Fixed points of $(\mathbb{F}^*,\Upsilon^*)$-weak contractions by generalized altering distances

G. V. R. Babu[1], P. A. Kameswari[2] and P. Mounika[3*]

Abstract: We define $(\mathbb{F}^*,\Upsilon^*)$-contraction and $(\mathbb{F}^*,\Upsilon^*)$-weak contraction where Υ^* is a generalized altering distance function and prove the existence and uniqueness of fixed points of these maps in complete metric spaces. Further, we extend it to $(\mathbb{F}^*,\Upsilon^*)$-contraction in orbits by using Λ-orbitally continuity. Our results generalize the result of Wardowski, Theorem 2.1, **[10]**.

Mathematics Subject Classification: 47H10, 54H25.

Keywords: $\mathbb{F}$-contraction, $\mathbb{F}^*$-weak contraction, generalized altering distance function, $(\mathbb{F}^*,\Upsilon^*)$-contraction, $(\mathbb{F}^*,\Upsilon^*)$-weak contraction, orbit, Λ-orbitally complete, orbitally continuous.

[1,2,3]Departmentof Mathematics, Andhra University, Visakhapatnam-530003, India.
[1]Email: gvr_babu@hotmail.com
[2] Email: panuradhakameswari@yahoo.in
[3*]Corresponding Author Email: mounika.palla15@gmail.com

I. INTRODUCTION

In the direction of generalization of contraction condition, Wardowski **[10]** introduced a new concept namely, $\mathbb{F}$-contraction as follows:

Definition 1.1.[10] Let $\mathcal{G}$ be the family of all functions $\mathbb{F}:(0,\infty)\to\mathbb{R}$ satisfying the following conditions:

$(\mathbb{F}_1)$: For any $\iota,\kappa\in(0,+\infty), \iota<\kappa$ implies $\mathbb{F}(\iota)<\mathbb{F}(\kappa)$

$(\mathbb{F}_2)$: $\lim_{n\to+\infty}\iota_n=0 \Leftrightarrow \lim_{n\to+\infty}\mathbb{F}(\iota_n)=-\infty$, for any $\{\iota_n\}\subset(0,+\infty)$.

$(\mathbb{F}_3)$: There exists a number $k\in(0,1)$ such that $\lim_{\iota\to0^+}\iota^k\mathbb{F}(\iota)=0$.

We denote $\mathcal{G}=\{\mathbb{F}:(0,\infty)\to\mathbb{R}/\mathbb{F}\text{satisfies } (\mathbb{F}_1)-(\mathbb{F}_3)\}$.

Example 1.1. [10] The following functions belong to $\mathcal{G}$. For $\iota>0$,

i) $\mathbb{F}(\iota)=-\frac{1}{\sqrt{\iota}}$

ii) $\mathbb{F}(\iota)=\iota+\ln\iota$

iii) $\mathbb{F}(\iota)=\ln\iota$

iv) $\mathbb{F}(\iota)=\ln(\iota^2+\iota)$.

We denote $\mathcal{G}^*$, the family of all functions $\mathbb{F}^*$ which satisfy the conditions $(\mathbb{F}_1)$ and $(\mathbb{F}_2)$. Here we observe that $\mathcal{G}\subset\mathcal{G}^*$.

Example 1.2. The following functions belong to $\mathcal{G}^*$, but not to $\mathcal{G}$. For $\iota>0$,

i) $\mathbb{F}(\iota)=-\frac{1}{\iota}+\ln\iota+\iota$

ii) $\mathbb{F}(\iota)=-\frac{1}{\sqrt{\iota}}+\ln\iota$.

Definition 1.2. [10] Let (Ξ,ϱ) be a metric space. Let $\Lambda:\Xi\to\Xi$. If there exist $\Gamma>0$ and $\mathbb{F}\in\mathcal{G}$ such that

(1.1) $$\varrho(\Lambda\curlywedge,\Lambda\wp)>0\Rightarrow\Gamma+\mathbb{F}(\varrho(\Lambda\curlywedge,\Lambda\wp))\le\mathbb{F}(\varrho(\curlywedge,\wp))$$

for all $\curlywedge,\wp$ in Ξ, then Λ is said to be an $\mathbb{F}$-*contraction.*

Wardowski **[10]** observed that every $\mathbb{F}$-contraction is a continuous mapping.

