# CAPUTO-FABRIZIO FRACTIONAL ORDER RESPONSE IN THERMOELASTIC THICK CIRCULAR PLATE WITH HEAT SOURCE

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**Abstract**
In this present work, mathematical modeling is done for fractional thermoelastic problem of a thick circular plate occupying the space $D = \{(x, y, z) \in R^3 : 0 \le (x^2 + y^2)^{1/2} \le b, -h \le z \le h\},$ where $r = (x^2 + y^2)^{1/2}$. Plate is subjected to heat source as liner function of temperature and convective heat exchange boundaries applied at lower and upper surfaces. The equation of heat conduction involves Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$. Finite Hankel, Marchi-Fasulo and Laplace transform method is employed to evaluate the analytical solution of the problem. The obtained expression of temperature, displacements and stresses are evaluated numerically and illustrated graphically by considering material properties of aluminum metal.

**Keywords:** Thick plate, Caputo-Fabrizio fractional derivative, Integral Transforms, Temperature, Thermal stresses.

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## 1. Introduction

Mathematical modeling based on fractional theory found more accurate than the traditional order models and useful in modeling of many physical problems. Thermoelastic modeling involving Fractional order derivatives presents memory impact and history as well as describe nonlocal effect. In many fields like mechanics, bioengineering, biology and geophysics etc. applications of fractional order derivatives appeared. From recent years fractional calculus grows very fast because of its application in real-word physical processes and researchers continuously working hard for the development of more suitable fractional operator. Povstenko and Kyrylych [6] discussed symmetric distribution of stresses in fractional framework by considering Caputo derivative in an infinite plane with heat flux. Kumar and Kamdi [7] determined analytical solution of two dimensional thermoelastic problem with convective boundary and fractional derivative using integral transformation and illustrated the numerical results successfully. Sherief and Hussein [8] discussed the effect of fractional thermoelasticity in axisymmetric two-dimensional problem for an infinite space solid sphere in Laplace transform domain. Geetanjali et al. [9] investigate the variable impact of diffusivity on 2 dimensional half space with concentration thermal loading, also they convert nonlinear heat equations to liner one by using Kirchhoff's variable technique.

Bassiouny [10] investigated the thermo-dynamical temperature of a isotropic homogeneous semi-infinite medium in the frame work of fractional generalized thermoelasticity and explained the stress, strain and heat conductivity. Kaur and Singh [11] determined components of displacement, temperature in transform domain by constructing mathematical model with fractional order strain in thick circular plate. Lata [12] investigated problem of two temperature thermoelasticity and investigated thermal interactions in fractional theory for thick isotropic plate by using Hankel transform and direct approach also analytical expressions of stress, temperature distribution are computed. Guangyin et al. [13] discussed the thermal stresses due to action of short laser pulse and investigated its behavior for semi-infinite space with certain boundaries. Thakare and Warbhe [14] solved two dimensional problem of thermoelastic nonhomogeneous thick hollow cylinder by considering fractional order derivative concept and subjected to Convective heating on surfaces by employing Integral transform method. Verma et al. [15] investigated successfully memory impact in hollow cylinder by using Laplace, Hankel and Fourier transform method under coupled and uncoupled theory of heat and moisture. Ezzat et al. [16, 17] discussed the viscoelastic effect of fractional order problem with thermal conductivity in half space subjected to arbitrary heating by Laplace transform technique. Xiong, C., Guo [18] analyzed magneto thermal problem of one-dimensional fractional order rod with the effects of the temperature-dependent properties.

Ahmed et al. [19] investigated the nonlocal parameters effect and magnetic field behavior in thermal isotropic semi-infinite field by using Caputo-Fabrizio differential operator. Abouelregal [20] developed thermoelastic model by considering Caputo-Fabrizio fractional differential in isotropic solid and discussed the effect of laser pulse by using Laplace transform domain. Recently, Sherief and Raslan [21] discussed the wave's propogation behavior by considering new Caputo Fabrizio operator for an infinite elastic space using Integral transformation and represents graphical results.

