PYTHAGOREAN NEUTROSOPHIC REFINED SETS ON DYNAMICAL SYSTEMS

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Abstract

The evolution of functions has been collated in Pythagorean Neutrosophic Refined sets in this article using a dynamical approach, leading rise to Pythagorean Neutrosophic Refined Orbit Topological space. With the help of appropriate cases, several of the fundamental characteristics and theorems were proven.

Keywords: $\mathcal{PNR}$ orbit open set, $\mathcal{PNR}$ orbit closed set, $\mathcal{PNR}$ orbit topology, $\mathcal{PNR}$ orbit closure, $\mathcal{PNR}$ orbit interior.

1.Introduction

Since the concept of the fuzzy set was initially put forward by Zadeh[1], the fuzzy concept has encroached upon practically all fields of Mathematics.The fuzzy set concept is accepted because it better manages uncertainty.As a result, Atanassov[2] developed an extension of fuzzy sets called intuitionistic fuzzy sets that deals with the degree of membership and non-membership.Neutosophic sets, which are a generalization of intuitionistic fuzzy sets, were developed by Smarandache[3-5].

The notions of intuitionistic fuzzy orbit topological space by Priscilla and Irudayam [6] and fuzzy orbit set by Majeed and El-Sheikh [7] were introduced. Madhumathi was the first to present the idea of a neutrosophic orbit [8].With the proof of certain fundamental work, the Pythagorean Neutrosophic Refined Orbit Topological space was also introduced in this study.

2.Preliminaries

DEFINITION: 2.1*[9]*

Let U be a Universe. A Pythagorean Neutrosophic Refined Set can be defined as follows:

P_{PNR}={ < x , $(T_P^1(X), T_P^2(X), T_P^3(X), \dots\dots T_P^K(X))$, (($I_P^1(X), I_P^2(X), I_P^3(X), \dots\dots I_P^k X)$) ,

($F_P^1(X), F_P^2(X), F_P^3(X), \dots\dots F_P^k(X)$)>: x ∈ U}

Where $T_P^1(X), T_P^2(X), T_P^3(X), \dots\dots T_P^k(X)$: U → [0,1] ,

$I_P^1(X), I_P^2(X), I_P^3(X), \dots\dots I_P^k(X)$: U → [0,1] and

$F_P^1(X), F_P^2(X), F_P^3(X), \dots\dots F_P^k(X)$: U → [0,1] such that

And $0 \leq (T_P^k(X))^2 + (I_P^k(X))^2 + (F_P^k(X))^2 \leq 2$

for j = 1,2,3,…p and for any x ∈ U .$T_P^k(X)$is the degree of membership sequence, $I_P^k(X)$is the degree of indeterminacy membership sequence and $F_P^k(X)$is the degree of non-membership sequence.

DEFINITION: 2.2*[9]*

Let X be a non empty set in U,

P_{PNR} = { < x , $(T_P^1(X), T_P^2(X), T_P^3(X), \ldots\ldots T_P^K(X))$, $(I_P^1(X), I_P^2(X), I_P^3(X), \ldots\ldots I_P^k X))$,

($F_P^1(X), F_P^2(X), F_P^3(X), \ldots\ldots F_P^k(X)$)>: x ∈ U}

Q_{PNR} = { < x , $(T_Q^1(X), T_Q^2(X), T_Q^3(X), \ldots\ldots T_Q^K(X))$, $\left(I_Q^1(X), I_Q^2(X), I_Q^3(X), \ldots\ldots I_Q^k X\right))$,

($F_Q^1(X), F_Q^2(X), F_Q^3(X), \ldots\ldots F_Q^k(X)$)>: x ∈ U}

are Pythagorean Neutrosophic Refined sets (PNRS) in U.

The union of P_{PNR} and Q_{PNR} is defined as Follows :

$P_{PNR} \cup Q_{PNR}$ = { < x, s($(T_P^1(X), T_Q^1(X)), (T_P^2(X), T_Q^2(X), \ldots\ldots (T_P^k(X), T_Q^k(X))$),

t($\left(I_P^1(X), I_Q^2(X), (I_P^2(X), I_Q^2(X), \ldots\ldots (I_P^k(X), I_Q^k(X)\right)$,

t(($F_P^1(X), F_Q^1(X)), (F_P^2(X), F_Q^2(X)), \ldots\ldots (F_P^k(X), F_Q^k(X)$)> : x ∈ U}.

The intersection of P_{PNR} and Q_{PNR} is defined as Follows:

$P_{PNR} \cap Q_{PNR}$ = { < x, t($(T_P^1(X), T_Q^1(X)), (T_P^2(X), T_Q^2(X), \ldots\ldots (T_P^k(X), T_Q^k(X))$),

s($\left(I_P^1(X), I_Q^2(X), (I_P^2(X), I_Q^2(X), \ldots\ldots (I_P^k(X), I_Q^k(X)\right)$,

s(($F_P^1(X), F_Q^1(X)), (F_P^2(X), F_Q^2(X)), \ldots\ldots (F_P^k(X), F_Q^k(X)$)> : x ∈ U}.

DEFINITION:2.3*[10]*

A Pythagorean Neutrosophic Refined topology (PNRT) is a non-empty set X is a family τ of a Pythagorean Neutrosophic Refined sets in X satisfying the following conditions

a. (PNRT 1) $0_{PNRS}, I_{PNRS} \in \tau$
b. (PNRT 2)$\cup G_{PNRS_i} \in \tau$ for every $\{G_{PNRS_i}; i \in j\} \subseteq \tau$
c. (PNRT 3) $P_{PNRS_1} \cap P_{PNRS_2} \in \tau$ for any $P_{PNRS_1}, P_{PNRS_2} \in \tau$

In this case (X,τ) is called a Pythagorean Neutrosophic Refined topological space.

Definition:2.4*[10]*

1) PNRcl(A_{PNR}) = ∩ { $A_{PNR} \subseteq P_{PNR}$, where P_{PNR} is a collection of Pythagorean Neutrosophic Refined closed sets in X(PNRCS)}
2) PNRint(A_{PNR}) = ∪ {$Q_{PNRS} \subseteq A_{PNR}$, where Q_{PNR} is a collection of Pythagorean Neutrosophic Refined open sets in X(PNROS)}

Definition:2.5*[11]*

Orbit of a point u of U under the mapping f is $O^{\blacksquare}{}_f(u)$ = {u , f(u), f^2(u),…….}

Definition:2.6*[8]*

Let U be a non empty set and f: U→ U be any mapping . Let a be any neutrosophic set in U. The Nuetrosophic orbit $O^{\blacksquare}{}_{f}(a)$ under the mapping f , $O^{\blacksquare}{}_{fT}(a)$ ={ (a, f(a), $f^2(a)$,…….},$O^{\blacksquare}{}_{fI}(a)$ ={ (a, f(a), $f^2(a)$,…….},$O^{\blacksquare}{}_{fF}(a)$ ={ (a, f(a), $f^2(a)$,…….} for $a \in$ U.