Theorem 1.1. (Theorem 2.1, **[10]**) Let (Ξ,ϱ) be a complete metric space and let $\Lambda:\Xi\to\Xi$ be an $\mathbb{F}$-contraction. Then Λ has a unique fixed point

$\curlywedge^* \in \Xi$ and for every $\curlywedge_0 \in \Xi, \{\Lambda^n \curlywedge_0\}_{n\in\mathbb{N}}$ is convergent to $\curlywedge^*$.

For more works on $\mathbb{F}$-contractions and related results on existence of fixed points, we refer **[6],[11]**.

Further, in 2020, Alfaqih, Imdad and Gubran **[1]**, introduced the following class of functions.

Let $\mathcal{G}' = \{\mathbb{F}: (0,\infty) \to \mathbb{R} / \lim_{n\to\infty} \mathbb{F}(\iota_n) = -\infty \Rightarrow \lim_{n\to\infty} \iota_n = 0$ for any $\{\iota_n\} \subset (0,\infty)\}$.

Obviously, $\mathcal{G} \subset \mathcal{G}'$. But its converse is not true and it was shown in Example 2.1 and Example 2.2 **[1]**.

Definition 1.3. [1] Let (Ξ, ϱ) be a metric space. Let $\Lambda: \Xi \to \Xi$. If there exist $\Gamma > 0$ and $\mathbb{F} \in \mathcal{G}'$ such that

(1.2) $\quad \varrho(\Lambda \curlywedge, \Lambda\wp) > 0 \Rightarrow \Gamma + \mathbb{F}\big(\varrho(\Lambda \curlywedge, \Lambda\wp)\big) \leq \mathbb{F}(m(\curlywedge, \wp))$

where $(\curlywedge, \wp) = \max\{\varrho(\curlywedge, \wp), \varrho(\curlywedge, \Lambda \curlywedge), \varrho(\wp, \Lambda\wp)\}$, for all $\curlywedge, \wp$ in Ξ, then Λ is said to be an $\mathbb{F}'$-*weak contraction.*

Theorem 1.2. (Theorem 2.1, **[1]**) Let (Ξ, ϱ) be a complete metric space and $\Lambda: \Xi \to \Xi$ an $\mathbb{F}'$-weak contraction. If $\mathbb{F}'$ is continuous, then
a) Λ has a unique fixed point $\imath$ in Ξ,
b) $\lim_{n\to\infty} \Lambda^n \curlywedge = \imath$ for all $\curlywedge \in \Xi$.
Moreover, Λ is continuous at $\imath$ if and only if $\lim_{\Lambda\to\imath} m(\curlywedge, \imath) = 0$.

In 1984, Khan, Swaleh and Sessa **[4]** considered contraction condition with an altering distance function to prove the existence of fixed points in complete metric spaces.

Definition 1.4. [4] Let $\curlyvee: \mathbb{R}^+ \to \mathbb{R}^+ (\mathbb{R}^+ = [0,\infty))$ be a function. If $\curlyvee$ satisfies the conditions
$(\curlyvee_1)$ $\curlyvee$ is continuous
$(\curlyvee_2)$ $\curlyvee$ is monotonically increasing, and
$(\curlyvee_3)$ $\curlyvee(\jmath) = 0 \Leftrightarrow \jmath = 0$
then $\curlyvee$ is said to be an *altering distance function* or *control function.*

We denote the class of all altering distance functions by Υ.

For more details on altering distance functions and results based on altering distance functions, we refer Naidu **[5]**, Sastry and Babu **[7]** and **[8]**.

A function $\curlyvee$ that satisfies $(\curlyvee_1)$ and $(\curlyvee_3)$, we call $\curlyvee$ a *generalized altering distance function.*

We denote $\Upsilon^* = \{\curlyvee: \mathbb{R}^+ \to \mathbb{R}^+ / \curlyvee$ satisfies $(\curlyvee_1)$ and $(\curlyvee_3)\}$. Here we note that $\Upsilon \subset \Upsilon^*$.

Motivated by the works of Alfaqih, Imdad and Gubran **[1]**, we extend these results to find the existence and uniqueness of fixed points by using generalized altering distance functions.

In Section 2, we define $(\mathbb{F}^*, \curlyvee^*)$-contraction, where $\mathbb{F}^* \in \mathcal{G}^*$ and $\curlyvee^* \in \Upsilon^*$ and prove the existence and uniqueness of fixed points in complete metric spaces. We discuss the importance of $\curlyvee^*$ and provide examples in support of our results. In Section 3, we extend the result of Wardowski **[10]** to orbits, which generalizes the result of Wardowski **[10]**.