## 2. Caputo- Fabrizio Definition

Caputo and Fabrizio [4], differential operator of order $\alpha \in (0,1)$ for an absolutely continuous function $f(t)$ is defined as

$$ {}^{CF}_{0}D^{(\alpha)} f(t) = \frac{1}{1-\alpha}\int_0^t f'(t)\exp\left(-\alpha\frac{(t-\tau)}{1-\alpha}\right)d\tau, \qquad 0\le\alpha\le 1 \tag{1}$$

The advantage of above new Caputo-Fabrizio definition is that it contains nonsingular kernel.
Also, the Laplace transform of Caputo-Fabrizio derivative as in equation (1) is defined as

$$ L\left[{}^{CF}_{0}D^{(\alpha)} f(t)\right] = \frac{\partial^\alpha f}{\partial t^\alpha} = \frac{sL[f(t)]-f(0)}{s+\alpha(1-s)} \tag{2}$$

$$ L[f(r,z,t)] = \overline{f}(r,z,s) = \int_0^\infty e^{-st} f(r,z,t)\,dt \tag{3}$$

### 3. Formation of the Problem

We assumed an isotropic, homogeneous thick circular plate (geometry shown in Fig.1) occupying the space $D = \{(x, y, z) \in R^3 : 0 \le (x^2 + y^2)^{1/2} \le b,\ -h \le z \le h\}$, where $r = (x^2 + y^2)^{1/2}$ and subjected to heat source $g(r, z, t, \theta)$ as a linear function of temperature. Convective heat exchange boundaries are applied at lower and upper surface of plate which follows newton's law of cooling. In addition sectional heating $\exp(-\omega t)\delta(r - r_0)$ is also prescribed at the upper surface, whereas outer curved surface is thermally insulated.

here,

$$g(r, z, t, \theta) = \Phi(r, z, t) + \psi(t)\theta(r, z, t) \tag{4}$$

$$T(r, z, t) = \theta(r, z, t) e^{-\int_0^t \psi(\zeta) d\zeta} \tag{5a}$$

$$\chi(r, z, t) = \Phi(r, z, t) e^{-\int_0^t \psi(\zeta) d\zeta} \tag{5b}$$

### 3.1 Governing Equation of Heat Conduction

The heat conduction equation for a thick circular plate within the frame work of fractional order theory of thermoelasticity by utilizing Caputo-Fabrizio new definition in cylindrical coordinate system is

$$\kappa \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) + \frac{\partial^2 \theta}{\partial z^2} \right] + g(r, z, t, \theta) = \frac{\partial^\alpha \theta}{\partial t^\alpha} \quad ; \quad \alpha \in (0,1) \tag{6}$$

Where, $\theta(r, z, t)$ is the temperature function, $\kappa = \lambda / \rho C$, $\lambda$ being the thermal conductivity of the material, $\rho$ denotes the density and $C$ is the calorific capacity.

For the sake of simplicity let,

$$\chi(r, z, t) = \frac{\delta(r - r_0)\delta(z - z_0)}{2\pi r_0} \exp(-\omega t), \quad 0 \le r_0 \le b, \quad -h \le z_0 \le h, \quad \omega > 0 \tag{7}$$

On utilizing equations (4), (5a), (5b) and (7) to the heat conduction equation (6), we get

$$\kappa \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] + \chi(r, z, t) = \frac{\partial T}{\partial t} \tag{8}$$

where, $\kappa$ refers for the thermal diffusivity of the plate material (which is assumed to be constant).

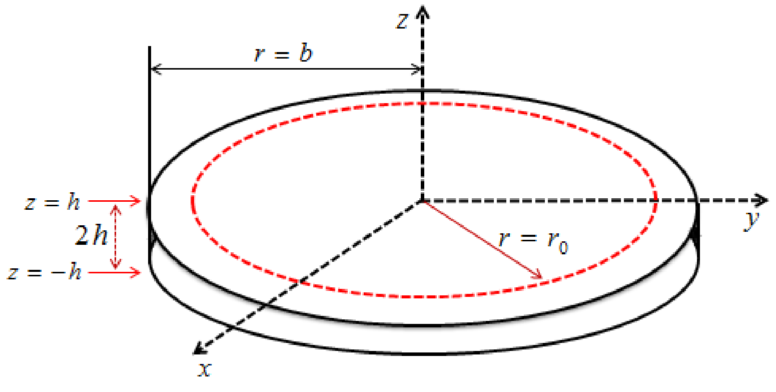


**Fig 1**: Geometry of thick circular plate with heat source

### 3.2 Initial and Boundary Conditions

The following initial and boundary conditions are prescribed on the surfaces of the thick plate