3.Pythagorean Neutrosophic Refined Orbit Set

Definition:3.1

Let $X_{\mathcal{PNR}}$ be a non empty set ,$h_{\mathcal{PNR}}: X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ be any mapping. Let ρbe any Pythagorean Neutrosophic Refined set in X. The Pythagorean Neutrosophic Refined Orbit $O^{\blacksquare}$ of ρ under the mapping $h_{\mathcal{PNR}}$ is defined as $O^{\blacksquare} h_{PNR}$ (ρ) = (ρ, $O^{\blacksquare} {h_{\mathcal{PNR}}}^{T}(\rho), O^{\blacksquare} {h_{\mathcal{PNR}}}^{I}(\rho), O^{\blacksquare} {h_{\mathcal{PNR}}}^{F}(\rho)$) where$O^{\blacksquare} {h_{\mathcal{PNR}}}^{T}(\rho)$ = $\{t(\rho, {h_{\mathcal{PNR}(T)}}^{1}(\rho), {h_{\mathcal{PNR}(T)}}^{2}(\rho), \ldots .. {h_{\mathcal{PNR}(T)}}^{n}(\rho))\}$, $O^{\blacksquare} {h_{\mathcal{PNR}}}^{I}(\rho) =$ $\{s(\rho, {h_{\mathcal{PNR}(I)}}^{1}(\rho), {h_{\mathcal{PNR}(I)}}^{2}(\rho), \ldots .. {h_{\mathcal{PNR}(I)}}^{n}(\rho))\}$,$O^{\blacksquare} {h_{\mathcal{PNR}}}^{F}(\rho)$) ={$s(\rho, {h_{\mathcal{PNR}(F)}}^{1}(\rho), {h_{\mathcal{PNR}(F)}}^{2}(\rho), \ldots ..$ ${h_{\mathcal{PNR}(F)}}^{n}(\rho)$) } for all $\rho \in X_{\mathcal{PNR}}, n \in Z^{+}$, where $(\rho, {h_{\mathcal{PNR}(T)}}^{1}(\rho), {h_{\mathcal{PNR}(T)}}^{2}(\rho), \ldots .. {h_{\mathcal{PNR}(T)}}^{n}(\rho))$, ${h_{\mathcal{PNR}(I)}}^{1}(\rho), {h_{\mathcal{PNR}(I)}}^{2}(\rho), \ldots .. {h_{\mathcal{PNR}(I)}}^{n}(\rho))$, ${h_{\mathcal{PNR}(F)}}^{1}(\rho), {h_{\mathcal{PNR}(F)}}^{2}(\rho), \ldots .. {h_{\mathcal{PNR}(F)}}^{n}(\rho)$ demonstrate the grades of membership, indeterminacy and non membership values under the mapping h_{PNR}.

Definition:3.2

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}})$ be a Pythagorean Neutrosophic Refined Topological space. Let $h_{\mathcal{PNR}}: X \to X$ be any mapping. The Pythagorean Neutrosophic Refined Orbit set under the mapping $h_{\mathcal{PNR}}$ is called Pythagorean Neutrosophic Refined Orbit open set. Its complement is called a Pythagorean Neutrosophic Refined Orbit closed set under the mapping $h_{\mathcal{PNR}}$.

Example:3.3

Let $X_{\mathcal{PNR}}, Y_{\mathcal{PNR}}$ = { a,b,c,d} . Let $\tau_{\mathcal{PNR}}$ = { $0_{\mathcal{PNR}}, 1_{\mathcal{PNR}}, U_{\mathcal{PNR}}, V_{\mathcal{PNR}}$ } where $U_{\mathcal{PNR}}, V_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to [0,1]$ are defined as :

$U_{\mathcal{PNR}}(a)$ = { (0.1,0.1,0.1,0.1),(0.3,0.3,0.3,0.3),(0.9,0.9,0.9,0.9)}

$U_{\mathcal{PNR}}(b)$ = { (0.2,0.2,0.3,0.4),(0.2,0.3,0.4,0.4),(0.8,0.8,0.7,0.6)}

$U_{\mathcal{PNR}}(c)$ = { (0.7,0.8,0.7,0.8),(0.5,0.5,0.5,0.5),(0.3,0.2,0.3,0.2)}

$U_{\mathcal{PNR}}(d)$ = { (0.3,0.2,0.3,0.3),(0.1,0.1,0.1,0.1),(0.7,0.8,0.7,0.7)}

$V_{\mathcal{PNR}}(a)$ = { (0.1,0.1,0.1,0.1),(0.5,0.5,0.5,0.5),(0.9,0.9,0.9,0.9)}

$V_{\mathcal{PNR}}(b)$ = { (0.1,0.1,0.1,0.1),(0.5,0.5,0.5,0.5),(0.9,0.9,0.9,0.9)}

$V_{\mathcal{PNR}}(c)$ = { (0.1,0.1,0.1,0.1),(0.5,0.5,0.5,0.5),(0.9,0.9,0.9,0.9)}

$V_{\mathcal{PNR}}(d)$ = { (0.1,0.1,0.1,0.1),(0.5,0.5,0.5,0.5),(0.9,0.9,0.9,0.9)}

Define $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ as $h_{\mathcal{PNR}}$(a) = a ,$h_{\mathcal{PNR}}$(b) = b, $h_{\mathcal{PNR}}$(c) = c. The Pythagorean Neutrosophic Refined Orbit set under the mapping $h_{\mathcal{PNR}}$ is defined as

$O^{\blacksquare} h_{\mathcal{PNR}}(U_{\mathcal{PNR}})$={$(t(U_{\mathcal{PNR}}, {h_{\mathcal{PNR}(T)}}^{1}(U_{\mathcal{PNR}}), {h_{\mathcal{PNR}(T)}}^{2}(U_{\mathcal{PNR}}), \ldots .. {h_{\mathcal{PNR}(T)}}^{n}(U_{\mathcal{PNR}}))$, $s(U_{\mathcal{PNR}}, {h_{\mathcal{PNR}(I)}}^{1}(U_{\mathcal{PNR}}), {h_{\mathcal{PNR}(I)}}^{2}(U_{\mathcal{PNR}}), \ldots .. {h_{\mathcal{PNR}(I)}}^{n}(U_{\mathcal{PNR}}))$, $s(U_{\mathcal{PNR}}, {h_{\mathcal{PNR}(F)}}^{1}(U_{\mathcal{PNR}}), {h_{\mathcal{PNR}(F)}}^{2}(U_{\mathcal{PNR}}), \ldots$ $.. {h_{\mathcal{PNR}(F)}}^{n}(U_{\mathcal{PNR}})$) } = $V_{\mathcal{PNR}}$.Then $V_{\mathcal{PNR}}$is a Pythagorean Neutrosophic Refined Orbit set under the mapping $h_{\mathcal{PNR}}$.

Remark:3.4

Every Pythagorean Neutrosophic Refined Orbit open set under the mapping $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ is Pythagorean Neutrosophic Refined open set in $X_{\mathcal{PNR}}$. But the converse is not true.Consider the above example, the $U_{\mathcal{PNR}}$ is a Pythagorean Neutrosophic Refined open set but its not a Pythagorean Neutrosophic Refined Orbit open set under the mapping $h_{\mathcal{PNR}}$.

In the following sessions consider $X_{\mathcal{PNR}}$as a countable non empty set and $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ is a mapping to obtain a fixed Pythagorean Neutrosophic Refined Orbit open set ($h_{\mathcal{PNR}}(\rho) = \rho$) for any Pythagorean Neutrosophic Refined Orbit open set ρ.

Theorem:3.5

Let ($X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}}$) be a Pythagorean Neutrosophic Refined Topological space with set and $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ be any bijective mapping. Then $h_{\mathcal{PNR}}(\rho) = \rho$ for any Pythagorean Neutrosophic Refined Orbit open set under the mapping $h_{\mathcal{PNR}}$.

Proof:

Let ($X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}}$) be a Pythagorean Neutrosophic Refined Topological space with set and $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ be any bijective mapping. Then we have 3 cases:

Case 1:

If $h_{\mathcal{PNR}}(u_i) = u_j$;u_i,$u_j \in X_{PNR}$ such that i ≠ j for all i,j∈ N. Suppose $X_{\mathcal{PNR}} = \{ u_1, u_2\}$ and $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ is defined as $h_{\mathcal{PNR}}(u_1) = u_2$, $h_{\mathcal{PNR}}(u_2) = u_1$.