II. Main results

Definition 2.1. Let Λ be a selfmap on a metric space (Ξ, ϱ). If there exist $\curlyvee^* \in \Upsilon^*$, $\mathbb{F}^* \in \mathcal{G}^*$ and $\Gamma > 0$ such that $\varrho(\Lambda \curlywedge, \Lambda\wp) > 0$ implies that

(2.1) $\quad \Gamma + \mathbb{F}^*\Big(\curlyvee^*\big(\varrho(\Lambda \curlywedge, \Lambda\wp)\big)\Big) \leq \mathbb{F}^*(\curlyvee^*(\varrho(\curlywedge, \wp)))$

for all $\curlywedge, \wp \in \Xi$, then we say that Λ is a $(\mathbb{F}^*, \curlyvee^*)$-*contraction*.

Example 2.1. Let $\Xi = [0,1]$ with the usual metric. We define $\Lambda: \Xi \to \Xi$ by $\Lambda \curlywedge = \frac{\curlywedge}{\curlywedge+2}$ and $\curlyvee^*: \mathbb{R}^+ \to \mathbb{R}^+$ by $\curlyvee^*(\jmath) = \jmath^2, \jmath \geq 0$. Then $\curlyvee^* \in \Upsilon^*$. We define $\mathbb{F}^* \in \mathcal{G}^*$ by $\mathbb{F}^*(\iota) = -\frac{1}{\sqrt{\iota}} + \ln \iota, \iota > 0$. We choose $\Gamma = \ln 2 > 0$. For this Γ, we have

$$\Gamma + \mathbb{F}^*\Big(\curlyvee^*\big(\varrho(\Lambda \curlywedge, \Lambda\wp)\big)\Big)$$
$$= \ln 2 + \mathbb{F}^*\left(\curlyvee^*\left(\varrho\left(\frac{\curlywedge}{\curlywedge+2}, \frac{\wp}{\wp+2}\right)\right)\right)$$
$$= \ln 2 + \mathbb{F}^*\left(\curlyvee^*\left(\left|\frac{\curlywedge}{\curlywedge+2} - \frac{\wp}{\wp+2}\right|\right)\right)$$
$$= \ln 2 + \mathbb{F}^*\left(\left|\frac{\curlywedge}{\curlywedge+2} - \frac{\wp}{\wp+2}\right|^2\right)$$
$$= \ln 2 + \mathbb{F}^*\left(\left|\frac{2|\curlywedge-\wp|}{(\curlywedge+2)(\wp+2)}\right|^2\right)$$

$$= \ln 2 - \frac{1}{\sqrt{\left|\frac{2|\lambda-\wp|}{(\lambda+2)(\wp+2)}\right|^2}} + \ln\left(\frac{2|\lambda-\wp|}{(\lambda+2)(\wp+2)}\right)^2$$
$$= \ln 2 - \frac{|(\lambda+2)(\wp+2)|}{2|\lambda-\wp|} + 2\ln(2|\lambda-\wp|) - 2\ln|(\lambda+2)(\wp+2)|$$
$$\leq \ln 2 - \frac{4}{2|\lambda-\wp|} + 2\ln 2 + 2\ln|\lambda-\wp| - 2\ln 4$$
$$= 2\ln|\lambda-\wp| - \frac{2}{|\lambda-\wp|} - \ln 2$$
$$< 2\ln|\lambda-\wp| - \frac{1}{\sqrt{|\lambda-\wp|^2}}$$
$$= \ln|\lambda-\wp|^2 - \frac{1}{\sqrt{|\lambda-\wp|^2}}$$
$$= \mathbb{F}^*(|\lambda-\wp|^2)$$
$$= \mathbb{F}^*(\upsilon^*(\varrho(\lambda,\wp))).$$

Therefore Λ satisfies the inequality (2.1), so that Λ is a $(\mathbb{F}^*,\upsilon^*)$-contraction.

Theorem 2.1. Let (Ξ,ϱ) be a complete metric space. Let $\Lambda:\Xi\to\Xi$ be a $(\mathbb{F}^*,\upsilon^*)$-contraction and $\mathbb{F}^*$ is continuous. Suppose that $\lambda_0 \in \Xi$. We define $\{\lambda_n\}$ in Ξ by $\lambda_{n+1} = \Lambda\lambda_n, n = 0,1,2,\ldots$. If Λ is continuous, then Λ has a unique fixed point λ^* in Ξ.