$$[T]_{t=0} = 0 \qquad \text{for all } 0 \le r \le b \,,\ -h \le z \le h \tag{9}$$

$$[T]_{r=b} = 0 \qquad \text{for all } -h \le z \le h \,,\ t > 0 \tag{10}$$

$$\left[T + k_1 \frac{\partial T}{\partial z}\right]_{z=h} = \exp(-\omega t)\delta(r - r_0) \qquad \text{for all } 0 \le r \le b \,,\ t > 0 \tag{11}$$

$$\left[T + k_2 \frac{\partial T}{\partial z}\right]_{z=-h} = 0 \qquad \text{for all } 0 \le r \le b \,,\ t > 0 \tag{12}$$

where, $\delta(r - r_0)$ denotes Dirac Delta function having $0 \le r_0 \le b$ ; $\omega > 0$ is a constants; $\exp(-\omega t)\delta(r - r_0)$ is the additional sectional heat available on its surface at $z = h$.

**3.3 Displacement and Thermal Stresses**

For the axisymmetric problem of two-dimensional thick circular plate the Navier's equations expressed as [5]

$$\nabla^2 u_r - \frac{u_r}{r^2} + \frac{1}{1-2\upsilon}\frac{\partial e}{\partial r} - \frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t \frac{\partial \theta}{\partial r} = 0 \,,$$

$$\nabla^2 u_z - \frac{1}{1-2\upsilon}\frac{\partial e}{\partial z} - \frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t \frac{\partial \theta}{\partial z} = 0 \tag{13}$$

where $u_r$ and $u_z$ are the corresponding displacement components along radial and axial directions respectively and the dilatation $e$ as

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$$

The expression of displacement function in terms of cylindrical coordinates are represented by the Goodier's thermoelastic displacement potential $\phi(r, z, t)$ and Michell's function $M$ as [5]

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \,, \tag{14}$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\upsilon)\nabla^2 M - \frac{\partial^2 M}{\partial^2 z} \tag{15}$$

in which Goodier's thermoelastic potential must satisfy the equation

$$\nabla^2 \phi = \frac{1+\upsilon}{1-\upsilon}\alpha_t \theta \tag{16}$$

and the Michell's function $M$ must satisfy the equation

$$\nabla^2 (\nabla^2 M) = 0 \tag{17}$$

Where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$

The radial, tangential, axial and shear stresses are represented by the use of the potential $\phi$ and Michell's function $M$ as

$$\sigma_{rr} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\upsilon \nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right)\right\}, \tag{18}$$

$$\sigma_{\theta\theta} = 2G\left\{\left(\frac{1}{r}\frac{\partial\phi}{\partial r} - \nabla^2\phi\right) + \frac{\partial}{\partial z}\left(\upsilon\nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right)\right\}, \tag{19}$$

$$\sigma_{zz} = 2G\left\{\left(\frac{\partial^2\phi}{\partial z^2} - \nabla^2\phi\right) + \frac{\partial}{\partial z}\left((2-\upsilon)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right\}, \tag{20}$$

$$\sigma_{rz} = 2G\left\{\frac{\partial^2\phi}{\partial r\partial z} + \frac{\partial}{\partial r}\left((1-\upsilon)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right\} \tag{21}$$

where $G$ and $\upsilon$ are the shear modulus and Poisson's ratio respectively.

The above equations (8) to (21) constitute the mathematical modeling of the problem under consideration.

## 4. Solution of the Problem

### 4.1 Solution of Heat Conduction Equation

Employing finite Hankel transform and Marchi-Fasulo transform stated in [1, 2] and [3] to the equation (8) and using the boundary conditions (10) to (12), one obtains

$$\frac{\partial^\alpha \overline{T}^*}{\partial t^\alpha} + \kappa\Lambda_{m,n}\overline{T}^* = H(\alpha_m, \beta_n)e^{-\omega t} \tag{22}$$

where $\Lambda_{m,n} = a_m^2 + \beta_n^2$

and

$$H(\alpha_m, \beta_n) = \left\{\frac{P_m(h)\kappa r_0}{k_1} + \frac{P_m(z_0)}{2\pi}\right\} k_0(\beta_n, r_0) \tag{23}$$