Let ρ be a Pythagorean Neutrosophic Refined Orbit open set under the mapping $h_{\mathcal{PNR}}$.Then there exists a Pythagorean Neutrosophic Refined set $\vartheta \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$such that , $O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta) =$
$\{(t(\vartheta, {h_{\mathcal{PNR}(T)}}^1(\vartheta), {h_{\mathcal{PNR}(T)}}^2(\vartheta),\ldots..{h_{\mathcal{PNR}(T)}}^n(\vartheta)), s(\vartheta, {h_{\mathcal{PNR}(I)}}^1(\vartheta), {h_{\mathcal{PNR}(I)}}^2(\vartheta),\ldots..{h_{\mathcal{PNR}(I)}}^n(\vartheta)),$
$s(\vartheta, {h_{\mathcal{PNR}(F)}}^1(\vartheta), {h_{\mathcal{PNR}(F)}}^2(\vartheta),\ldots..{h_{\mathcal{PNR}(F)}}^n(\vartheta))\} = \rho.$

$\vartheta = \{ (u_1, a_1, b_1, c_1) ,(u_2, a_2, b_2, c_2) \}; u_1, u_2 \in X_{\mathcal{PNR}}, a_1, b_1, c_1, a_2, b_2, c_2 \in Y_{\mathcal{PNR}},$

${h_{\mathcal{PNR}}}^1(\vartheta) =\{ (u_1, a_2, b_2, c_2) , (u_2, a_1, b_1, c_1) \}, {h_{\mathcal{PNR}}}^2(\vartheta) = \{ (u_1, a_1, b_1, c_1) ,(u_2, a_2, b_2, c_2) \}$

$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta) = \{ ((u_1, \inf(a_1, a_2, a_1,\ldots\ldots.), \sup(b_1, b_2, b_1, \ldots \ldots..), \sup(c_1, c_2, c_1, \ldots \ldots \ldots)), ((u_2, \sup(a_1, a_2, a_1,\ldots\ldots.), \inf(b_1, b_2, b_1, \ldots \ldots..), \inf(c_1, c_2, c_1, \ldots \ldots \ldots)) \}$

$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta) = \{ ((u_1, \min(a_1, a_2, a_1,\ldots\ldots.), \max(b_1, b_2, b_1, \ldots \ldots..), \max(c_1, c_2, c_1, \ldots \ldots \ldots)), ((u_2, \max(a_1, a_2, a_1,\ldots\ldots.), \min(b_1, b_2, b_1, \ldots \ldots..), \min(c_1, c_2, c_1, \ldots \ldots \ldots)) \} = \rho.$

Thus for each $u_i \in X_{\mathcal{PNR}}$, we get

$$h_{\mathcal{PNR}}(\rho)(u_j) = \begin{cases} \bigcup_{h_{\mathcal{PNR}}(u_i)=u_j} \rho(u_i) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) \neq \emptyset \\ (0,1,1) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) = \emptyset \end{cases}$$

From the definition and hypothesis of $h_{\mathcal{PNR}}$, we obtain $h_{\mathcal{PNR}}(\rho)(u_j) = \rho(u_j)$. Hence $h_{\mathcal{PNR}}(\rho) = (\rho)$.

Case 2:

If $h_{\mathcal{PNR}}(u_i) = u_j$; $u_i, u_j \in X_{\mathcal{PNR}}$ such that i = j for some i,j∈ N. Suppose $X_{\mathcal{PNR}} = \{u_1, u_2, u_3\}$ and $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ is defined as $h_{\mathcal{PNR}}(u) = u_1$, $h_{\mathcal{PNR}}(u_2) = u_3$, $h_{\mathcal{PNR}}(u_3) = u_2$. (i.e $h_{\mathcal{PNR}}(u_i) = u_j$, i=j=1 and $h_{\mathcal{PNR}}(u_i) = u_j$,i≠j, i,j ∈ {2,3}).

Let ρ be a Pythagorean Neutrosophic Refined Orbit open set under the mapping h_{PNR} .Then there exists a Pythagorean Neutrosophic Refined set $\vartheta \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$ such that ,

$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta)$={($t(\vartheta, {h_{\mathcal{PNR}(T)}}^1(\vartheta), {h_{\mathcal{PNR}(T)}}^2(\vartheta),\ldots..{h_{\mathcal{PNR}(T)}}^n(\vartheta))$, $s(\vartheta, {h_{\mathcal{PNR}(I)}}^1(\vartheta), {h_{\mathcal{PNR}(I)}}^2(\vartheta),\ldots..{h_{\mathcal{PNR}(I)}}^n(\vartheta))$, $s(\vartheta, {h_{\mathcal{PNR}(F)}}^1(\vartheta), {h_{\mathcal{PNR}(F)}}^2(\vartheta),\ldots..{h_{\mathcal{PNR}(F)}}^n(\vartheta))$} = ρ.

.ϑ = { (u_1, a_1, b_1, c_1) ,(u_2, a_2, b_2, c_2), (u_3, a_3, b_3, c_3) }; $u_1, u_2, u_3 \in X_{PNR}$, $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 \in Y_{\mathcal{PNR}}$,

${h_{\mathcal{PNR}}}^1(\vartheta)$ ={ (u_1, a_1, b_1, c_1) , (u_2, a_3, b_3, c_3) , (u_3, a_2, b_2, c_2)},${h_{PNR}}^2(\vartheta)$ = { (u_1, a_1, b_1, c_1) ,(u_2, a_2, b_2, c_2), (u_3, a_3, b_3, c_3) }

$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta)$ = { $((u_1, a_1, b_1, c_1)$, $(u_2$,inf$(a_2, a_3, a_2,\ldots\ldots.)$, sup$(b_2, b_3, b_2, \ldots\ldots..)$, sup$(c_2, c_3, c_2, \ldots\ldots\ldots))$, $((u_3$, inf$(a_3, a_2, a_3,\ldots\ldots.)$, sup$(b_3, b_2, b_3, \ldots\ldots..)$, sup$(c_3, c_2, c_3, \ldots\ldots\ldots))$ }

$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta)$ = { $((u_1, a_1, b_1, c_1)$, $(u_2$,min$(a_2, a_3, a_2,\ldots\ldots.)$, max$(b_2, b_3, b_2, \ldots\ldots..)$, max$(c_2, c_3, c_2, \ldots\ldots\ldots))$, $((u_3$, min$(a_3, a_2, a_3,\ldots\ldots.)$, max$(b_3, b_2, b_3, \ldots\ldots..)$, max$(c_3, c_2, c_3, \ldots\ldots\ldots))$ }= ρ

Thus for each $u_i \in X_{\mathcal{PNR}}$, we get

$$h_{\mathcal{PNR}}(\rho)(u_j) = \begin{cases} \cup_{h_{\mathcal{PNR}}(u_i)=u_j} \rho(u_i) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) \neq \emptyset \\ (0,1,1) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) = \emptyset \end{cases}$$

From the definition and hypothesis of h_{PNR}, we obtain

$$h_{\mathcal{PNR}}(\rho)(u_j) = \begin{cases} (a_i, b_i, c_i) & if\ i = j \\ (\inf(a_i), \sup(b_i), \sup(c_i)) & if\ i \neq j \end{cases}$$

Hence $h_{\mathcal{PNR}}(\rho) = (\rho)$.

Case 3:

If $h_{\mathcal{PNR}}$ is a identity mapping. Then every Pythagorean Neutrosophic Refined Orbit open set under this mapping results that, $h_{\mathcal{PNR}}(\rho) = (\rho)$. For every Pythagorean Neutrosophic Refined set $\rho \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$. Hence the proof.

Theorem :3.6

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}})$ be a Pythagorean Neutrosophic Refined Topological space with set and $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ be any constant mapping. Then $h_{\mathcal{PNR}}(\rho) = \rho$ for any Pythagorean Neutrosophic Refined Orbit open set under the mapping $h_{\mathcal{PNR}}$.

