A study of a two predator-one prey model considering the effects of chemical signals of predator and prey refuge

Dipam Das[1], Jyotishmita Kalita[2], Saponjeet Borah[3]

[1,2]Department Of Mathematics, Gauhati University, Assam, India
[3]Department Of Physics, Gauhati University, Assam, India

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Abstract

One of the most common factors influencing the stability of an ecosystem is how prey react to predator's chemical signals. In this paper, We investigate the dynamics of a two predator-one prey biosystem in which the two predators compete for food on both an intraspecific and interspecific scale. We further hypothesise that prey species growth and behaviour are influenced by predator's chemical signals. We also assume that prey utilise refuge to reduce predation risk. A comprehensive analysis was carried out to determine the ecologically sustainable equilibrium points of the biosystem. In addition, local stability analysis of the equilibrium points is carried out. Furthermore, bifurcation analysis is also done. Finally, to corroborate the results of the analytical investigation, we conducted comprehensive numerical simulations of the model.

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1. Introduction

One of the most commonly used models in mathematical ecology is the prey predator model, which studies the connection between prey and predator species. In recent decades, numerous researchers have explored predator-prey dynamics, drawing inspiration from the pioneering work of Lotka [1] and Volterra [2]. Mathematical models have been utilised to facilitate the comprehension of the intricate dynamics of these complex systems.

Functional responses constitute a crucial aspect of predator-prey modelling. The concept of the predator's response function, also known as the functional response, is employed within the realm of population dynamics to elucidate the rate at which a predator engages in the consumption of its prey. In theoretical ecology, a range of established response functions are commonly employed, including but not limited to the Holling type-I, II, III, IV, Hassel-Verley type response function, Beddington-DeAngelis-type response function, Ratio dependent type functional response, Crowley-Martin type functional response, etc. [3–10].

Competition is a prevalent phenomenon that emerges within an ecosystem comprising multiple predator species. In nature, two distinct forms of competition are commonly observed: interspecific competition, which takes place among predators of different species, and intraspecific competition, which occurs among predators of the same species. A number of researchers [11–14] have conducted investigations on diverse predator-prey models, taking into account interspecific competition, intraspecific competition, or both.

Predatory species possess the capability to exert a direct influence on the dynamics of prey populations through predation. Additionally, they can also induce alterations in other crucial parameters, including growth and reproduction rates of the prey species indirectly [15] . For example, it has been observed that the growth rates of clams (Bivalvia) decreases in the presence of whelks (Buccinidae), even in the absence of direct physical interaction. This implies that the decline in growth rates can be attributed to a behavioural response of the clams to indirect cues, specifically water-borne chemical signals emitted by the whelks [16]. Asian nest mussel (Arcuatula senhousia) exhibits behavioural modifications in reaction to chemical stimuli emitted by predators [17] which has a negative impact on its growth. Prey refuge is one of the most prevalent defensive strategies employed by prey against predators upon detecting chemical cues from the predators. The concept of prey refuge in nature pertains to the diverse array of shelters or concealment sites that prey species employ in order to seek protection and evade their predators. The utilisation of this particular strategy is of utmost importance for the survival of various prey species within diverse ecosystems. For instance, the refuge utilisation of Hyla tadpoles exhibits a notable increase upon exposure to water that had been conditioned with predatory fish [18]. The adoption of predator avoidance strategies by prey species, such as seeking refuge, is frequently associated with significant costs that can lead to reduced growth of the prey species [19]. Numerous scholars [20–23] have conducted extensive investigations on diverse ecological models that incorporate the phenomenon of prey refuge. However, as far as the authors are aware, no one has studied prey refuge and its negative effects on prey growth rate in conjunction with intraspecific and interspecific competitions among predators.

The present study examines a model comprising two predators and one prey. It is also postulated that there exists competition among the predators, both at the interspecific and intraspecific levels. Building upon the preceding discourse, we propose that the chemical cues emitted by the first predator exert an adverse influence on the growth of the prey species. The utilisation of refuge as a defensive strategy against predation is also considered. After contemplating all of these factors, we devise the model in section 2. The existence as well as local stability of the equilibrium points are discussed in sections 3 and 4 respectively. Then bifurcation analysis is done in section 5. The analytical findings in section 6 are verified through the utilisation of numerical simulations. Subsequently, the conclusions are presented in section 7.

2. Formulation of mathematical model

Let us consider, $s(t)$ denotes prey density, $p_1(t)$ denotes first predator density and $p_2(t)$ denotes second predator density. We consider logistic growth in prey. The linear functional response is being taken into consideration for both predator species. It is postulated that the two predator species engage in interspecific competition because of their shared prey. Furthermore, intraspecific competition among predator species is taken into consideration. Upon careful consideration of all relevant factors, our model has been refined

$$\begin{aligned}\frac{ds}{dt} &= r_1 s - r_1 s^2 - r_2 s p_1 - r_3 s p_2 \\ \frac{dp_1}{dt} &= r_4 r_2 s p_1 - k_1 p_1 p_2 - h_1 p_1^2 - d_1 p_1 \\ \frac{dp_2}{dt} &= r_5 r_3 s p_2 - k_2 p_2 p_2 - h_2 p_2^2 - d_2 p_2\end{aligned} \tag{1}$$

In this model (1), r_1 represents the intrinsic growth rate, r_2 and r_3 denote the coefficients of the intake rates of the first and second predators, respectively, d_2 and d_3 are the death rates of the first and second predators, respectively, k_1 and k_2 are the coefficients of interspecific competition, and h_1 and h_2 are the coefficients of intraspecific competition among first and second predators respectively.

By taking into account the detrimental effects of chemical cues emitted by the first predator on the growth of the prey species, as well as the prey refuge behaviour and deriving inspiration from the research conducted by Xu et al. [24], the model (1) becomes

$$\begin{aligned}\frac{ds}{dt} &= r_1 s(1 - m p_1) - r_1 s^2 - r_2 s(1 - m_1) p_1 - r_3 s(1 - m_1) p_2 \\ \frac{dp_1}{dt} &= r_4 r_2 s(1 - m_1) p_1 - k_1 p_1 p_2 - h_1 p_1^2 - d_1 p_1 \\ \frac{dp_2}{dt} &= r_5 r_3 s(1 - m_2) p_2 - k_2 p_2 p_2 - h_2 p_2^2 - d_2 p_2\end{aligned} \tag{2}$$

with initial conditions: $s(0) > 0$, $p_1(0) > 0$, $p_2(0) > 0$

Here, $0 < m_1 < 1$ denotes the degree or strength of prey refuge with respect to the first predator and $0 < m_2 < 1$ represents degree or strength of prey refuge with respect to the second predator. m denotes the rate of disturbance produced by an individual first predator through their chemical cues and $0 < m < 1$.

3. Existence of equilibrium points

In the bio-system (2), there exist five types of equilibrium points which are biologically feasible.