Next, applying Laplace transform to equation (22) by utilizing initial condition (9), and then on taking their inversion we get

$$\overline{T}^* = \frac{H(\alpha_m, \beta_n)}{(\kappa\Lambda_{m,n} - \omega)}\left\{\frac{l_1[l_2 - \omega]}{[l_3 - \omega l_4]}e^{-\omega t} + \frac{l_1[l_2 l_4 - l_3]}{l_4[\omega l_4 - l_3]}e^{-\frac{l_3}{l_4}t}\right\} \tag{24}$$

where, $(1-\alpha) = l_1$, $\frac{\alpha}{(1-\alpha)} = l_2$, $\kappa\Lambda_{m,n}\alpha = l_3$ and $(1+\kappa\Lambda_{m,n} l_1) = l_4$

Where $\overline{T}^*$ denotes the Marchi-Fasulo transform of $\overline{T}$ ,in which $\overline{T}$ denote the Hankel transform of *T*, m and n are the Marchi-Fasulo transform and Hankel transform parameters respectively.

Further applying inversion of Marchi-Fasulo transform and Hankel transform to the equation (24), one obtains the temperature distribution as follows:

$$T(r,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\wp_{m,n}\left\{\frac{l_1[l_2 - \omega]}{[l_3 - \omega l_4]}e^{-\omega t} + \frac{l_1[l_2 l_4 - l_3]}{l_4[\omega l_4 - l_3]}e^{-\frac{l_3}{l_4}t}\right\}P_m(z)k_0(\beta_n, r) \tag{25}$$

where $\wp_{m,n} = \frac{H(\alpha_m, \beta_n)}{\lambda_m(\kappa\Lambda_{m,n} - \omega)}$,

Taking into account the first equation i.e equation (5a), the temperature distribution is finally represented by

$$\theta(r,z,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\wp_{m,n}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t}+\frac{l_1[l_2 l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}P_m(z)k_0(\beta_n,r)$$
$$\times\exp\left[\int_0^t\psi(\zeta)d\zeta\right] \tag{26}$$

The function given in equation (26) represents the temperature at every instant and at all points of thick circular plate within the frame work of fractional thermoelasticity.

### 4.2 Thermoelastic Solution for Displacement Function

The expression for Goodier's thermoelastic displacement potential $\phi$ calculated by using equation (26) in (16) and represented as

$$\phi(r,z,t)=-\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t}+\frac{l_1[l_2 l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}P_m(z)k_0(\beta_n,r)$$
$$\times\exp\left[\int_0^t\psi(\zeta)d\zeta\right] \tag{27}$$

Also, the Michell's function *M* are assumed so as to satisfy the governed condition of equation (17) as

$$M(r,z,t)=-\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t}+\frac{l_1[l_2 l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}k_0(\beta_n,r)$$

$$\times\exp\left[\int_0^t\psi(\zeta)d\zeta\right][\cosh(\beta_n z)+z\sinh(\beta_n z)] \tag{28}$$

Now substituting displacement potential $\phi$ and Michell's function *M* from equation (27) and (28) in equations (14) and (15), one obtains the expression for displacement functions as

$$u_r=-\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t}+\frac{l_1[l_2 l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}\times\exp\left[\int_0^t\psi(\zeta)d\zeta\right]$$
$$\times\left(\frac{\sqrt{2}\ J_1(\beta_n r)}{b\,J_1(\beta_n b)}\right)\left\{p_m(z)-[(\beta_n+1)\sinh(\beta_n z)+z\,\beta_n\cosh(\beta_n z)]\right\} \tag{29}$$

$$u_z=-\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t}+\frac{l_1[l_2 l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}k_0(\beta_n,r)\exp\left[\int_0^t\psi(\zeta)d\zeta\right]$$
$$\times\{-a_m[Q_m\sin(a_m z)+W_m\cos(a_m z)]+[z-\beta_n-2\upsilon]\beta_n\cosh(\beta_n z)$$
$$+[(2\upsilon-1)\beta_n-2\upsilon](\beta_n z)\sinh(\beta_n z)\} \tag{30}$$