Proof. Let $\lambda_0 \in \Xi$. We define the sequence $\lambda_{n+1} = \Lambda\lambda_n$ for $= 0,1,2,\ldots$.

If $\lambda_{n+1} = \lambda_n$ for some n, then we have $\Lambda\lambda_n = \lambda_n$. By choosing $\imath = \lambda_n$, we have $\Lambda\imath = \imath$, and the conclusion of the theorem follows.

We now assume, without loss of generality, that $\lambda_n \neq \lambda_{n+1}$, for every $n\in\mathbb{N}$.

By taking $\lambda = \lambda_n$ and $\wp = \lambda_{n-1}$ in (2.1), we have
$$\Gamma + \mathbb{F}^*(\upsilon^*(\varrho(\Lambda\lambda_n,\Lambda\lambda_{n-1}))) \leq \mathbb{F}^*(\upsilon^*(\varrho(\lambda_n,\lambda_{n-1})))$$
and hence
$$\mathbb{F}^*(\upsilon^*(\varrho(\lambda_{n+1},\lambda_n))) \leq \mathbb{F}^*\left(\upsilon^*\left(\varrho(\lambda_n,\lambda_{n-1})\right)\right) - \Gamma$$
$$\leq \mathbb{F}^*\left(\upsilon^*\left(\varrho(\lambda_{n-1},\lambda_{n-2})\right)\right) - 2\Gamma$$
$$\vdots$$
$$\leq \mathbb{F}^*\left(\upsilon^*\left(\varrho(\lambda_1,\lambda_0)\right)\right) - n\Gamma.$$

On letting $n\to\infty$, it follows that $\lim_{n\to\infty}\mathbb{F}^*(\upsilon^*(\varrho(\lambda_{n+1},\lambda_n))) = -\infty$.

By using $(\mathbb{F}_2)$, we have $\lim_{n\to\infty}\upsilon^*(\varrho(\lambda_{n+1},\lambda_n)) = 0$.

This implies that $\upsilon^*(\lim_{n\to\infty}\varrho(\lambda_{n+1},\lambda_n)) = 0$.

Hence, by applying (υ_3), we have $\lim_{n\to\infty}\varrho(\lambda_{n+1},\lambda_n) = 0$.

Now, if $\{\lambda_n\}$ is not Cauchy, then by Lemma 1.4 of [2], there exist $\varsigma > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $\varrho(\lambda_{m_k},\lambda_{n_k}) \geq \varsigma$ and $\varrho(\lambda_{m_k-1},\lambda_{n_k}) < \varsigma$ and
$\lim_{k\to\infty}\varrho(\lambda_{m_k},\lambda_{n_k}) = \varsigma$,
$\lim_{k\to\infty}\varrho(\lambda_{m_k-1},\lambda_{n_k-1}) = \varsigma$ and
$\lim_{k\to\infty}\varrho(\lambda_{m_k-1},\lambda_{n_k}) = \varsigma$.

By taking $\lambda = \lambda_{m_k}$ and $\wp = \lambda_{n_k}$ in (2.1), we have
$$\Gamma + \mathbb{F}^*(\upsilon^*(\varrho(\lambda_{m_k},\lambda_{n_k}))) = \Gamma + \mathbb{F}^*(\upsilon^*(\varrho(\Lambda\lambda_{m_k-1},\Lambda\lambda_{n_k-1}))) \leq \mathbb{F}^*(\upsilon^*(\varrho(\lambda_{m_k-1},\lambda_{n_k-1}))).$$

Since $\mathbb{F}^*$ and υ^* are continuous and on letting $k\to\infty$, we have
$$\Gamma + \mathbb{F}^*(\upsilon^*(\varsigma)) \leq \mathbb{F}^*(\upsilon^*(\varsigma)),$$
a contradiction.

Therefore $\{\lambda_n\}$ is Cauchy.

Since Ξ is complete, we have $\lim_{n\to\infty}\lambda_n = \lambda^*$, for some λ^* in Ξ.

Since Λ is continuous, we have
$$\lambda^* = \lim_{n\to\infty}\lambda_{n+1} = \lim_{n\to\infty}\Lambda\lambda_n = \Lambda(\lim_{n\to\infty}\lambda_n) = \Lambda\lambda^*$$
Therefore $\Lambda\lambda^* = \lambda^*$.