Trivial equilibrium point, $E_0 \equiv (0,0,0)$, axial equilibrium point or predators free equilibrium point, $E_1 \equiv (1,0,0)$, second predator free equilibrium point , $E_2 \equiv (B_1, B_2, 0)$, first predator free equilibrium point, $E_3 \equiv (C_1, 0, C_2)$, coexisting equilibrium point or interior equilibrium point , $E^* \equiv (s^*, p_1^*, p_2^*)$. Where

$$B_1 = -\frac{h_1 r_1 - d_1 m r_1 - d_2 r_2 + d_1 m_1 r_2}{h_1 r_1 + m r_1 r_2 r_4 - m m_1 r_1 r_2 r_4 + r_2^2 r_4 - 2 m_1 r_2^2 r_4 + m_1^2 r_2^2 r_4},$$

$$B_2 = -\frac{-d_1 r_1 - r_1 r_2 r_4 - m_1 r_1 r_2 r_4}{-h_1 r_1 - m r_1 r_2 r_4 + m m_1 r_1 r_2 r_4 - r_2^2 r_4 + 2 m_1 r_2^2 r_4 - m_1^2 r_2^2 r_4},$$

$$C_1 = -\frac{h_2 r_1 - d_2 r_3 + d_2 m_1 r_2}{h_2 r_1 + r_3^2 r_5 - 2 m_2 r_3^2 r_5 + m_2^2 r_3^2 r_5}, \quad C_2 = -\frac{d_2 r_1 - r_1 r_3 r_5 + m_2 r_1 r_3 r_5}{h_2 r_1 + r_3^2 r_5 - 2 m_2 r_3^2 r_5 + m_2^2 r_3^2 r_5},$$

$$s^* = -\frac{\left(-h_1 h_2 + k_1^2\right)\left(-k_1 r_1 - d_1\left(r_3 - m_2 r_3\right)\right) - \left(-d_1 h_2 + d_2 k_1\right)\left(k_1\left(m r_1 + r_2 - m_1 r_2\right) - h_1\left(r_3 - m_2 r_3\right)\right)}{\left(-h_1 h_2 + k_1^2\right)\left(k_1 r_1 - \left(r_3 - m_2 r_3\right)\left(-r_2 r_4 + m_1 r_2 r_4\right)\right) - \left(k_1\left(m r_1 + r_2 - m_1 r_2\right) - h_1\left(r_3 - m_2 r_3\right)\right)\left(-h_2\left(-r_2 r_4 + m_1 r_2 r_4\right) + k_1\left(-r_3 r_5 + m_2 r_3 r_5\right)\right)},$$

$$p_1^* = -\frac{\left(d_1 h_2 r_1 - d_2 k_1 r_1 - h_2 r_1 r_2 r_4 + h_2 m_1 r_1 r_2 r_4 - d_2 r_2 r_3 r_4 + d_2 m_1 r_2 r_3 r_4 + d_2 m_2 r_2 r_3 r_4 - d_2 m_1 m_2 r_2 r_3 r_4 + k_1 r_1 r_3 r_5 - k_1 m_2 r_1 r_3 r_5 + d_1 r_3^2 r_5 - 2 d_1 m_2 r_3^2 r_5\right)}{\Delta_1},$$

$p_2^* = -\dfrac{\Delta_2}{\Delta_3}$ and

$$\Delta_1 = (h_1 h_2 r_1 - k_1^2 r_1 + h_2 m r_1 r_2 r_4 - h_2 m m_1 r_1 r_2 r_4 + h_2 r_2^2 r_4 - 2 h_2 m_1 r_2^2 r_4 + h_2 m_1^2 r_2^2 r_4 - k_1 r_2 r_3 r_4 + k_1 m_1 r_2 r_3 r_4 + k_1 m_2 r_2 r_3 r_4 -$$
$$k_1 m_1 m_2 r_2 r_3 r_4 - k_1 m r_1 r_3 r_5 + k_1 m m_2 r_1 r_3 r_5 - k_1 r_2 r_3 r_5 + k_1 m_1 r_2 r_3 r_5 + k_1 m_2 r_2 r_3 r_5 - k_1 m_1 m_2 r_2 r_3 r_5 + h_1 r_3^2 r_5 - 2 h_1 m_2 r_3^2 r_5 + h_1 m_2^2 r_3^2 r_5),$$

$$\Delta_2 = (d_2 h_1 r_1 - d_1 k_1 r_1 + k_1 r_1 r_2 r_4 + d_2 m r_1 r_2 r_4 - k_1 m_1 r_1 r_2 r_4 - d_2 m m_1 r_1 r_2 r_4 + d_2 r_2^2 r_4 - 2 d_2 m_1 r_2^2 r_4 + d_2 m_1^2 r_2^2 r_4 -$$
$$h_1 r_1 r_3 r_5 - d_1 m r_1 r_3 r_5 + h_1 m_2 r_1 r_3 r_5 + d_1 m m_2 r_1 r_3 r_5 - d_1 r_2 r_3 r_5 + d_1 m_1 r_2 r_3 r_5 + d_1 m_2 r_2 r_3 r_5 - d_1 m_1 m_2 r_2 r_3 r_5),$$

$$\Delta_3 = (h_1 h_2 r_1 - k_1^2 r_1 + h_2 m r_1 r_2 r_4 - h_2 m m_1 r_1 r_2 r_4 + h_2 r_2^2 r_4 - 2 h_2 m_1 r_2^2 r_4 + h_2 m_1^2 r_2^2 r_4 - k_1 r_2 r_3 r_4 + k_1 m_1 r_2 r_3 r_4 + k_1 m_2 r_2 r_3 r_4 -$$
$$k_1 m_1 m_2 r_2 r_3 r_4 - k_1 m r_1 r_3 r_5 + k_1 m m_2 r_1 r_3 r_5 - k_1 r_2 r_3 r_5 + k_1 m_1 r_2 r_3 r_5 + k_1 m_2 r_2 r_3 r_5 - k_1 m_1 m_2 r_2 r_3 r_5 + h_1 r_3^2 r_5 - 2 h_1 m_2 r_3^2 r_5 + h_1 m_2^2 r_3^2 r_5),$$

Existance Condition:

(a) E_0 : It always exist.

(b) E_1 : It always exist.