### 4.3 Thermoelastic Solution for Stresses

The expression for thermal stresses can be calculated by utilizing the values of displacement potential $\phi$ from equation (27) and Michell's function *M* from equation (28) in equations (18) to (21), as

$$\sigma_{rr}=2\sqrt{2}G\left(-\frac{1+\upsilon}{1-\upsilon}\right)a_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\frac{1}{b\,J_1(b\beta_n)}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t}+\frac{l_1[l_2 l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}$$

$$\times \exp\left[\int_0^t \psi(\zeta)d\zeta\right] \times \left\{\left[\beta_n\, J_0(\beta_n r)[-(1+2\nu+\beta_n)] + \frac{J_1(\beta_n r)}{r}(\beta_{n+1})\right]\sinh(\beta_n z)\right.$$

$$-\beta_n\left[\beta_n J_0(\beta_n r) - \frac{J_1(\beta_n r)}{r}\right] z\cosh(\beta_n\, z)$$

$$\left.-\left(a_{m^2}\,\frac{J_0(\beta_n r)}{\beta_n} - \frac{J_1(\beta_n r)}{r}\right)p_m(z)\right\} \tag{31}$$

$$\sigma_{\theta\theta} = 2\sqrt{2}G\left(-\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\frac{1}{b\,J_1(b\beta_n)}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t} + \frac{l_1[l_2\,l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}$$

$$\times \exp\left[\int_0^t \psi(\zeta)d\zeta\right] \times \left\{\left[\beta_n(-2\nu)J_0(\beta_n r) - \frac{J_1(\beta_n r)}{r}(\beta_{n+1})\right]\sinh(\beta_n z)\right.$$

$$\left.-\frac{J_1(\beta_n r)}{r}z\cosh(\beta_n z)\;-\left(\left(\beta_n{}^2 + a_m{}^2\right)\frac{J_0(\beta_n r)}{\beta_n} - \frac{J_1(\beta_n r)}{r}\right)p_m(z)\right\} \tag{32}$$

$$\sigma_{zz} = 2\sqrt{2}G\left(-\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\frac{1}{b\,J_1(b\beta_n)}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t} + \frac{l_1[l_2\,l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}$$

$$\times \exp\left[\int_0^t \psi(\zeta)d\zeta\right]\{[J_0(\beta_n r)\beta_n[-[(1-2\nu-\beta_n)]\sinh(\beta_n z)$$

$$+(\beta_n z)\cosh(\beta_n z) - p_m(z)]]\} \tag{33}$$

$$\sigma_{rz} = 2\sqrt{2}G\left(-\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\frac{1}{b\,J_1(b\beta_n)}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t} + \frac{l_1[l_2\,l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}$$

$$\times \exp\left[\int_0^t \psi(\zeta)d\zeta\right]\{-a_m\;[Q_m\sin(a_m z) + W_m\cos(a_m z)]$$

$$-z\,\beta_n^2\sinh(\beta_n z)]\;-\beta_n^2(1-\nu)\cosh(\beta_n z)]\} \tag{34}$$

**5. Special Case**

Set $\psi(\zeta) = \delta(\zeta - \zeta_0), \qquad 0 < \zeta_0 < t$ (35)

$$\Rightarrow \int_0^t \psi(\zeta)d\zeta = 1 \tag{36}$$

Substituting the value of equation (36) into equation (26) and (31) to (34), one obtains the expressions for the temperature and stresses respectively as follows:

$$\theta(r,z,t) = e\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\wp_{m,n}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t} + \frac{l_1[l_2\,l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}P_m(z)\,k_0(\beta_n, r) \tag{37}$$

$$\sigma_{rr} = 2e\sqrt{2}G\left(-\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\frac{1}{b\,J_1(b\beta_n)}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t} + \frac{l_1[l_2\,l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}$$

$$\times\left\{\left[\beta_n J_0(\beta_n r)\left[-(1+2\nu+\beta_n)\right]+\frac{J_1(\beta_n r)}{r}(\beta_{n+1})\right]\sinh(\beta_n z)\right.$$

$$-\beta_n\left[\beta_n J_0(\beta_n r)-\frac{J_1(\beta_n r)}{r}\right]z\cosh(\beta_n z)$$

$$\left.-\left(a_{m^2}\frac{J_0(\beta_n r)}{\beta_n}-\frac{J_1(\beta_n r)}{r}\right)p_m(z)\right\} \tag{38}$$

$$\sigma_{\theta\theta}=2e\sqrt{2}G\left(-\frac{1+\upsilon}{1-\upsilon}\right)a_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\frac{1}{b\,J_1(b\beta_n)}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t}+\frac{l_1[l_2 l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}$$