Suppose that $\Lambda\wp^* = \wp^*$ and $\lambda^* \neq \wp^*$.

We now consider
$$\mathbb{F}^*\left(\upsilon^*\left(\varrho(\lambda^*,\wp^*)\right)\right) = \mathbb{F}^*(\upsilon^*(\varrho(\Lambda\lambda^*,\Lambda\wp^*))) < \Gamma + \mathbb{F}^*(\upsilon^*(\varrho(\Lambda\lambda^*,\Lambda\wp^*))) \leq \mathbb{F}^*(\upsilon^*(\varrho(\lambda^*,\wp^*))),$$
a contradiction.

Therefore $\lambda^* = \wp^*$.

Hence λ^* is the unique fixed point of Λ.

This completes the proof of the theorem.

In the following, we show the importance of $\upsilon^* \in \Upsilon^*$ in Theorem 2.1.

Example 2.2. Let $\Xi = \{1,2,3,\ldots\}$ with the usual metric. We define $\Lambda:\Xi\to\Xi$ by $\Lambda\lambda = \lambda^2$. We define $\upsilon^* \in \Upsilon^*$ by $\upsilon^*(\jmath) = \begin{cases} \jmath^2, & 0 \leq \jmath \leq 1 \\ \frac{1}{\jmath}, & \jmath \geq 1 \end{cases}$ and $\mathbb{F}^* \in \mathcal{G}^*$ by $\mathbb{F}^*(\iota) = -\frac{1}{\iota} + \ln\iota + \iota, \iota > 0$. We choose $\Gamma = 1 > 0$. For this Γ, we consider
$$\Gamma + \mathbb{F}^*\left(\upsilon^*\left(\varrho(\Lambda\lambda,\Lambda\wp)\right)\right) = 1 + \mathbb{F}^*(\upsilon^*(|\lambda^2 - \wp^2|)) = 1 + \mathbb{F}^*\left(\frac{1}{|\lambda^2-\wp^2|}\right)$$

$= 1 - |\curlywedge^2 - \wp^2| + \ln\left(\frac{1}{|\curlywedge^2-\wp^2|}\right) + \frac{1}{|\curlywedge^2-\wp^2|}$
$\leq 1 - 2|\curlywedge - \wp| + \ln\left(\frac{1}{|\curlywedge-\wp|}\right) + \frac{1}{|\curlywedge-\wp|}$
$= 1 - |\curlywedge - \wp| - |\curlywedge - \wp| + \ln\left(\frac{1}{|\curlywedge-\wp|}\right) + \frac{1}{|\curlywedge-\wp|}$
$\leq -|\curlywedge - \wp| + \ln\left(\frac{1}{|\curlywedge-\wp|}\right) + \frac{1}{|\curlywedge-\wp|}$
$= \mathbb{F}^*\left(\frac{1}{|\curlywedge-\wp|}\right)$
$= \mathbb{F}^*(\Upsilon^*(\varrho(\curlywedge, \wp)))$.

Thus Λ satisfies the inequality (2.1), and satisfies the hypotheses of Theorem 2.1 and '1' is the unique fixed point of Λ.

If $\Upsilon^*(\jmath) = \jmath$ in the inequality (2.1), we have

$\Gamma + \mathbb{F}^*(\varrho(\Lambda\curlywedge, \Lambda\wp)) = \Gamma + \mathbb{F}^*(|\curlywedge^2 - \wp^2|)$
$= \Gamma - \frac{1}{|\curlywedge^2-\wp^2} + \ln(|\curlywedge^2 - \wp^2) + |\curlywedge^2 - \wp^2|$
$\nleq -\frac{1}{|\curlywedge-\wp|} + \ln(|\curlywedge - \wp|) + |\curlywedge - \wp|$
$= \mathbb{F}^*(\varrho(\curlywedge, \wp))$, so that Λ fails to satisfy the inequality (1.1) and hence Theorem 1.1 is not applicable.

Therefore Theorem 2.1 generalizes Wardowski's theorem, Theorem 1.1 .