(c) E_2 : It exist iff $r_4 > -\dfrac{d_1}{(-r_2 + m_1 r_2)}$

(d) E_3 : It exist iff $r_5 > -\frac{d_2}{(-r_3 + m_2 r_3)}$

(e) E^* : It exist iff $r_3 < \frac{-k_1 m r_1 - k_1 r_2 + k_1 m_1 r_2}{-h_1 + h_1 m_2}$, $r_4 > -\frac{d_1}{(-r_2 + m_1 r_2)}$, $h_2 > \frac{k_1^2 r_1 + d_1 k_1 r_3 - d_1 m_2 k_1 r_3}{h_1 r_1 + d_1 m r_1 + d_1 r_2 - d_1 m_1 r_2}$,

$d_2 < \Delta_4$,

$$\Delta_4 < r_5 < \frac{-d_1 h_2 r_1 + d_2 k_1 r_1 + h_2 r_1 r_2 r_4 - h_2 m_1 r_1 r_2 r_4 + d_2 r_2 r_3 r_4 - d_2 m_1 r_2 r_3 r_4 - d_2 m_2 r_2 r_3 r_4 + d_2 m_1 m_2 r_2 r_3 r_4}{k_1 r_1 r_3 - k_1 m_2 r_1 r_3 + d_1 r_3^2 - 2 d_1 m_2 r_3^2 + d_1 m_2^2 r_3^2}$$

Where,

$$\Delta_5 = \frac{h_1 h_2 r - 1 - k_1^2 r_1 + d_1 h_2 m r_1 + d_1 h_2 r_2 - d_1 h_2 m_1 r_2 - d_1 k_1 r_3 + d_1 k_1 m_2 r_3}{k_1 m r_1 + k_1 r_2 - k_1 m_1 r_2 - h_1 r_3 + h_1 m_2 r_3},$$

$$\Delta_6 = \frac{d_2 h_1 r_1 + d_1 k_1 r_1 - k_1 r_1 r_2 r_4 - d_2 m r_1 r_2 r_4 + k_1 m_1 r_1 r_2 r_4 + d_2 m m_1 r_1 r_2 r_4 - d_2 r_2^2 r_4 + 2 d_2 m_1 r_2^2 r_4 - d_2 m_1^2 r_2^2 r_4}{-h_1 r_1 r_3 - d_1 m r_1 r_3 + h_1 m_2 r_1 r_3 + d_1 m m_2 r_1 r_3 - d_1 r_2 r_3 + d_1 m_1 r_2 r_3 + d_1 m_2 r_2 r_3 - d_1 m - 1 m_2 r_2 r_3}$$

4. Stability Analysis of the equilibrium point

Theorem 1. *The bio-system (2) is unstable around* E_0.

Proof. The jacobian matrix of the system (2) around E_0 is

$$J(E_0) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}$$

$r_1 > 0, -d_1 < 0$, and $-d_2 < 0$ are the eigenvalues of the matrix $J_{(E_0)}$ where two of them are of opposite signs. Hence, the system (2) shows unstability around the trivial equilibrium point E_0.

Theorem 2. *The system (2) is locally asymptotically stable around the axial equilibrium point* E_1 *iff two conditions:* $r_3 < -\frac{d_2}{(-r_5 + m_2 r_5)}$ *and* $r_2 < -\frac{d_1}{(-r_4 + m_1 r_4)}$ *hold simultaneously.*

Proof. The jacobian matrix of the system (2) around E_1 is

$$J(E_1) = \begin{bmatrix} -r_1 & -m r_1 - (1 - m_1) r_2 & -(1 - m_2) r_3 \\ 0 & -d_1 + (1 - m_1) r_2 r_4 & 0 \\ 0 & 0 & -d_2 + (1 - m_2) r_3 r_5 \end{bmatrix}$$

The eigenvalues of $J_{(E_1)}$ are $-r_1$, $-d_1 + (1 - m_1) r_2 r_4$, $-d_2 + (1 - m_2) r_3 r_5$ when the two conditions: $r_3 < -\frac{d_2}{(-r_5 + m_2 r_5)}$ and $r_2 < -\frac{d_1}{(-r_4 + m_1 r_4)}$ are both true simultaneously, then real part of all the eigenvalues becomes negative. Hence, the system (2) is locally asymptotically stable around the axial equilibrium point E_1 iff two conditions: $r_3 < -\frac{d_2}{(-r_5 + m_2 r_5)}$ and $r_2 < -\frac{d_1}{(-r_4 + m_1 r_4)}$ hold simultaneously.

Theorem 3. *The system (2) is locally asymptotically stable around the second predator free equilibrium point E_2 iff $\Gamma_i > 0$, $i = 1,2,3$ and $\Gamma_1\Gamma_2 - \Gamma_3 > 0$ are met simultaneously.*

Proof. The jacobian matrix of the system (2) around E_2 is

$$J(E_2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Where

$$a_{11} = -\frac{r_1(-(h_1 + d_1 m)r_1 + d_1(-1 + m_1)r_2)}{h_1 r_1 - (-1 + m_1)r_2(mr_1 + r_2 - m_1 r_2)r_4},\ a_{12} = \frac{(mr_1 + r_2 - m_1 r_2)\big(h_1 r_1 + d_1(mr_1 + r_2 - m_1 r_2)\big)}{-h_1 r_1 + (-1 + m_1)r_2(mr_1 + r_2 - m_1 r_2)r_4}$$

$$a_{13} = \frac{(1 - m_2)(-h_1 r_1 - d_1 m r_1 - d_1 r_2 + d_1 m_1 r_2)r_3}{h_1 r_1 + m r_1 r_2 r_4 - m m_1 r_1 r_2 r_4 + r_2^2 r_4 - 2m_1 r_2^2 r_4 + m_1^2 r_2^2 r_4},\ a_{21} = -\frac{(1 - m_1)r_2 r_4(-d_1 r_1 + r_1 r_2 r_4 - m_1 r_1 r_2 r_4)}{\varphi_1},$$

$$a_{22} = \frac{h_1 r_1(d_1 + (-1 + m_1)r_2 r_4)}{h_1 r_1 - (-1 + m_1)r_2(mr_1 + r_2 - m_1 r_2)r_4},\quad a_{23} = \frac{k_1(-d_1 r_1 + r_1 r_2 r_4 - m_1 r_1 r_2 r_4)}{-h_1 r_1 - m r_1 r_2 r_4 + m m_1 r_1 r_2 r_4 - r_2^2 r_4 + 2m_1 r_2^2 r_4 - m_1^2 r_2^2 r_4},$$

$$a_{31} = 0, a_{32} = 0,\ a_{33} = \frac{-d_2 h_1 r_1 + d_2(-1 + m_1)r_2(mr_1 + r_2 - m_1 r_2)r_4 + k_1 r_1(d_1 + (-1 + m_1)r_2 r_4) - (-1 + m_2)\varphi_2}{h_1 r_1 - (-1 + m_1)r_2(mr_1 + r_2 - m_1 r_2)r_4}$$

where, $\varphi_1 = -h_1 r_1 - m r_1 r_2 r_4 + m m_1 r_1 r_2 r_4 - r_2^2 r_4 + 2m_1 r_2^2 r_4 - m_1^2 r_2^2 r_4, \varphi_2 = \big(h_1 r_1 + d_1(mr_1 + r_2 - m_1 r_2)\big)r_3 r_5$

Now, the characteristic equation of $J_{(E_1)}$ is given by

$\sigma^3 + \Gamma_1\sigma^2 + \Gamma_2\sigma + \Gamma_3 = 0$,

Where,

$\Gamma_1 = -(a_{11} + a_{22} + a_{33}),\ \Gamma_2 = -(a_{12}a_{21} - a_{11}a_{22} + a_{11}a_{33}),\ \Gamma_3 = -(a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}),$

Now, by using Routh-Hurtwitz criteria for stability, the system (2) is stable around E_2 iff $\Gamma_i > 0$; where i = 1, 2, 3 and $\Gamma_1\Gamma_2 > \Gamma_3$.