$$\times\left\{\left[\beta_n(-2\nu)J_0(\beta_n r)-\frac{J_1(\beta_n r)}{r}(\beta_{n+1})\right]\sinh(\beta_n z)\right.$$

$$\left.-\frac{J_1(\beta_n r)}{r}z\cosh(\beta_n z)\;-\left(\left(\beta_n{}^2+a_m{}^2\right)\frac{J_0(\beta_n r)}{\beta_n}-\frac{J_1(\beta_n r)}{r}\right)p_m(z)\right\} \tag{39}$$

$$\sigma_{zz}=2e\sqrt{2}G\left(-\frac{1+\upsilon}{1-\upsilon}\right)a_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\frac{1}{b\,J_1(b\beta_n)}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t}+\frac{l_1[l_2 l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}$$

$$\times\left\{\left[J_0(\beta_n r)\beta_n\left[-\left[(1-2\nu-\beta_n)\right]\sinh(\beta_n z)+(\beta_n z)\cosh(\beta_n z)-p_m(z)\right]\right]\right\} \tag{40}$$

$$\sigma_{rz}=2e\sqrt{2}G\left(-\frac{1+\upsilon}{1-\upsilon}\right)a_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\wp_{m,n}}{\Lambda_{m,n}}\frac{1}{b\,J_1(b\beta_n)}\left\{\frac{l_1[l_2-\omega]}{[l_3-\omega l_4]}e^{-\omega t}+\frac{l_1[l_2 l_4-l_3]}{l_4[\omega l_4-l_3]}e^{-\frac{l_3}{l_4}t}\right\}$$

$$\times\{-a_m\,[Q_m\sin(a_m z)+W_m\cos(a_m z)]$$

$$-z\,\beta_n^2\sinh(\beta_n z)]\;-\beta_n^2(1-\nu)\cosh(\beta_n z)]\} \tag{41}$$

## 6. Numerical Computation

### 6.1 Material Properties

For numerical computations, the material properties of Aluminum metal is assumed as

| | |
|---|---|
| Modulus of Elasticity, $E$ (dynes/cm$^2$) | $6.9 \times 10^{11}$ |
| Shear modulus, $G$ (dynes/cm$^2$) | $2.7 \times 10^{11}$ |
| Poisson ratio, $\upsilon$ | 0.281 |
| Thermal expansion coefficient, $\alpha_t$ (cm/cm-$^0$C) | $25.5 \times 10^{-6}$ |
| Thermal diffusivity, $\kappa$ (cm$^2$/sec) | 0.86 |
| Outer radius, $b$ (cm) | 3 |
| Thickness, $h$ (cm) | 1 |

**Table 1**: Material properties and parameters used in this study

Also, let $k_1 = 0.86$ $k_2 = 1$, $r_0 = 1.5$, $z_0 = 0.5$ and $\omega = 0.5$ in equations (5.3.5) and (5.4.3) to (5.4.7) .

### 6.2 Graphical Representation

Following plotting as shown in **Fig. 2–8** represent the variation $\theta$, $u_r$ , $u_z$ , $\sigma_{rr}$ , $\sigma_{\theta\theta}$ and $\sigma_{rz}$ at t=0.05 for three different values of fractional order parameter $\alpha = \{0.5, 0.95, 1\}$. In these analysis, red line represent $\alpha = 0.5$, dashed brown lines represent $\alpha = 0.95$, while doted blue lines represent $\alpha = 1.0$ (theory of thermoelastic diffusion).