Proof:

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}})$ be a Pythagorean Neutrosophic Refined Topological space and ρ be a Pythagorean Neutrosophic Refined Orbit open set under the mapping $h_{\mathcal{PNR}}$.Then from the definition, there exists a Pythagorean Neutrosophic Refined set ϑ = { (u_k, a_k, b_k, c_k) ; $u_k \in X_{\mathcal{PNR}}$ and $a_k, b_k, c_k \in Y_{\mathcal{PNR}}$ for all k ∈ N, such that $O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta) = \rho$. Since $h_{\mathcal{PNR}}$ is constant mapping , this implies there exists a fixed element $u_k \in X_{\mathcal{PNR}}$ such that $h_{\mathcal{PNR}}(u_i) = (u_k)$, for all $u_i \in X_{\mathcal{PNR}}$ and i∈ N

From the definition 3.1 we get,

$$h_{\mathcal{PNR}}(\vartheta)(u_j) = \begin{cases} \bigcup_{h_{\mathcal{PNR}}(u_i)=u_k} \vartheta(u_j) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) \neq \emptyset \\ (0,1,1) & otherwise \end{cases}$$

Thus

$$h_{\mathcal{PNR}}(\vartheta)(u_j) = \begin{cases} (sup_i(\vartheta(u_i)), inf_i(\vartheta(u_i)), inf_i(\vartheta(u_i))) & if\ u_j = u_k, \\ (0,1,1) & if\ u_j \neq u_k \end{cases}$$

ϑ = { (u_1, a_1, b_1, c_1) ,(u_2, a_2, b_2, c_2),……… (u_k, a_k, b_k, c_k) };

$h_{\mathcal{PNR}}(\vartheta)$={($u_1$,(0,1,1)),($u_{2,}$(0,1,1)),($u_3$,(0,1,1)),…………($u_{k,}$($sup_i\ \vartheta(u_i), inf_i\ \vartheta(u_i), inf_i\ \vartheta(u_i)$))},

${h_{\mathcal{PNR}}}^2(\vartheta)$= {($u_1$,(0,1,1)),($u_{2,}$(0,1,1)),($u_3$,(0,1,1)),…………($u_{k,}$($sup_i\ \vartheta(u_i), inf_i\ \vartheta(u_i), inf_i\ \vartheta(u_i)$))},

${h_{\mathcal{PNR}}}^3(\vartheta)$= {($u_1$,(0,1,1)),($u_{2,}$(0,1,1)),($u_3$,(0,1,1)),…………($u_{k,}$($sup_i\ \vartheta(u_i), inf_i\ \vartheta(u_i), inf_i\ \vartheta(u_i)$))}

thus we generalize by, $O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta)$ = (ϑ, $O^{\blacksquare} {h_{\mathcal{PNR}}}^T(\vartheta), O^{\blacksquare} {h_{\mathcal{PNR}}}^I(\vartheta), O^{\blacksquare} {h_{\mathcal{PNR}}}^F(\vartheta)$) where
$O^{\blacksquare} {h_{\mathcal{PNR}}}^T(\vartheta)$={$t(\vartheta, {h_{\mathcal{PNR}(T)}}^1(\vartheta), {h_{\mathcal{PNR}(T)}}^2(\vartheta)$,…..${h_{\mathcal{PNR}(T)}}^n(\vartheta)$)},$O^{\blacksquare} {h_{\mathcal{PNR}}}^I(\vartheta)$ =
{$s(\vartheta, {h_{\mathcal{PNR}(I)}}^1(\vartheta), {h_{\mathcal{PNR}(I)}}^2(\vartheta)$,…..${h_{\mathcal{PNR}(I)}}^n(\vartheta)$)},$O^{\blacksquare} {h_{\mathcal{PNR}}}^F(\vartheta)$) ={$s(\vartheta, {h_{\mathcal{PNR}(F)}}^1(\vartheta), {h_{\mathcal{PNR}(F)}}^2(v)$,…..
${h_{\mathcal{PNR}(F)}}^n(v)$) }

$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta)$={($u_1$,(0,1,1)),($u_{2,}$(0,1,1)),($u_3$,(0,1,1)),…………($u_{k,}$(min($a_k, sup_i\ \vartheta(u_i)$),max($b_k, inf_i\ \vartheta(u_i)$), max($c_k, inf_i\ \vartheta(u_i)$)},

$$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta) = \begin{cases} (u_k, (0,1,1)) & \text{if } i \neq k \\ (u_{k,}(\min(a_k, sup_i\ \vartheta(u_i)), \max(b_k, inf_i\ \vartheta(u_i)), \max(c_k, inf_i\ \vartheta(u_i))) & \text{if } i = k \end{cases}$$

$= \rho$

From the definition of $h_{\mathcal{PNR}}$, thus we get $h_{\mathcal{PNR}}(\rho) = (\rho)$.

Remark:3.7

The condition $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ is bijective or constant is necessary condition to obtain fixed Pythagorean Neutrosophic Refined Orbit open sets for any Pythagorean Neutrosophic Refined Orbit open set ρ under the mapping $h_{\mathcal{PNR}}$.

Example:3.8

Let $X_{\mathcal{PNR}}$ = { u_1,u_2,u_3, u_4 } and $\mathfrak{G}_{\mathcal{PNR}}$) = {$0_{\mathcal{PNR}}, 1_{\mathcal{PNR}}, \rho$} where $\rho \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$ and defined as,ρ = {(u_1,(0.1,0.3,0.9)),(u_2, (0.1,0.3,0.9)),(u_3,(0.2,0.6,0.8)),(u_4,(0.2,0.6,0.8))}.Let $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ be a mapping defined as $h_{\mathcal{PNR}}(u_1) = u_2$, $h_{\mathcal{PNR}}(u_2) = u_1$,$h_{\mathcal{PNR}}(u_3) = u_4$,$h_{\mathcal{PNR}}(u_4) = u_3$.$h_{\mathcal{PNR}}$ is not a bijective mapping. Let $\vartheta \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$ be difined as, ϑ = {(u_1,(0.5,0.1,0.5)),(u_2, (0.1,0.3,0.9)),(u_3,(0.8,0.2,0.2)),(u_4,(0.2,0.6,0.8))}.

$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta)$ = (ϑ, $O^{\blacksquare} {h_{\mathcal{PNR}}}^T(\vartheta), O^{\blacksquare} {h_{\mathcal{PNR}}}^I(\vartheta), O^{\blacksquare} {h_{\mathcal{PNR}}}^F(\vartheta)$) where
$O^{\blacksquare} {h_{\mathcal{PNR}}}^T(\vartheta)$={$t(\vartheta, {h_{\mathcal{PNR}(T)}}^1(\vartheta), {h_{\mathcal{PNR}(T)}}^2(\vartheta)$,…..${h_{\mathcal{PNR}(T)}}^n(\vartheta)$)},$O^{\blacksquare} {h_{\mathcal{PNR}}}^I(\vartheta)$ =

$\{s(\vartheta, h_{\mathcal{PNR}(I)}{}^{1}(\vartheta), h_{\mathcal{PNR}(I)}{}^{2}(\vartheta),\ldots..h_{\mathcal{PNR}(I)}{}^{n}(\vartheta))\}$,$O^{\blacksquare} h_{\mathcal{PNR}}{}^{F}(\vartheta))$ =$\{s(\vartheta, h_{\mathcal{PNR}(F)}{}^{1}(\vartheta), h_{\mathcal{PNR}(F)}{}^{2}(v),\ldots..$ $h_{\mathcal{PNR}(F)}{}^{n}(v))$ }

$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta)$={($u_1$,t(0.1,0.5,0.1,…),s(0.3,0.1,0.3,…),s(0.9,0.5,0.9,….)),(u_2,t(0.5,0.1,0.5,….),s(0.1,0.3,0.1,...),s(0.5,0.9,0.5,…)),(u_3,t(0.2,0.8,0.2,….)s(0.6,0.2,0.6,…),s(0.8,0.2,0.8,…)),(u_4,t(0.8,0.2,0.8,….),s(0.2,0.6,0.2,….),s(0.2,0.8,0.2,….))

$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta)$ = {(u_1,(0.1,0.3,0.9)),(u_2, (0.1,0.3,0.9)),(u_3,(0.2,0.6,0.8)),(u_4,(0.2,0.6,0.8))} = ρ.

Thus we Proved the Pythagorean Neutrosophic Refined open set ρ is Pythagorean Neutrosophic Refined Orbit open set ρ under the mapping $h_{\mathcal{PNR}}$.But $h_{\mathcal{PNR}}(\rho) \neq (\rho)$.

Result:3.9

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}})$ be a Pythagorean Neutrosophic Refined Topological space with set and $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \rightarrow X_{\mathcal{PNR}}$ be either constant or bijective mapping and ρ is a Pythagorean Neutrosophic Refined Orbit open set under $h_{\mathcal{PNR}}$, then $h_{\mathcal{PNR}}(\rho) = (\rho)$.