Theorem 4. *The system (2) is locally asymptotically stable around the first predator free equilibrium point E_3 iff $\gamma_i > 0$; i=1, 2, 3 and $\gamma_1\gamma_2 - \gamma_3 > 0$ are satisfied simultaneously.*

Proof. The jacobian matrix of the system (2) around E_3 is

$$J(E_3) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Where,

$$b_{11} = -\frac{r_1(-h_2 r_1 + d_2(-1 + m_2)r_3)}{h_2 r_1 + (-1 + m_2)^2 r_3^2 r_5}, b_{12} = -\frac{(mr_1 + r_2 - m_1 r_2)(h_2 r_1 - d_2(-1 + m_2)r_3)}{h_2 r_1 + (-1 + m_2)^2 r_3^2 r_5}, b_{13} = \frac{(1 - m_2)r_3(-h_2 r_1 - d_2 r_3 + d_2 m_2 r_3)}{h_2 r_1 + r_3^2 r_5 - 2m_2 r_3^2 r_5 + m_2^2 r_3^2 r_5},$$

$$b_{21} = 0, b_{22} = \frac{-d_1 h_2 r_1 + d_2 k_1 r_1 - h_2(-1 + m_1)r_1 r_2 r_4 + d_2(-1 + m_1)(-1 + m_2)r_2 r_3 r_4 + k_1(-1 + m_2)r_1 r_3 r_5 - d_1(-1 + m_2)^2 r_3^2 r_5}{h_2 r_1 + (-1 + m_2)^2 r_3^2 r_5}, b_{23} = 0,$$

$$b_{31} = -\frac{(1 - m_2)r_3 r_5(d_2 r_1 - r_1 r_3 r_5 + m_2 r_1 r_3 r_5)}{h_2 r_1 + r_3^2 r_5 - 2m_2 r_3^2 r_5 + m_2^2 r_3^2 r_5}, b_{32} = \frac{k_1(d_2 r_1 - r_1 r_3 r_5 + m_2 r_1 r_3 r_5)}{h_2 r_1 + r_3^2 r_5 - 2m_2 r_3^2 r_5 + m_2^2 r_3^2 r_5}, b_{33} = \frac{h_2 r_1(d_2 + (-1 + m_2)r_3 r_5)}{h_2 r_1 + (-1 + m_2)^2 r_3^2 r_5}$$

Now, the characteristic equation of $J_{(E_3)}$ is given by

$\Lambda^3 + \gamma_1\Lambda^2 + \gamma_2\Lambda + \gamma_3 = 0,$

Where,

$\gamma_1 = -(b_{11} + b_{22} + b_{33}),\ \gamma_2 = -(-b_{11}b_{22} + b_{13}b_{31} - b_{11}b_{33} - b_{22}b_{33}),\ \gamma_3 = -(-b_{13}b_{22}b_{31} + b_{11}b_{22}b_{33}),$

Now, by using Routh-Hurtwitz criteria for stability , the system (2) is stable around E_3 iff $\gamma_i > 0$; i=1, 2, 3 and $\gamma_1\gamma_2 > \gamma_3$.

Theorem 5. The system (2) is locally asymptotically stable around the coexisting equilibrium point E^* iff $\kappa_i > 0$; i=1, 2, 3 and $\kappa_1\kappa_2 - \kappa_3 > 0$ are true simultaneously.

Proof. The jacobian matrix of the system (2) around E^* is

$$J(E^*) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Where,

$$\tau_{11} = \frac{r_1\left(-k_1^2r_1+d_1h_2(mr_1+r_2-m_1r_2)-d_2k_1(mr_1+r_2-m_1r_2)+d_1k_1(-1+m_2)r_3+h_1(h_2r_1-d_2(-1+m_2)r_3)\right)}{-h_1h_2r_1+k_1^2r_1+h_2(-1+m_1)r_2(mr_1+r_2-m_1r_2)r_4-h_1(-1+m_2)^2r_3^2r_5+k_1(-1+m_2)r_3\left(-mr_1r_5+(-1+m_1)r_2(r_4+r_5)\right)},$$

$$\tau_{12} = \frac{(mr_1+r_2-m_1r_2)\left(-k_1^2r_1+d_1h_2(mr_1+r_2-m_1r_2)-d_2k_1(mr_1+r_2-m_1r_2)+d_1k_1(-1+m_2)r_3+h_1(h_2r_1-d_2(-1+m_2)r_3)\right)}{-h_1h_2r_1+k_1^2r_1+h_2(-1+m_1)r_2(mr_1+r_2-m_1r_2)r_4-h_1(-1+m_2)^2r_3^2r_5+k_1(-1+m_2)r_3\left(-mr_1r_5+(-1+m_1)r_2(r_4+r_5)\right)},$$

$$\tau_{13} = \frac{(1-m_2)r_3\left((-h_1h_2+k_1^2)(-k_1r_1-d_1(r_3-m_2r_3))-(-d_1h_2+d_2k_1)\left(k_1(mr_1+r_2-m_1r_2)-h_1(r_3-m_2r_3)\right)\right)}{(-h_1h_2+k_1^2)\left(k_1r_1-(r_3-m_2r_3)(-r_2r_4+m_1r_2r_4)\right)-\left(k_1(mr_1+r_2-m_1r_2)-h_1(r_3-m_2r_3)\right)\left(-h_2(-r_2r_4+m_1r_2r_4)+k_1(-r_3r_5+m_2r_3r_5)\right)},$$

$$\tau_{21} = -\frac{(-1+m_1)r_2r_4\left(d_1h_2r_1-d_2k_1r_1+h_2(-1+m_1)r_1r_2r_4-d_2(-1+m_1)(-1+m_2)r_2r_3r_4-k_1(-1+m_2)r_1r_3r_5+d_1(-1+m_2)^2r_3^2r_5\right)}{k_1^2r_1+h_2(-1+m_1)r_2(mr_1+r_2-m_1r_2)r_4-h_1\left(h_2r_1+(-1+m_2)^2r_3^2r_5\right)+k_1(-1+m_2)r_3\left(-mr_1r_5+(-1+m_1)r_2(r_4+r_5)\right)},$$

