k}) + d(s_{n_k}, Ts_{n_k})) - d(A,B))$$

It is being followed that

$$\psi(d(Ts_{m_k}, Ts_{n_k}) \leq k\psi((d(s_{m_k}, s_{n_k} + d(s_{n_k}, Ts_{n_{k+1}}) + d(s_{m_k}, Ts_{m_{k+1}})) - d(A,B))$$

From (4), (5), (6) and (7) and letting $k \to \infty$in the preceding inequity and by the conditions of $\psi$ we get$\psi(\varepsilon) \leq \psi(\varepsilon)$a contradiction by the condition of$\psi$.
Hence $\{s_n\}$ is Cauchy.
For$\{s_n\} \subset A$ and $A$ is closed subset of the CMS$(S,d)$, there exists $s*$ in $A$ such that $s_n \to s^*$.
Putting s= $s_n$and t= $t*$ and since

$d(s_n, Ts^*) \leq d(s_n, s^*) + d(s*, Ts_n)$and
$d(s^*, Ts_n) \leq d(s^*, Ts^*)$+$d(Ts*, Ts_n)$
We get$\psi(d(s_{n+1}, Ts^*) - d(A,B) \leq \psi d(Ts_n, Ts^*))$

$$\leq k\psi((d(s_n, s^*) + d(s_n, Ts_n) + d(s^*, Ts^*)) - d(A,B))$$

Taking the limit as $n \to \infty$ in the preceding inequity and by the conditions of $\psi$we get
$\psi(d(s^*, Ts^*) - d(A, \leq k\psi(d(s^*, Ts^*) - d(A,B))$
Implies that $d(s^*, Ts^*) = d(A,B)$
Hence $s^*$ is BPP of $T$.
For the uniqueness
Assume that v and w are the two closest points$p \neq q$,
Then putting s= $v$ and t= $w$ in (1) we obtain

$$\psi(d(T_v, T_w)) \leq k\psi((d(v,w) + d(v, T_v) + d(w, T_w)) - d(A,B)) \qquad (9)$$

that is $\psi(d(v,w)) \leq \psi(d(v,w))$
a contradiction by the property $\psi$.
Therefore v= $w$
The proof is now complete.

**Result: 2.2**
Let$(S,d)$be a CMS in B's notion and let$(A,B)$ be two subsets of MSs.t$A_0$ is nonempty satisfying$T(A_0) \subset B_0$. Suppose$T: A \to B$ be a map, where k∈ (0,1) and a function $\psi: [0,\infty) \to [0,\infty)$ satisfying belowproperties
for every $\{s_n\} \subset (0,\infty)$
$\lim_{n\to\infty} \psi(s_n) = 0$ iff$\lim_{n\to\infty} s_n = 0$.
$\psi$ is continuous such that
$\psi(d(Ts,Tt)) \leq k\psi(d(s,t) - d(A,B))$(10)
T has the BPP.

**Proof:** The proof of 2.1suffices to show that the sequence $\{s_n\}$ is Cauchy.
Assume that $\{s_n\}$does not go through the motions of Cauchy.
Thereexists∊> 0, for
Sub sequences can be derived. $\{s_{m_k}\}$,$\{s_{n_k}\}$where$n_k > m_k > k$, s.t $d(s_{m_k}, s_{n_k}) \geq \varepsilon$
Similar to$m_k$, we can select$n_k$ such a way that, it is smallest integer with $n_k > m_k$ and satisfies$d(s_{m_k}, s_{n_k}) \geq \varepsilon$ then $d(s_{m_k}, s_{n_{k-1}}) < \varepsilon$
The inequality of rectangular is derived, thus get

$$\varepsilon \leq d(s_{m_k}, s_{n_k})$$
$$\leq d(s_{n_k}, s_{n_{k-2}}) + d(s_{n_{k-2}}, s_{n_{k-1}}) + d(s_{n_{k-1}}, s_{m_k})$$
$$\leq d(s_{n_k}, s_{n_{k-2}}) + d(s_{n_{k-2}}, s_{n_{k-1}}) + \varepsilon$$