Theorem:3.10

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}})$ be a Pythagorean Neutrosophic Refined Topological space with set and $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \rightarrow X_{\mathcal{PNR}}$ be any mapping . If ρ is Pythagorean Neutrosophic Refined Orbit open set then,$O^{\blacksquare} h_{\mathcal{PNR}}(\rho)$ = ρ.

Proof:

From the dentition3.1 and the result 3.9the proof is obvious.

Theorem :3.11

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}})$ be a Pythagorean Neutrosophic Refined Topological space with set and $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \rightarrow X_{\mathcal{PNR}}$ be any mapping. If ρ_1, ρ_2 are Pythagorean Neutrosophic Refined Orbit open sets under the mapping $h_{\mathcal{PNR}}$, then $O^{\blacksquare} h_{\mathcal{PNR}}(\rho_1 \cap \rho_2)$ = $O^{\blacksquare} h_{\mathcal{PNR}}(\rho_1) \cap O^{\blacksquare} h_{\mathcal{PNR}}(\rho_2)$

Proof:

We have 3 cases to prove the theorem,

Case1:

Suppose $h_{\mathcal{PNR}}$ is a bijective mapping and If $h_{\mathcal{PNR}}(u_i) = u_j$; u_i,u_j $\in X_{\mathcal{PNR}}$such that i ≠ j for all i,j∈ N. Let ρ_1 and ρ_2 are Pythagorean Neutrosophic Refined Orbit open sets under the mapping $h_{\mathcal{PNR}}$. Then $\exists \vartheta_1, \vartheta_2 \in Y_{PNR}$ defined as ϑ_1 = {(u_i, a_i, b_i,c_i)}, ϑ_2 = {(u_i, d_i, e_i,f_i)}, where $u_i \in X_{\mathcal{PNR}}$ and a_i, b_i,c_i,d_i, e_i,f_i,∈ $Y_{\mathcal{PNR}}$ such that $O^{\blacksquare} h_{\mathcal{PNR}}(\rho_1) = \rho_1$and $O^{\blacksquare} h_{\mathcal{PNR}}(\rho_2) = \rho_2$

From the theorem3.5 ,case1, we have

$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta_1)$ = { ((u_i, (inf(a_i),sup(b_i),sup(c_i)), i ∈ N}= ρ_1;$O^{\blacksquare} h_{\mathcal{PNR}}(\vartheta_2)$={(($u_i$,(inf($d_i$),sup($e_i$),sup($f_i$)), i ∈ N}= ρ_2

$\rho_1 \cap \rho_2$ = {u_i, min (a_i,d_i), max (b_i, e_i), max (c_i, f_i), i ∈ N }

$\rho_1 \cap \rho_2$={u_i, l,m,n} where l = min (a_i,d_i), m = max (b_i, e_i), n = (c_i, f_i).

Thus we genaralise, for all $u_i \in X_{\mathcal{PNR}}$,

$$h_{\mathcal{PNR}}\,(\rho_1 \cap \rho_2)\,(u_j) = \begin{cases} \bigcup_{h_{\mathcal{PNR}}(u_i)=u_j} \rho_1 \cap \rho_2(u_i) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) \neq \emptyset \\ (0,1,1) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) = \emptyset \end{cases}$$

= (l,m,n)

Hence $h_{PNR}\ (\rho_1 \cap \rho_2) = \rho_1 \cap \rho_2$ which implies ,${h_{\mathcal{PNR}}}^2(\rho_1 \cap \rho_2) = {h_{\mathcal{PNR}}}^3(\rho_1 \cap \rho_2) = \rho_1 \cap \rho_2$.Then from the definition and theorem $O^{\blacksquare}h_{\mathcal{PNR}}(\rho_1 \cap \rho_2) = \rho_1 \cap \rho_2 = O^{\blacksquare}h_{\mathcal{PNR}}(\rho_1) \cap O^{\blacksquare}h_{\mathcal{PNR}}(\rho_2)$

Case 2:

Suppose $h_{\mathcal{PNR}}$ is a bijective mapping and If $h_{\mathcal{PNR}}(u_i) = u_j$; u_i,$u_j \in X_{PNR}$ such that i = j for all i,j∈ N. Let ρ_1 and ρ_2 are Pythagorean Neutrosophic Refined Orbit open sets under the mapping $h_{\mathcal{PNR}}$. Then $\exists \vartheta_1, \vartheta_2 \in Y_{\mathcal{PNR}}$ defined as $\vartheta_1 = \{(u_i, a_i, b_i, c_i)\}$, $\vartheta_2 = \{(u_i, d_i, e_i, f_i)\}$, where $u_i \in X_{\mathcal{PNR}}$ and $a_i, b_i, c_i, d_i, e_i, f_i, \in Y_{\mathcal{PNR}}$ such that $O^{\blacksquare}h_{\mathcal{PNR}}(\rho_1) = \rho_1$and $O^{\blacksquare}h_{\mathcal{PNR}}(\rho_2) = \rho_2$

From the theorem3.5 ,case2, we have

$O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta_1)$ = {(u_i, (inf(a_i),sup(b_i),sup(c_i))), i ∈ N,;(i.e $h_{PNR}\ (u_i) = u_j$, i=j=1 and $h_{\mathcal{PNR}}(u_i) = u_j$,i≠j, i,j ∈ {2,3, ... })}= ρ_1

$O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta_2)$={($u_i$,(inf($d_i$),sup($e_i$),sup($f_i$))), i ∈ N;(i.e $h_{\mathcal{PNR}}(u_i) = u_j$, i=j=1 and $h_{\mathcal{PNR}}(u_i) = u_j$,i≠j, i,j ∈ {2,3, ... })}= ρ_2

Then $\rho_1 \cap \rho_2 = \begin{cases} \left(u_i, \left(\min(a_i, d_i), \max(b_i, e_i), \max(c_i, f_i)\right)\right); & if\ h_{\mathcal{PNR}}(u_i) = u_j, \mathrm{i = j}, \\ (u_i, (\min(a, d), \max(b, e), \max(c, f))); & if\ h_{\mathcal{PNR}}(u_i) = u_j, \mathrm{i \neq j} \end{cases}$

$$h_{\mathcal{PNR}}\,(\rho_1 \cap \rho_2)\,(u_j) = \begin{cases} \bigcup_{h_{\mathcal{PNR}}(u_i)=u_j} \rho_1 \cap \rho_2(u_i) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) \neq \emptyset \\ (0,1,1) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) = \emptyset \end{cases}$$

Hence $h_{\mathcal{PNR}}\ (\rho_1 \cap \rho_2) = \rho_1 \cap \rho_2$, which implies ${h_{\mathcal{PNR}}}^2(\rho_1 \cap \rho_2) = {h_{\mathcal{PNR}}}^3(\rho_1 \cap \rho_2) = \rho_1 \cap \rho_2$.Then from the definition 3.1 and theorem3.5$O^{\blacksquare}h_{\mathcal{PNR}}(\rho_1 \cap \rho_2) = \rho_1 \cap \rho_2 = O^{\blacksquare}h_{\mathcal{PNR}}(\rho_1) \cap O^{\blacksquare}h_{\mathcal{PNR}}(\rho_2)$.