$$\tau_{22} = -\frac{h_1\left(d_1h_2r_1-d_2k_1r_1+h_2(-1+m_1)r_1r_2r_4-d_2(-1+m_1)(-1+m_2)r_2r_3r_4-k_1(-1+m_2)r_1r_3r_5+d_1(-1+m_2)^2r_3^2r_5\right)}{k_1^2r_1+h_2(-1+m_1)r_2(mr_1+r_2-m_1r_2)r_4-h_1\left(h_2r_1+(-1+m_2)^2r_3^2r_5\right)+k_1(-1+m_2)r_3(-mr_1r_5+(-1+m_1)r_2(r_4+r_5)},$$

$$\tau_{23} = \frac{k_1\left(-d_1h_2r_1+d_2k_1r_1-h_2(-1+m_1)r_1r_2r_4+d_2(-1+m_1)(-1+m_2)r_2r_3r_4+k_1(-1+m_2)r_1r_3r_5-d_1(-1+m_2)^2r_3^2r_5\right)}{k_1^2r_1+h_2(-1+m_1)r_2(mr_1+r_2-m_1r_2)r_4-h_1\left(h_2r_1+(-1+m_2)^2r_3^2r_5\right)+k_1(-1+m_2)r_3\left(-mr_1r_5+(-1+m_1)r_2(r_4+r_5)\right)},$$

$$\tau_{31} = \frac{(-1+m_2)r_3r_5\left(d_2h_1r_1-d_2(-1+m_1)r_2(mr_1+r_2-m_1r_2)r_4-k_1r_1(d_1+(-1+m_1)r_2r_4)+(-1+m_2)\left(h_1r_1+d_1(mr_1+r_2-m_1r_2)\right)r_3r_5\right)}{h_1h_2r_1-k_1^2r_1+h_2(-1+m_1)r_2(-mr_1+(-1+m_1)r_2)r_4+h_1(-1+m_2)^2r_3^2r_5-k_1(-1+m_2)r_3\left(-mr_1r_5+(-1+m_1)r_2(r_4+r_5)\right)},$$

$$\tau_{32} = \frac{k_1\left(-d_2h_1r_1+d_2(-1+m_1)r_2(mr_1+r_2-m_1r_2)r_4+k_1r_1(d_1+(-1+m_1)r_2r_4)-(-1+m_2)\left(h_1r_1+d_1(mr_1+r_2-m_1r_2)\right)r_3r_5\right)}{-h_1h2r1+k1^2r1+h2(-1+m1)r2(mr_1+r_2-m_1r_2)r_4-h_1(-1+m_2)^2r_3^2r_5+k_1(-1+m_2)r_3\left(-mr_1r_5+(-1+m_1)r_2(r_4+r_5)\right)},$$

$$\tau_{33} = \frac{h_2\left(d_2h_1r_1-d_2(-1+m_1)r_2(mr_1+r_2-m_1r_2)r_4-k_1r_1(d_1+(-1+m_1)r_2r_4)+(-1+m_2)\left(h_1r_1+d_1(mr_1+r_2-m_1r_2)\right)r_3r_5\right)}{h_1h_2r_1-k_1^2r_1+h_2(-1+m_1)r_2(-mr_1+(-1+m_1)r_2)r_4+h_1(-1+m_2)^2r_3^2r_5-k_1(-1+m_2)r_3\left(-mr_1r_5+(-1+m_1)r_2(r_4+r_5)\right)}$$

Now, the characteristic equation of $J_{(E^*)}$ is given by

$$\lambda^3 + \kappa_1\lambda^2 + \kappa_2\lambda + \kappa_3 = 0,$$

$\kappa_1 = -(c_{11} + c_{22} + c_{33}),\ \kappa_2 = -(c_{11}c_{33} + c_{11}c_{22} + c_{22}c_{33} - c_{12}c_{21} - c_{13}c_{31} - c_{32}c_{32}),$

$\kappa_3 = -(c_{11}(c_{22}c_{33} - c_{23}c_{32}) + c_{12}(c_{23}c_{31} - c_{21}c_{33}) + c_{13}(c_{21}c_{32} - c_{22}c_{31})),$

By Routh-Hurwitz criterion for stability, E^* is locally asymptotically stable if and only if $\kappa_1 > 0,\ \kappa_2 > 0,\ \kappa_3 > 0$ and $\kappa_1\kappa_2 - \kappa_3 > 0$

5. Transcritical bifurcations

Theorem 6. A transcritical bifurcation occurs along the parametric surface $-d_1+(1-m_1)r_2r_4=0$ around the axial equilibrium point E_1 in the biosystem (2).

Proof. We use Sotomayor's theorem [25] to demonstrate the occurance of a transcritical bifurcation occurs along the parametric surface $-d_1+(1-m_1)r_2r_4=0$ around E_1.
The Jacobian matrix around the axial equilibrium point E_1 is

$$J(E_1)=\begin{bmatrix}-r_1 & -mr_1-(1-m_1)r_2 & -(1-m_2)r_3\\ 0 & -d_1+(1-m_1)r_2r_4 & 0\\ 0 & 0 & -d_2+(1-m_2)r_3r_5\end{bmatrix}$$

The eigenvalues of the Jacobian matrix $J(E_1)$ are $-r_1$, $-d_1+(1-m_1)r_2r_4$ and $-d_2+(1-m_2)r_3r_5$. One of the three eigenvalues of $J(E_1)$ becomes zero when $d_1=r_2r_4-m_1r_2r_4=d_1^{tb}$. Considering d_1 as the bifurcating parameter and $d_1=d_1^{tb}$, the Jacobian matrix $J(E_1)$ becomes

$$J(E_1)=\begin{bmatrix}-r_1 & -mr_1-(1-m_1)r_2 & -(1-m_2)r_3\\ 0 & -d_1+(1-m_1)r_2r_4 & 0\\ 0 & 0 & -d_2+(1-m_2)r_3r_5\end{bmatrix}$$

Let us consider, M be an eigenvector corresponding to the zero eigenvalue of $J(E_1)$)and N be an eigenvector corresponding to the zero eigenvalue of $J(E_1)^T$. Then,
$M=(m^1,m^2,m^3)^t=\left(-\frac{(mr_1+r_2-m_1r_2)}{r_1},1,0\right)^t$ and $N=(n^1,n^2,n^3)^t=(0,1,0)^t$
And

$$R_{d_1}(E_1;d_1^{tb})=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$D\left(R_{d_1}(E_1;d_1^{tb})\right)M=\begin{bmatrix}0&0&0\\0&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}-\dfrac{(mr_1+r_2-m_1r_2)}{r_1}\\1\\0\end{bmatrix}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$$

$$D^2\left(R_{d_1}(E_1;d_1^{tb})\right)(M,M)=\begin{bmatrix}-\dfrac{2(h_1r_1+(1-m_1)r_2(mr_1+r_2-m_1r_2)r_4)}{r_1}\\0\end{bmatrix}$$