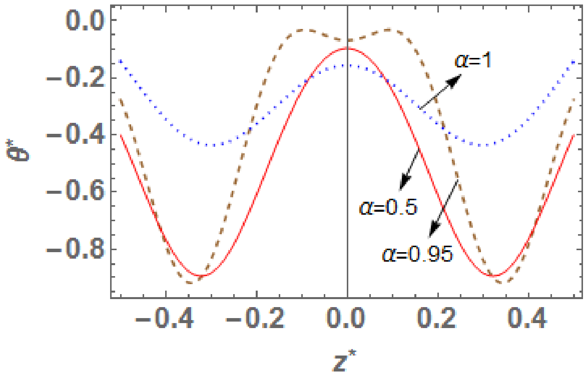


**Fig 2**: Dimensionless temperature distribution along dimensionless thickness for different fractional parameters

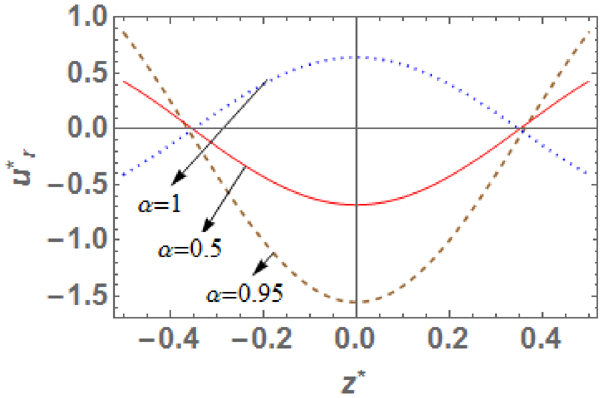


**Fig 3**: Dimensionless radial displacement distribution along dimensionless thickness for different fractional parameters

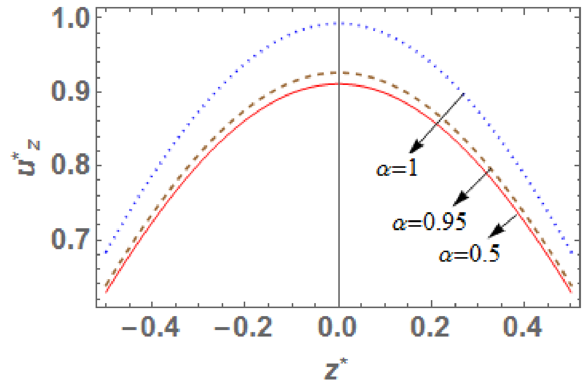


**Fig 4**: Dimensionless axial displacement distribution along dimensionless thickness for different fractional parameters