Case 3:

If $h_{\mathcal{PNR}}$ is constant mapping, let ρ_1 and ρ_2 are Pythagorean Neutrosophic Refined Orbit open sets under the mapping $h_{\mathcal{PNR}}$. Then $\exists \vartheta_1, \vartheta_2 \in Y_{\mathcal{PNR}}$ defined as $\vartheta_1 = \{(u_i, a_i, b_i, c_i)\}$, $\vartheta_2 = \{(u_i, d_i, e_i, f_i)\}$, where $u_i \in X_{\mathcal{PNR}}$ and $a_i, b_i, c_i, d_i, e_i, f_i, \in Y_{PNR}$ such that $O^{\blacksquare}h_{\mathcal{PNR}}(\rho_1) = \rho_1$and $O^{\blacksquare}h_{\mathcal{PNR}}(\rho_2) = \rho_2$,

From the theorem 3.5 we have,

$$O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta_1) = \begin{cases} \left(u_i, (0,1,1)\right) & if\ i \neq k \\ (u_{k,}(\min(a_k, sup_i\vartheta_1(u_i)), \max(b_k, inf_i\vartheta_1(u_i)), \max(c_k, inf_i\vartheta_1(u_i))) & \text{if } i = k \end{cases}$$

$= \rho_1$

$$O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta_2) = \begin{cases} \left(u_i, (0,1,1)\right) & if\ i \neq k \\ (u_{k,}(\min(d_k, sup_i\vartheta_2(u_i)), \max(e_k, inf_i\vartheta_2(u_i)), \max(f_k, inf_i\vartheta_2(u_i))) & \text{if } i = k \end{cases}$$

$= \rho_2$

$$\rho_1 \cap \rho_2 = \begin{cases} (u_i,(0,1,1)) & if\ i \neq k \\ (u_k,(\min((a_k, sup_i\vartheta_1(u_i)),(d_k, sup_i\vartheta_2(u_i)), \max((b_k, inf_i\vartheta_1(u_i)),(e_k, inf_i\vartheta_2(u_i)), \\ \max(c_k, inf_i\vartheta_1(u_i))(f_k, inf_i\vartheta_2(u_i))) & \text{if } i = k \end{cases}$$

$h_{\mathcal{PNR}}(\rho_1 \cap \rho_2)=\rho_1 \cap \rho_2$; ${h_{\mathcal{PNR}}}^2(\rho_1 \cap \rho_2)= \rho_1 \cap \rho_2$;${h_{\mathcal{PNR}}}^3(\rho_1 \cap \rho_2)= \rho_1 \cap \rho_2$,…….

From the definition 3.1 and theorem3.5 we get ,$O^{\blacksquare}h_{\mathcal{PNR}}(\rho_1 \cap \rho_2)$ =$\rho_1 \cap \rho_2 = O^{\blacksquare}h_{\mathcal{PNR}}(\rho_1) \cap O^{\blacksquare}h_{\mathcal{PNR}}(\rho_2)$.

Theorem:3.12

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}})$ be a Pythagorean Neutrosophic Refined Topological space with set and $h_{\mathcal{PNR}} : X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ be any mapping . If $\{\rho_\delta\}, \delta \in \Delta_{\mathcal{PNR}}$ is any family of Pythagorean Neutrosophic Refined Orbit open set under $h_{\mathcal{PNR}}$,then,$O^{\blacksquare}h_{\mathcal{PNR}}(\cup_\delta\{\rho_\delta\})$ =$\cup_\delta O^{\blacksquare}h_{\mathcal{PNR}}(\{\rho_\delta\})$.

Proof :

Case 1:

Suppose $h_{\mathcal{PNR}}$ is a bijective mapping and If $h_{\mathcal{PNR}}(u_i) = u_j$; u_i,$u_j \in X_{\mathcal{PNR}}$such that i ≠ j for all i,j∈ N. Let $\{\rho_\delta\}, \delta \in \Delta_{\mathcal{PNR}}$ is any family of Pythagorean Neutrosophic Refined Orbit open sets under the mapping $h_{\mathcal{PNR}}$. Then $\exists\vartheta_\delta \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}, \delta \in \Delta_{\mathcal{PNR}}$defined as $\vartheta_\delta = \{(u_i, a_{i_\delta}, b_{i_\delta}, c_{i_\delta})\}$, where $u_i \in X_{\mathcal{PNR}}$ and $a_{i_\delta}, b_{i_\delta}, c_{i_\delta} \in Y_{\mathcal{PNR}}$ such that $O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta_\delta) = \vartheta_\delta, \delta \in \Delta_{\mathcal{PNR}}$

From theorem 3.5,case 1 we get,

$O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta_\delta)$ = { $((u_i, (l_\delta, m_\delta, n_\delta)), \delta \in \Delta_{\mathcal{PNR}}\}= \rho_\delta$, where $l_\delta =$ (inf$(a_{i_\delta}), m_\delta =$ sup$(b_{i_\delta}), n_\delta =$ sup(c_{i_δ}). Thus $(\cup_\delta\{\rho_\delta\})$ ={ $((u_i$, (sup(a_{i_δ}),inf(b_{i_δ}),inf(c_{i_δ}))), i ∈ $N, \delta \in \Delta_{\mathcal{PNR}}\}= \rho_\delta$,

$$h_{\mathcal{PNR}}\ (\cup_\delta\{\rho_\delta\})\ (u_j) = \begin{cases} \cup_{h_{\mathcal{PNR}}(u_i)=u_j}\ (\cup_{h_{\mathcal{PNR}}(u_i)=u_j}\{\rho_\delta\})\ (u_i) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) \neq \emptyset \\ (0,1,1) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) = \emptyset \end{cases}$$

$$= (l_\delta, m_\delta, n_\delta) = \cup_\delta\{\rho_\delta\}$$

Thus $h_{\mathcal{PNR}}$ $(\cup_\delta\{\rho_\delta\}) = \cup_\delta\{\rho_\delta\}$, which implies ${h_{\mathcal{PNR}}}^2$ $(\cup_\delta\{\rho_\delta\}) = \cup_\delta\{\rho_\delta\}$, ${h_{\mathcal{PNR}}}^3$ $(\cup_\delta\{\rho_\delta\}) = \cup_\delta\{\rho_\delta\}$,,….

From the definition 3.1 and theorem 3.5we get,

$O^{\blacksquare}h_{\mathcal{PNR}}(\cup_\delta\{\rho_\delta\})$ =$\cup_\delta\{\rho_\delta\}$ = $\cup_\delta O^{\blacksquare}h_{\mathcal{PNR}}(\{\rho_\delta\})$

Case 2:

Suppose $h_{\mathcal{PNR}}$ is a bijective mapping and If $h_{\mathcal{PNR}}(u_i) = u_j$; u_i,$u_j \in X_{\mathcal{PNR}}$such that i = j for all i,j∈ N. Let $\{\rho_\delta\}, \delta \in \Delta_{\mathcal{PNR}}$ is any family of Pythagorean Neutrosophic Refined Orbit open sets under the mapping $h_{\mathcal{PNR}}$. Then $\exists\vartheta_\delta \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}, \delta \in \Delta_{\mathcal{PNR}}$ defined as $\vartheta_\delta = \{(u_i, a_{i_\delta}, b_{i_\delta}, c_{i_\delta})\}$, where $u_i \in X_{\mathcal{PNR}}$ and $a_{i_\delta}, b_{i_\delta}, c_{i_\delta} \in Y_{\mathcal{PNR}}$ such that $O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta_\delta) = \vartheta_\delta, \delta \in \Delta_{\mathcal{PNR}}$

From the theorem3.5 ,case2, we have

$O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta_\delta)$ = {$(u_i$, (inf(a_{i_δ}),sup(b_{i_δ}),sup(c_{i_δ}))), i ∈ N,;(i.e $h_{\mathcal{PNR}}(u_i) = u_j$, i=j=1 and $h_{\mathcal{PNR}}(u_i) = u_j$,i≠j, i,j ∈ {2,3,‥})}

$$h_{\mathcal{PNR}}\ (\cup_\delta\{\rho_\delta\})\ (u_j) = \begin{cases} (\cup_{h_{\mathcal{PNR}}(u_i)=u_j}\{\rho_\delta\})(u_i) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) \neq \emptyset \\ (0,1,1) & if\ (h_{\mathcal{PNR}})^{-1}(u_j) = \emptyset \end{cases}$$

Hence $h_{\mathcal{PNR}}$ $(\cup_\delta\{\rho_\delta\})$ = $\cup_\delta\{\rho_\delta\}$, which implies ${h_{\mathcal{PNR}}}^2(\cup_\delta\{\rho_\delta\})$ = ${h_{\mathcal{PNR}}}^3(\cup_\delta\{\rho_\delta\})$ = $\cup_\delta\{\rho_\delta\}$.Then from the definition 3.1 and theorem 3.5 we get, $O^{\blacksquare}h_{\mathcal{PNR}}(\cup_\delta\{\rho_\delta\})$ =$\cup_\delta\{\rho_\delta\}$ = $\cup_\delta\, O^{\blacksquare}h_{\mathcal{PNR}}(\{\rho_\delta\})$