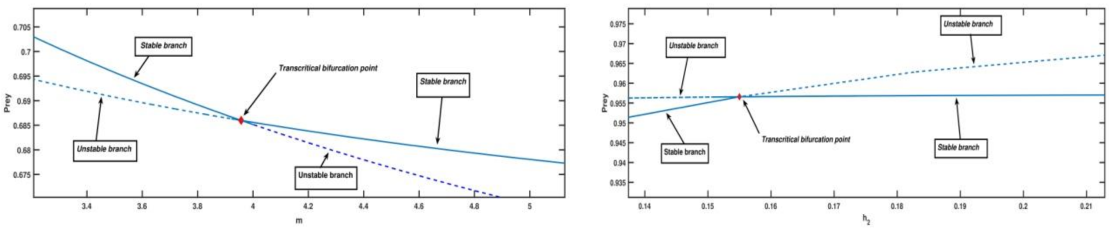
Therefore,

$$N^T\left(R_{d_1}(E_1;d_1^{tb})\right)=0,$$

$$N^T\left(D\left(R_{d_1}(E_1;d_1^{tb})\right)M\right)=-1\neq 0,$$

$$N^T\left(D^2\left(R_{d_1}(E_1;d_1^{tb})\right)(M,M)\right)=-\frac{2(h_1r_1+(1-m_1)r_2(mr_1+r_2-m_1r_2)r_4)}{r_1}\neq 0.$$

Hence, the existence of a transcritical bifurcation around E_1 at $d_1 = d_1^{tb}$ is confirmed with the help of Sotomayor's theorem [25]. However, we can also use other parameters as bifurcating parameters.



(a)Occurance of transcritical bifurcation Around E_2 at $m = m^{tb}$

(b) Occurance of transcritical bifurcation Around E_3 at $h_2 = h_2^{tb}$

Figure 1: Different bifurcation scenarios in the biosystem (2).

6. Numerical Simulations

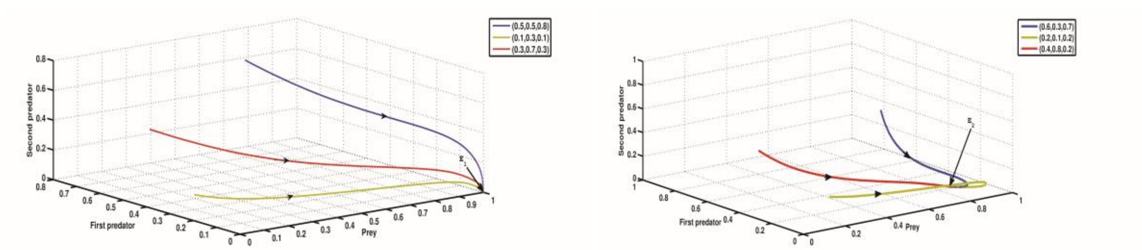
In this section of the paper, numerical simulations are used to justify the theoretical findings presented in the preceding sections. The software packages MATHEMATICA, MATCONT [26] and MATLAB serve as the tools in order to carry out these numerical simulations. Given the lack of actual data for all the parameters of the system (2), some hypothetical parameter values are used to quantitatively validate all of the analytical findings. These hypothetical parameter values are as follows:

Parameter	value
r_1	0.5
d_1	0.05
d_2	0.01
r_2	0.3
r_3	0.27
r_4	0.39629
r_5	0.28307
m	0.2
k_1	0.2
k_2	0.3

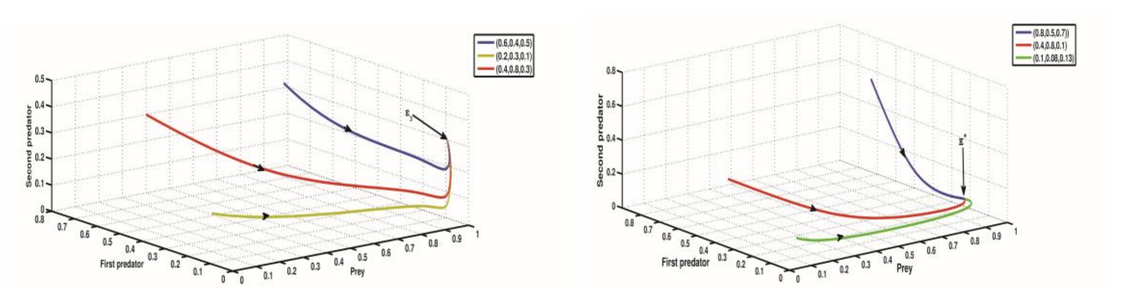
Table 1: Assigned parameter values of the system (2)

6.1 Numerical verification of equilibrium points

To verify the existence as well as stability of the axial equilibrium point E_1 , we consider $m_1 = 0.783,\ m_2 = 0.93,\ h_1 = 0.5,\ h_2 = 0.07$ and the rest parameter values from the above table. For these paramete values, the eigenvalues of the jacobian matrix $J(E_1)$ are $-0.5 < 0,\ -0.025 < 0,\ -0.005 < 0.$ Since, all the eigenvalues of $J(E_1)$ are real , distinct and negative, E_1 is stable. It can be seen in figure (2a). Taking parameter values as $m_1 = 0.26,\ m_2 = 0.51,\ h_1 = 0.12,\ h_2 = 0.5$ and the rest parameter values from the above table, then $J(E_2)$ has eigenvalues -0.420658, -0.0375193, -0.019408 which are all real , distinct and negative. It confirms the local stability of the second predator free boundary equilibrium point E_2 as shown in figure (2b). Morever, considering parameter values as $m_1 = 0.26,\ m_2 = 0.51,\ h_1 = 0.5,\ h_2 = 0.07$ and the rest from the above table. The eigenvalues of the jacobian matrix $J(E_3)$ are given by -0.450846, -0.292974, -0.032353 which confirms the local stability of the first predator free boundary equilibrium point E_3 which can be easily seen from figure (2c). To show the local stability of the most desirable equilibrium point E^* , we consider parameter values as $r_4 = 1.6,\ r_5 = 2,\ h_1 = 2.23,\ h_2 = 1.6,\ m_1 = 0.2613161406911,\ m_2 = 0.5102946395031973$ and the rest are from the above table. The eigenvalues of the jacobian matrix $J(E^*)$ are given by -0.439313, -0.13601, -0.0238935 which confirms the fact that the co-existant equilibrium point E^* is locally stable. It is shown in figure (2d).