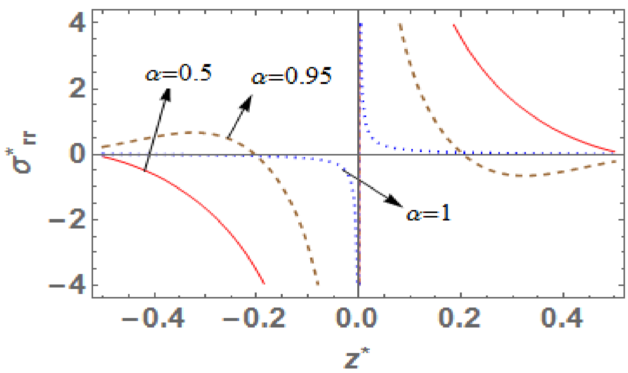


**Fig 5**: Radial stress distribution along dimensionless thickness for different fractional parameters

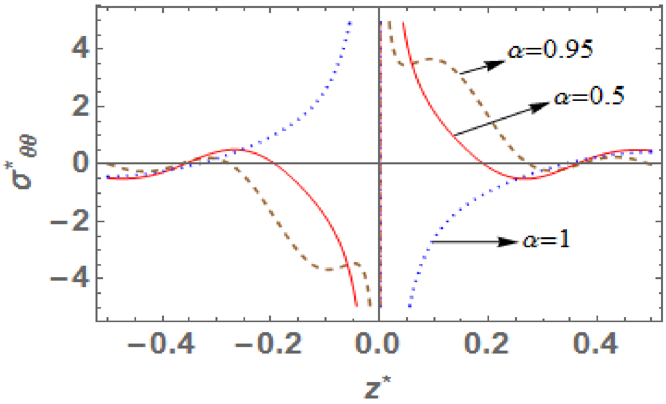


**Fig 6**: Tangential stress distribution along dimensionless thickness for different fractional parameters

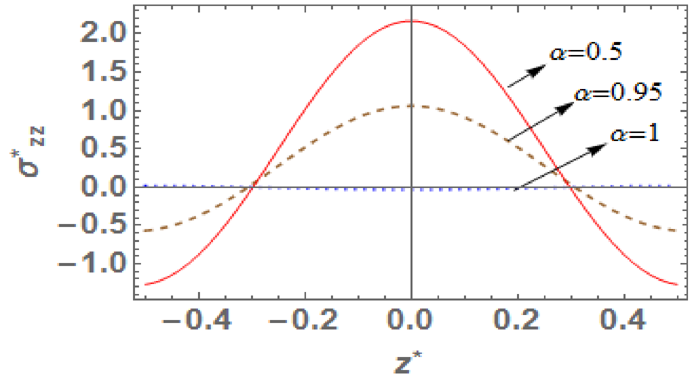


**Fig 7**: Axial stress distribution along dimensionless thickness for different fractional parameters

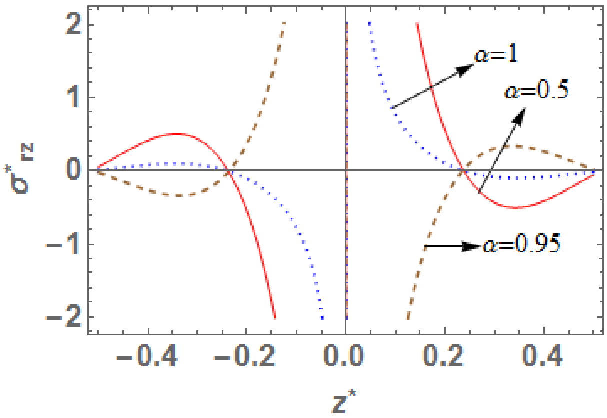


**Fig 8**: Shear stress distribution along dimensionless thickness for different fractional parameters

**Fig**. **2** shows the dimensionless temperature distribution along the dimensionless thickness direction of the thick circular plate. It is observed that due to the thickness of the plate, a steep decrease in temperature was found as expected at the beginning of the transient period for different fractional parameters. At the upper face temperature is non-zero due to the additional sectional heat. Also it is noted that temperature distribution found directly proportional of fractional parameters which plays important role in design of new materials.

**Fig**. **3** and **4** shows the dimensionless radial and axial displacement distribution along dimensionless thickness for different fractional parameters respectively. In both the case effect of dependency of displacement function on different value fractional parameters can be seen easily. Large variation at the mid occurs it may be due to the effect of partial heating.

**Fig 5, 6, 7 and 8** respectively shows dimensionless radial, axial, tangential and shear stress distribution along thickness direction of the thick circular plate for different fractional parameters. It is observed that minimum stress distribution occur at the beginning throughout the thickness direction while location of maximum stress response is noted at the mid due to the influence of partial healing. Axial stresses are compressive in the mid whereas tensile at lower and upper plane surface of circular plate, whereas shear stresses are tensile at lower and upper surface while tensile at the mid. Radial and axial stress distribution are compressive in nature throughout the plate region. In all the above plotting variation of stresses are totally dependent on different values of fractional parameters.

## 7. Convergence of the Series Solution

$$T(r,z,t) = \sum_{n=1}^{\infty} A_n$$

Where,
$$A_n = \sum_{m=1}^{\infty} \wp_{m,n} \left\{ \frac{l_1 [l_2 - \omega]}{[l_3 - \omega l_4]} e^{-\omega t} + \frac{l_1 [l_2 l_4 - l_3]}{l_4 [\omega l_4 - l_3]} e^{-\frac{l_3}{l_4} t} \right\} P_m(z) k_0(\beta_n, r)$$

$T(r,z,t)$ converges everywhere as $\sum_{n=1}^{\infty} A_n$ converges. ie.

$$\text{as } \lim_{n\to\infty} -\left( \frac{\sqrt{2}\ J_0(\beta_{n+1} r)}{b\,\beta_{n+1} J_1(b_n b)} \right) \to 0$$

## Conclusion

The expression for temperature distribution, displacement and stress functions are obtained for a thermoelastic thick circular plate with the impact of Caputo-Fabrizio differential operator and heat source as the linear function of the temperature. Also addition sectional heating response is noted on the upper surface of plate. Method of Integral transformations have been used to obtain the solution of heat equation and all the results are expressed in infinite series in form of Bessel's functions. Numerical computations are done by considering material properties of aluminum metal. Numerical analysis indicates that for $\alpha = 0.5$ the infinite velocity behaviour of waves propogation are observed while for $\alpha = 1$ and $\alpha = 0.95$ (near to 1) the finite behaviour of wave propogation is observed which is as per the observations for generalized thermoelastic diffusion theory. Also it is observed speed of waves depends on the values of fractional parameters. Hence, it can be concluded that thick bodies under the influence of heat source and impact of memory plays an important role in design of new structural materials.

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