Case 3:

Suppose $h_{\mathcal{PNR}}$ is a constant mapping and If $h_{\mathcal{PNR}}(u_i) = u_j$; $u_i,u_j \in X_{\mathcal{PNR}}$ such that i = j for all i,j$\in N$. Let $\{\rho_\delta\}, \delta \in \Delta_{\mathcal{PNR}}$ is any family of Pythagorean Neutrosophic Refined Orbit open sets under the mapping $h_{\mathcal{PNR}}$. Then $\exists \vartheta_\delta \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}, \delta \in \Delta_{\mathcal{PNR}}$ defined as $\vartheta_\delta = \{(u_i, a_{i_\delta}, b_{i_\delta}, c_{i_\delta})\}$, where $u_i \in X_{\mathcal{PNR}}$ and $a_{i_\delta}, b_{i_\delta}, c_{i_\delta} \in Y_{\mathcal{PNR}}$ such that $O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta_\delta) = \vartheta_\delta$, $\delta \in \Delta_{\mathcal{PNR}}$

From the theorem (2) we have,

$$O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta_\delta) = \begin{cases} (u_i, (0,1,1)) & if\ i \neq k \\ (u_{k,}(\min(a_{i_\delta}, sup_i\vartheta_\delta(u_i)), \max(b_{i_\delta}, inf_i\vartheta_\delta(u_i)), \max(c_{i_\delta}, inf_i\vartheta_\delta(u_i))) & \text{if } i = k \end{cases}$$

$$\text{Thus,} \cup_\delta\{\rho_\delta\} = \begin{cases} (u_i, (0,1,1)) & if\ i \neq k \\ (u_{k,}(\min(\min(a_{i_\delta}, sup_i\vartheta_\delta(u_i)), \max(\max(b_{i_\delta}, inf_i\vartheta_\delta(u_i)), & \\ \max(\max(c_{i_\delta}, inf_i\vartheta_\delta(u_i)))) & \text{if } i = k \end{cases}$$

Clearly, $\cup_\delta\{\rho_\delta\}$ is a point in $X_{\mathcal{PNR}}$.Hence$h_{\mathcal{PNR}}$ $(\cup_\delta\{\rho_\delta\})$ = $\cup_\delta\{\rho_\delta\}$, which implies ${h_{\mathcal{PNR}}}^2(\cup_\delta\{\rho_\delta\})$ = ${h_{\mathcal{PNR}}}^3(\cup_\delta\{\rho_\delta\})$ = $\cup_\delta\{\rho_\delta\}$.Then from the definition and theorem we get, $O^{\blacksquare}h_{\mathcal{PNR}}(\cup_\delta\{\rho_\delta\})$ =$\cup_\delta\{\rho_\delta\}$ = $\cup_\delta\, O^{\blacksquare}h_{\mathcal{PNR}}(\{\rho_\delta\})$.

4.Pythagorean Neutrosophic Refined Orbit Topological space

Theorem:4.1

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}})$ be a Pythagorean Neutrosophic Refined Topological space. Let $h_{\mathcal{PNR}}: X_{\mathcal{PNR}} \to X_{\mathcal{PNR}}$ be a mapping. Let $\mathfrak{G}_{\mathcal{PNR}_O}$ denote the family of all $\mathcal{PNR}$ Orbit open sets under the mapping h_{PNR}.Then $\mathfrak{G}_{\mathcal{PNR}_O}$ is a $\mathcal{PNR}$ Topology on $X_{\mathcal{PNR}}$ coaser than $\mathfrak{G}_{\mathcal{PNR}}$.

Proof:

i)We know that $0_{\mathcal{PNR}}$ and $1_{\mathcal{PNR}}$ are $\mathcal{PNR}$ Orbit open sets under the mapping h_{PNR} because $\exists \rho = 0_{\mathcal{PNR}}$ and $\vartheta = 1_{\mathcal{PNR}}$ such that $O^{\blacksquare}h_{\mathcal{PNR}}(\rho) = 0_{\mathcal{PNR}} \in \mathfrak{G}_{\mathcal{PNR}}$ and $O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta) = 1_{\mathcal{PNR}} \in \mathfrak{G}_{\mathcal{PNR}}$. Thus $0_{\mathcal{PNR}} \in \mathfrak{G}_{\mathcal{PNR}_O}$ and $1_{\mathcal{PNR}} \in \mathfrak{G}_{\mathcal{PNR}_O}$

ii)Letu_1, u_2 are Pythagorean Neutrosophic Refined Orbit open sets under the mapping h_{PNR} , To Prove $u_1 \cap u_2$ is also a Pythagorean Neutrosophic Refined Orbit open set under the mapping h_{PNR} , have to find a $\mathcal{PNR}\ set\ \vartheta \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$ such that , $O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta) = u_1 \cap u_2 \in \mathfrak{G}_{\mathcal{PNR}}$.

By choosing $\vartheta = u_1 \cap u_2$ from the theorem and also from the preposition we get,$O^{\blacksquare}h_{\mathcal{PNR}}(\vartheta)$ = $O^{\blacksquare}h_{\mathcal{PNR}}(u_1 \cap u_2) = O^{\blacksquare}h_{\mathcal{PNR}}(u_1) \cap O^{\blacksquare}h_{\mathcal{PNR}}(u_2) = u_1 \cap u_2$. Since every $\mathcal{PNR}$ Orbit open set is $\mathcal{PNR}$ open set in $X_{\mathcal{PNR}}$, Thus $u_1 \cap u_2 \in \mathfrak{G}_{\mathcal{PNR}}$. Then $u_1 \cap u_2 \in \mathfrak{G}_{\mathcal{PNR}_O}$.

iii) Let $\{\rho_\delta\}, \delta \in \Delta_{\mathcal{PNR}}$ is any family of Pythagorean Neutrosophic Refined Orbit open set under $h_{\mathcal{PNR}}$, Let $\vartheta = (\cup_\delta\{\rho_\delta\})$, then from the theorem $O^{\blacksquare}h_{PNR}(\vartheta) = O^{\blacksquare}h_{PNR}(\cup_\delta\{\rho_\delta\}) = \cup_\delta\, O^{\blacksquare}h_{PNR}(\{\rho_\delta\}) = (\cup_\delta\{\rho_\delta\}) \in \mathfrak{G}_{\mathcal{PNR}}$. Thus $\{\rho_\delta\}, \delta \in \Delta_{\mathcal{PNR}} \in \mathfrak{G}_{\mathcal{PNR}_O}$

Thus we proved $\mathfrak{G}_{\mathcal{PNR}_O}$ is a $\mathcal{PNR}$ Topology on $X_{\mathcal{PNR}}$ coaser than $\mathfrak{G}_{\mathcal{PNR}}$

Definition:4.2

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}})$ be a Pythagorean Neutrosophic Refined Topological space. Let $h_{\mathcal{PNR}}: X_{\mathcal{PNR}} \rightarrow X_{\mathcal{PNR}}$ be a mapping . Then $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}_0})$ is called Pythagorean Neutrosophic Refined Orbit Topological space associated with $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}})$ when it satisfies the following axioms;

1) $0_{\mathcal{PNR}} \in \mathfrak{G}_{\mathcal{PNR}_O}$ and $1_{\mathcal{PNR}} \in \mathfrak{G}_{\mathcal{PNR}_O}$
2) $Q_1 \cap Q_2 \in \mathfrak{G}_{\mathcal{PNR}_O}$, for any $Q_1, Q_2 \in \mathfrak{G}_{\mathcal{PNR}_O}$
3) $\cup\{Q_\delta\} \in \mathfrak{G}_{\mathcal{PNR}_O}$, where $\{Q_\delta\}$, $\delta \in \Delta_{\mathcal{PNR}}$be any arbitrary family $\mathcal{PNR}$ orbit open sets