(a)The equilibrium point E_1 is locally stable (b)The equilibrium point E_2 is locally stable



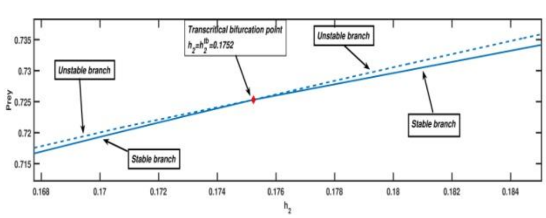
(c)The equilibrium point E_3 is locally stable (d)The equilibrium point E^* is locally stable

Figure 2: Phase portraits showing local stability of all the equilibrium points

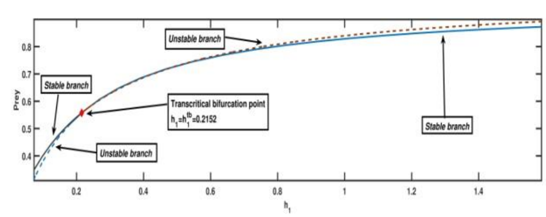
6.2 Role of the intraspecific parameters h_1 and h_2 in the system

For the purpose of demonstrating the significance of intraspecific competition parameters h1 and h2, we take parameter values $h_1 = 0$, $h_2 = 0$, $r_4 = 1.6$, $r_5 = 2$, $m_1 = 0.2613161406911$, $m_2 = 0.5102946395031973$ and the rest parameter values from the above table. Then the system (2) exhibits bistability phenomenon. The jacobian matrix $J(E_2)$ has three eigenvalues which are given by $v_1 = 0.239802$, $v_2 = 0.0352541 + 0.142238\mathrm{i}$, $v_3 = 0.0352541 + 0.142238\mathrm{i}$. The real parts of v_1, v_2, v_3 are all negative and hence, E_2 is locally stable. On the other hand, the eigenvalues of the jacobian matrix $J(E_3)$ are $v_4 = 0.764304$, $v_5 = -0.00945391 + 0.0687135\mathrm{i}$, $v_6 = -0.00945391 \text{-} 0.0687135\mathrm{i}$. Since all these eigenvalues have negative real part, the first predator free boundary equilibrium point E_3 is locally stable. Hence, for these parameter values, both the boundary equilibrium points E_2 and E_3 are locally stable at the same time which is clearly visible from figure (3c). However, for these parameter values , the coexistent equilibrium point E^* becomes unstable as all the eigenvalues of $J(E^*)$ are $v_7 = \text{-}0.174509 + 0.189758\mathrm{I},$, $v_8 = \text{-}0.174509 - 0.189758\mathrm{I},$, $v_9 = 0.112248 > 0$. To illustrate the role of parameter h_1 individually , we consider $r_4 = 1.6$, $r_5 = 2$, $h_2 = 1.6$, $m_1 = 0.2613161406911$, $m_2 = 0.5102946395031973$ and all other parameter values as same as in the table above. Variating only the parameter h_1, it is found that at relatively low value of $h_1 = 0.2152 = h_1^{tb}$, a transcritical bifurcation takes place around the coexisting equilibrium point E^* which is shown in figure (3b).

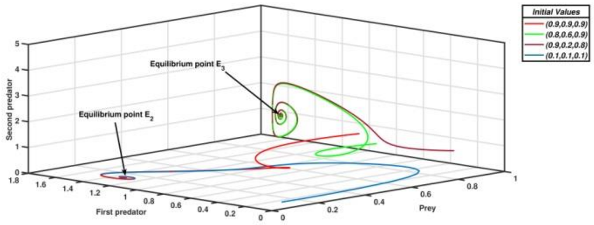
In a similar fashion, we take $h_1 = 2.23$and all other parameter values are as same as those previously discussed in the preceeding paragraph to demonstrate the significance of the parameter h_2 independently . Varying the value of h_2, the system (2) manifests another transcritical bifurcation around the coexisting equilibrium point E^* at $h_2 = h_2^{tb} = 0.1752$. Figure (3a) represents this particular scenario.



(a) Occurance of transcritical bifurcation at $h_2 = 0.1752$



(b) Occurance of transcritical bifurcation at $h_1 = 0.2152$

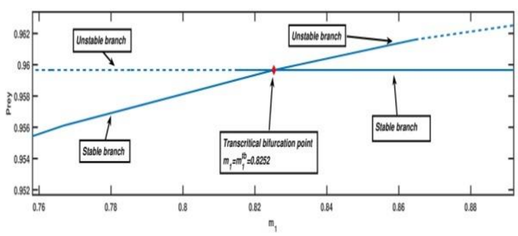


(C) Occurance of bistability phenomenon when $h_1 = 0$ and $h_2 = 0$

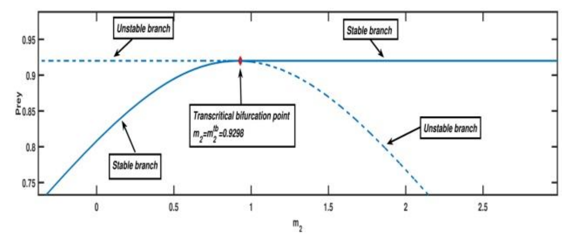
Figure 3: Importance of parameters h_1 and h_2 in the system

6.3 Role of the Prey refuge parameters m_1 and m_2

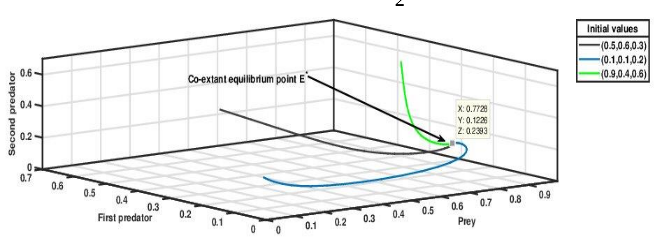
Prey refuge parameters plays an important role in the bio-system (2). We take into consideration the values of the parameters $r_1 = 0.5$, $r_2 = 0.3$, $r_3 = 0.27$, $m = 0.2$ $d_1 = 0.05$, $d_2 = 0.01$, $r_4 = 0.39$, $r_5 = 0.28$, $k_1 = 0.2$, $k_2 = 0.3$, $r_4 = 1.6$, $r_5 = 2$, $h_1 = 2.23$, $h_2 = 1.6$. Taking $m_2 = 0.51$ alongwith these above parameter values and changing the value of m_1, a transcritical bifurcation occurs at $m_1 = 0.8252$ which can be seen from figure(4a). Again in the same way, fixing $m_1 = 0.26$ and changing m_2 taking all other parameter values as same as above, another transcritical bifurcation appears at $m_2 = 0.9298$. Figure(4b) illustrates the same. When there is no prey refuge in the system (2), i.e., $m_1 = 0$ and $m_2 = 0$ alongwith same other parameter values as discussed above, coexisting equilibrium is achieved as shown in figure(4c).