Thus $\lim_{k\to\infty} d(s_{m_k}, s_{n_k}) = \varepsilon$
Again we get$d(s_{n_k}, s_{m_k}) \leq d(s_{n_k}, s_{n_{k-1}}) + d(s_{n_{k-1}}, s_{m_{k-1}}) + d(s_{m_{k-1}}, s_{m_k})$.
$d(s_{n_{k-1}}, s_{m_{k-1}}) \leq d(s_{n_k}, s_{n_{k-1}}) + d(s_{m_k}, s_{n_k}) + d(s_{m_{k-1}}, s_{m_k})$.

Letting $k \to \infty$,
$\lim_{k\to\infty} d(s_{n_{k-1}}, s_{m_{k-1}}) = \varepsilon$.
Using inequality (10) we have
$\psi(d(s_{m_k}, s_{n_k})) \leq k\psi(\, d(s_{m_{k-1}}, s_{n_{k-1}}) - d(A,B))$
letting$k \to \infty$since $\psi$ is continuous $\psi(\varepsilon) \leq k\psi(\varepsilon) < \psi(\varepsilon)$a contradiction.
Thus $\{s_n\}$ is Cauchy.

**Example: 2.1**

Let $S = \{\frac{1}{n} : n \in N\} \cup \{4,5\}$ and $d: S \times S \to [0, \infty)$ defined as

$$d(s,t) \begin{cases} 0, & s = t \\ 4, & s,t \in \{\frac{1}{n} : n \in N\} \\ \frac{1}{4n}, & s = \frac{1}{n}, t \in \{4,5\} \\ 4, & s = 4, t = 5 \; or s = 5, t = 4 \end{cases}$$

We observe that

$$d(\frac{1}{4}, \frac{1}{5}) > d(\frac{1}{4}, 4) + d(4, \frac{1}{5})$$

Hence $d(s,t)$ is non metric.

We demonstrate that $d(s,t)$ is BM. If s= $t$, obvious.

For s= $\frac{1}{n}$, t= $\frac{1}{m}$ and

$$p, q \neq s, tp = 4 \; and q = 5$$

We get$d(\frac{1}{n}, \frac{1}{m})$≤ $d(\frac{1}{n}, 4) + d(4,5) + d(5, \frac{1}{m})$; $4 \leq \frac{1}{4n} + 4 + \frac{1}{4n}$

If s= 4 and t= $\frac{1}{n}$ (p,q)≠ 4,$\frac{1}{n}$)

Then we get

$d(4, \frac{1}{n}) \leq d(4,5) + d(5, \frac{1}{m}) + d(\frac{1}{m}, \frac{1}{n})$; $\frac{1}{4n} \leq 4 + \frac{1}{4n} + 4$

If s= 4 and t= $5(p, q \neq 4,5)$

Then we get

$$d(4,5) \leq d(4, \frac{1}{n}) + d(\frac{1}{n}, \frac{1}{m}) + d(\frac{1}{m}, 5); 4 \leq \frac{1}{4n} + 4 + \frac{1}{4m}$$

So we conclude that $d(s,t)$ is BM.

On the other hand,

$$\lim_{n\to\infty} d(\frac{1}{n}, 4) = \lim_{n\to\infty} d(\frac{1}{n}, 5) = \lim_{n\to\infty} \frac{1}{4n} = 0$$

Thus the limit is distinct.

Though $\left\{\frac{1}{n}\right\}$converges, not Cauchy.

For$\lim_{k\to\infty} d(\frac{1}{n}, \frac{1}{n+k}) = 4 \neq 0, \forall k.$

Therefore we conclude that this is not Hausdroff.

## 2. Conclusion

In this paper we have discussed on best proximity points BMS and enumerated an example also. In future we have a plan to extend this result for b-metric space.

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