Example:4.3

1) Let $X_{\mathcal{PNR}}$be any non empty countable set , then $\mathfrak{G}_{\mathcal{PNR}_O}$ = ($0_{\mathcal{PNR}}$, $1_{\mathcal{PNR}}$) is a $\mathcal{PNR}$ orbit topology on $X_{\mathcal{PNR}}$.
2) Let $X_{\mathcal{PNR}}$be any non empty countable set , if $h_{\mathcal{PNR}}: X_{\mathcal{PNR}} \rightarrow X_{\mathcal{PNR}}$ is the identity mapping, then $\mathfrak{G}_{\mathcal{PNR}_O} = \mathfrak{G}_{\mathcal{PNR}}$

Definition:4.4

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}_0})$ is called Pythagorean Neutrosophic Refined Orbit Topological space and $\vartheta \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$. Then $\mathcal{PNR}$ orbit closure of ϑ,$\mathcal{C}l^*(\vartheta)$ is the intersection of all $\mathcal{PNR}$ orbit closed supersets under the mapping $h_{\mathcal{PNR}}$, $\mathcal{C}l^*(\vartheta) = \cap\{\sigma \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}} / \sigma \supseteq \vartheta, 1_{\mathcal{PNR}} - \sigma \in \mathfrak{G}_{\mathcal{PNR}_O}\}$.$\mathcal{C}l^*(\vartheta)$ is the smallest $\mathcal{PNR}$ orbit closed set which contains ϑ under the mapping $h_{\mathcal{PNR}}$.

Definition:4.5

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}_0})$ is called Pythagorean Neutrosophic Refined Orbit Topological space and $\vartheta \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$. Then $\mathcal{PNR}$ orbit interior of ϑ,$\mathcal{I}nt^*(\vartheta)$ is the union of all $\mathcal{PNR}$ orbit open subsets under the mapping $h_{\mathcal{PNR}}$, $\mathcal{I}nt^*(\vartheta) = \cup\{\sigma \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}} / \sigma \subseteq \vartheta, \sigma \in \mathfrak{G}_{\mathcal{PNR}_O}\}$.$\mathcal{I}nt^*(\vartheta)$ is the largest $\mathcal{PNR}$ orbit open set which contained in ϑ under the mapping $h_{\mathcal{PNR}}$.

Theorem:4.6

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}_0})$ is called Pythagorean Neutrosophic Refined Orbit Topological space and $\vartheta, \sigma \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$.Then $\mathcal{I}nt^*(\vartheta) \subseteq int_{PNR}(\vartheta) \subseteq \vartheta \subseteq Cl_{PNR}(\vartheta) \subseteq \mathcal{C}l^*(\vartheta)$

Proof:

It is obvious, because every $\mathcal{PNR}$ orbit closed set is $\mathcal{PNR}$ closed under the mapping $h_{\mathcal{PNR}}$, similarly every $\mathcal{PNR}$ orbit open set is $\mathcal{PNR}$ open under the mapping $h_{\mathcal{PNR}}$.

Theorem:4.7

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}_0})$ is called Pythagorean Neutrosophic Refined Orbit Topological space and $\vartheta, \sigma \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$.Then

1) $\mathcal{I}nt^*(0_{\mathcal{PNR}}) = 0_{\mathcal{PNR}}$ and $\mathcal{I}nt^*(1_{\mathcal{PNR}}) = 1_{\mathcal{PNR}}$
2) $\mathcal{I}nt^*(\vartheta) \subseteq \vartheta$
3) $\mathcal{I}nt^*(\vartheta \cup \sigma) = \mathcal{I}nt^*(\vartheta) \cup \mathcal{I}nt^*(\sigma)$
4) If $\vartheta \subseteq \sigma$ then $\mathcal{I}nt^*(\vartheta) \subseteq \mathcal{I}nt^*(\sigma)$
5) $\mathcal{I}nt^*(\mathcal{I}nt^*(\vartheta)) = \mathcal{I}nt^*(\vartheta)$
6) If ϑ is a $\mathcal{PNR}$ orbit open set if and only if $\vartheta = \mathcal{I}nt^*(\vartheta)$ under the mapping $h_{\mathcal{PNR}}$

Theorem:4.8

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}_0})$ is called Pythagorean Neutrosophic Refined Orbit Topological space and $\vartheta, \sigma \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$.Then

1) $\mathcal{C}l^*(0_{\mathcal{PNR}}) = 0_{\mathcal{PNR}}$ and $\mathcal{C}l^*(1_{\mathcal{PNR}}) = 1_{\mathcal{PNR}}$
2) $\vartheta \subseteq \mathcal{C}l^*(\vartheta)$
3) $\mathcal{C}l^*(\vartheta \cup \sigma) = \mathcal{C}l^*(\vartheta) \cup \mathcal{C}l^*(\sigma)$
4) If $\vartheta \subseteq \sigma$ then $\mathcal{C}l^*(\vartheta) \subseteq \mathcal{C}l^*(\sigma)$
5) $\mathcal{C}l^*(\mathcal{C}l^*(\vartheta)) = \mathcal{C}l^*(\vartheta)$
6) If ϑ is a $\mathcal{PNR}$ orbit closed set if and only if $\vartheta = \mathcal{C}l^*(\vartheta)$ under the mapping $h_{\mathcal{PNR}}$

Theorem:4.9

Let $(X_{\mathcal{PNR}}, \mathfrak{G}_{\mathcal{PNR}_0})$ is called Pythagorean Neutrosophic Refined Orbit Topological space and $\vartheta \in {Y_{\mathcal{PNR}}}^{X_{\mathcal{PNR}}}$.Then ,

1) $1_{\mathcal{PNR}} - \mathcal{I}nt^*(\vartheta) = \mathcal{C}l^*(1_{\mathcal{PNR}} - \vartheta)$
2) $1_{\mathcal{PNR}} - \mathcal{C}l^*(\vartheta) = \mathcal{I}nt^*(1_{\mathcal{PNR}} - \vartheta)$

Proof:

From Proposition(2) , we know that $\mathcal{I}nt^*(\vartheta) \subseteq \vartheta$, taking complement on both sides,$1_{\mathcal{PNR}} - \vartheta \subseteq 1_{\mathcal{PNR}} - \mathcal{I}nt^*(\vartheta)$. Thus $1_{\mathcal{PNR}} - \mathcal{I}nt^*(\vartheta)$ is $\mathcal{PNR}$ orbit closed set and by Proposition (4)$\mathcal{C}l^*(1_{\mathcal{PNR}} - \vartheta) \subseteq \mathcal{C}l^*(1_{\mathcal{PNR}} - \mathcal{I}nt^*(\vartheta)) = 1_{\mathcal{PNR}} - \mathcal{I}nt^*(\vartheta)$.Thus we proved $1_{\mathcal{PNR}} - \mathcal{I}nt^*(\vartheta) = \mathcal{C}l^*(1_{\mathcal{PNR}} - \vartheta)$.Conversely , by proposition(2) we have $(1_{\mathcal{PNR}} - \vartheta) \subseteq \mathcal{C}l^*(1_{\mathcal{PNR}} - \vartheta)$, taking complement $1_{\mathcal{PNR}} - \mathcal{C}l^*(1_{\mathcal{PNR}} - \vartheta) \subseteq \vartheta$ .Thus $\mathcal{C}l^*(1_{\mathcal{PNR}} - \vartheta)$ is $\mathcal{PNR}$ orbit closed set. Then $1_{\mathcal{PNR}} - \mathcal{C}l^*(1_{\mathcal{PNR}} - \vartheta)$ is $\mathcal{PNR}$ orbit open set. From proposition(6), we get $1_{\mathcal{PNR}} - \mathcal{C}l^*(1_{\mathcal{PNR}} - \vartheta) \subseteq \mathcal{I}nt^*(\vartheta)$

Conclusion:

This led to the development of Pythagorean Neutrosophic Refined Orbit Topological Space and the evaluation of certain fundamental theorems and properties. Additionally, the prerequisites for determining the orbit of the PNR sets have been established.

Abbreviations: PNR – Pythagorean Neutrosophic Refined

PNRT - Pythagorean Neutrosophic Refined Topology

Int – interior

Cl - Closure

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