(a) Occurance of transcritical bifurcation At $m_1 = 0.8252$



(b) Occurance of transcritical bifurcation at $m_2 = 0.9298$



(c) Coexistence is conceivable when $m_1 = 0$ and $m_2 = 0$

Figure 4: Significance of the parameters m_1 and m_2 within the system

7. Discussions

The present study examines a biosystem comprising three distinct species, wherein two predators engage in predation activities targeting a common prey species. An aspect of predator-prey interaction that has received relatively limited attention in academic literature pertains to the impact of predator's chemical signals on prey species. Drawing inspiration from this, the effects of predators's chemical cues have been duly considered in the present study. Also, commonly occuring natural phenomenon like prey refuge and competition both interspecific as well as intraspecific are considered in this model. The biosystem (2) manifests five distinct types of equilibrium points. It is found that in this bio-system, the three species will never go extinct together as long as there are any populations of either one of them exist. The predator free equilibrium, the boundary equilibrium points and the coexisting equilibrium , all exhibit asymptotic stability under certain parametric conditions. From a biological standpoint, this suggests that without predators, prey species can survive and reproduce to their full potential. Also, without any one of the predators, the other predator and prey may survive. Most crucially, all three species may persist side-by-side in the ecosystem. Figure (2) represents these scenarios very well. Furthermore, the significance of prey refuge in the system (2) is also discussed. It has been observed that coexistence can occur even in the absence of prey refuge. Changes in the levels of prey refuge can trigger extinction of either of the predators. It can be seen from figure (4). It is also addressed how the intraspecific competition among the two predators species affects the system (2) as shown in figure (3). In absence of intraspecific competition among both the predators gives rise to the scenario where either one of the predator species vanishes from the system. Prey refuge and intraspecific competition amongst predator species, therefore, have an important part to play in determining the population dynamics of the biosystem (2) . Further research could entail a more comprehensive exploration of the biosystem (2) through the examination of potential manifestations of various different types of bifurcation.

References

1. Lotka, A.J., 1925. Elements of physical biology. Williams and Wilkins.
2. Volterra, V., 1927. Variazioni e fluttuazioni nel numero di individui in specie animali conviventi, Memorie del Regio Comitato Talassografico Italiano mem. CXXXI (ripubb. in V. Volterra, Opere matematiche. Memorie e note, voi. V, Roma, Ac cademia Nazionale dei Lincei, 1962). Kostitsin VA (1938), R´emarques sur l'action toxique du milieu, Acad´emie des Sciences, Comptes Rendus, 207.
3. Savitri, D., Suryanto, A. and Kusumawinahyu, W.M., 2019, June. A Dynamics Behaviour of Two Predators and One Prey Interaction with Competition Between Predators. In IOP Conference Series: Materials Science and Engineering (Vol. 546, No. 5, p. 052069). IOP Publishing

4. Sarwardi, S., Mandal, P.K. and Ray, S., 2013. Dynamical behaviour of a two-predator model with prey refuge. Journal of biological physics, 39(4), pp.701-722.
5. Didiharyono, D., 2016. Stability analysis of one prey two predator model with Holling type III functional response and harvesting. Journal of Math Sciences, 1(2 October), pp.50-54.
6. Beddington, J.R., 1975. Mutual interference between parasites or predators and its effect on searching efficiency. The Journal of Animal Ecology, pp.331-340.
7. DeAngelis, D.L., Goldstein, R.A. and O'Neill, R.V., 1975. A model for tropic interaction. Ecology, 56(4), pp.881-892.
8. Crowley, P.H. and Martin, E.K., 1989. Functional responses and interference within and between year classes of a dragonfly population. Journal of the North American Benthological Society, 8(3), pp.211-221.
9. Arditi, R. and Ginzburg, L.R., 1989. Coupling in predator-prey dynamics: ratio-dependence. Journal of theoretical biology, 139(3), pp.311-326.
10. Hassell, M.P. and Varley, G.C., 1969. New inductive population model for insect parasites and its bearing on biological control. Nature, 223(5211), pp.1133-1137.
11. Hsu, S.B., 1982. On a resource based ecological competition model with interference. Journal of Mathematical Biology, 12(1), pp.45-52
12. Wang, Z., Xie, Y., Lu, J. and Li, Y., 2019. Stability and bifurcation of a delayed generalized fractionalorder prey–predator model with interspecific competition. Applied Mathematics and Computation, 347, pp.360-369.
13. Manna, K., Volpert, V. and Banerjee, M., 2020. Dynamics of a diffusive two-prey-one-predator model with nonlocal intra-specific competition for both the prey species. Mathematics, 8(1), p.101.
14. Ali, N., Haque, M., Venturino, E. and Chakravarty, S., 2017. Dynamics of a three species ratio-dependent food chain model with intra-specific competition within the top predator. Computers in biology and medicine, 85, pp.63-74.
15. Nelson, E. H., Matthews, C. E., Rosenheim, J. A. (2004). Predators reduce prey population growth by inducing changes in prey behavior. Ecology, 85(7), 1853-1858.
16. Nakaoka, M. (2000). Nonlethal effects of predators on prey populations: predator-mediated change in bivalve growth. Ecology, 81(4), 1031-1045.
17. Castorani, M. C., Hovel, K. A. (2016). Native predator chemical cues induce anti-predation behaviors in an invasive marine bivalve. Biological Invasions, 18, 169-181.
18. Petranka, J. W., Kats, L. B., Sih, A. (1987). Predator-prey interactions among fish and larval amphibians: use of chemical cues to detect predatory fish. Animal Behaviour, 35(2), 420-425.
19. Smee, D. L., Weissburg, M. J. (2006). Hard clams (Mercenaria mercenaria) evaluate predation risk using chemical signals from predators and injured conspecifics. Journal of chemical ecology, 32, 605-619.
20. Kar, T. K. (2005). Stability analysis of a prey–predator model incorporating a prey refuge. Communications in Nonlinear Science and Numerical Simulation, 10(6), 681-691.

21. Zhang, H., Cai, Y., Fu, S., Wang, W. (2019). Impact of the fear effect in a prey-predator model incorporating a prey refuge. Applied Mathematics and Computation, 356, 328-337
22. Ma, Z., Li, W., Zhao, Y., Wang, W., Zhang, H., Li, Z. (2009). Effects of prey refuges on a predator–prey model with a class of functional responses: the role of refuges. Mathematical biosciences, 218(2), 73-79.
23. Ko, W., Ryu, K. (2006). Qualitative analysis of a predator–prey model with Holling type II functional response incorporating a prey refuge. Journal of Differential Equations, 231(2), 534-550.
24. Xu, W., Wu, D., Gao, J. and Shen, C., 2022. Mechanisms of stable species coexistence in food chain systems: Strength of odor disturbance and group defense. Chaos, Solitons and Fractals: X, 8, p.100073.
25. Perko, L., 2013. Differential equations and dynamical systems (Vol. 7). Springer Science and Business Media.
26. Dhooge, A., Govaerts, W., Kuznetsov, Y.A., Meijer, H.G.E. and Sautois, B., 2008. New features of the software MatCont for bifurcation analysis of dynamical systems. Mathematical and Computer Modelling of Dynamical Systems, 14(2), pp.147-175.