Definition 2.2. Let Λ be a selfmap on a metric space (Ξ, ϱ). If there exist $\Upsilon^* \in \Upsilon^*$, $\mathbb{F}^* \in \mathcal{G}^*$ and $\Gamma > 0$ such that $\varrho(\Lambda\curlywedge, \Lambda\wp) > 0$ implies that

(2.2) $\quad \Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(\Lambda\curlywedge, \Lambda\wp))) \leq \mathbb{F}^*(m_{\Upsilon^*}(\curlywedge, \wp))$

where $m_{\Upsilon^*}(\curlywedge, \wp) = \max\{\Upsilon^*(\varrho(\curlywedge, \wp)),$ *ϒ*ϱ⅄,Λ⅄, ϒ*(ϱ(℘,Λ℘))*, for all *⅄,℘∈Ξ*, then we say that Λ is an $(\mathbb{F}^*, \Upsilon^*)$-*weak contraction*.

If Υ^* is the identity map, then we call Λ is an $\mathbb{F}^*$-weak contraction.

Here we note that every $(\mathbb{F}^*, \Upsilon^*)$-contraction is a $(\mathbb{F}^*, \Upsilon^*)$-weak contraction. But its converse is not true due to the following example.

Example 2.3. Let $\Xi = [0,1]$ with the usual metric. We define $\Lambda: \Xi \to \Xi$ by $\Lambda\curlywedge = \begin{cases} \frac{1}{2} & if\ \curlywedge \in [0,1) \\ \frac{1}{4} & if\ \curlywedge = 1 \end{cases}$. We define $\Upsilon^* \in \Upsilon^*$ by $\Upsilon^*(\jmath) = \begin{cases} \jmath^2, & 0 \leq \jmath \leq 1 \\ \frac{1}{\jmath}, & \jmath \geq 1 \end{cases}$ and we define $\mathbb{F}^* \in \mathcal{G}^*$ by $\mathbb{F}^*(\iota) = -\frac{1}{\sqrt{\iota}} + \ln\iota, \iota > 0$. We choose $\Gamma = 2\ln 3 > 0$. For this Γ, we verify that Λ satisfies the inequality (2.2). Let $\curlywedge \in [0,1)$ and $\wp = 1$. We now consider,

$\Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(\Lambda\curlywedge, \Lambda\wp)))$
$= 2\ln 3 + \mathbb{F}^*\left(\Upsilon^*\left(\varrho\left(\frac{1}{2}, \frac{1}{4}\right)\right)\right)$
$= 2\ln 3 + \mathbb{F}^*\left(\Upsilon^*\left(\left|\frac{1}{2} - \frac{1}{4}\right|\right)\right)$
$= 2\ln 3 + \mathbb{F}^*\left(\left(\frac{1}{4}\right)^2\right)$
$= 2\ln 3 - \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2}} + \ln\left(\frac{1}{4}\right)^2$
$= \ln 3^2 - 4 + \ln\frac{1}{16}$
$= \ln 9 - 4 + \ln 1 - \ln 16$
$\leq -\frac{4}{3} + \ln 9 - \ln 16$
$\leq \begin{cases} \mathbb{F}^*(\Upsilon^*(\varrho(\wp, \Lambda\wp))) & if\ \curlywedge \geq \frac{1}{4} \\ \mathbb{F}^*(\Upsilon^*(\varrho(\curlywedge, \wp))) & if\ \curlywedge \leq \frac{1}{4} \end{cases}$
$\leq \mathbb{F}^*(\max\{\Upsilon^*(\varrho(\curlywedge, \wp)), \Upsilon^*(\varrho(\curlywedge, \Lambda\curlywedge)), \Upsilon^*(\varrho(\wp, \Lambda\wp))\}$ for all $\curlywedge, \wp \in \Xi$
$= \mathbb{F}^*(m_{\Upsilon^*}(\curlywedge, \wp))$.

Therefore Λ is a $(\mathbb{F}^*, \Upsilon^*)$-weak contraction.

But its converse is not true. For, if $\curlywedge = \frac{3}{4}$ and $\wp = 1$, then

$\Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(\Lambda\curlywedge, \Lambda\wp))) = \Gamma + \mathbb{F}^*\left(\left(\frac{1}{4}\right)^2\right)$
$\nleq \mathbb{F}^*\left(\left(\frac{1}{4}\right)^2\right)$
$= \mathbb{F}^*(\Upsilon^*(\curlywedge, \wp))$, for any $\Gamma > 0$, $\mathbb{F}^* \in \mathcal{G}^*$ and $\Upsilon^* \in \Upsilon^*$.

Hence Λ is not a $(\mathbb{F}^*, \Upsilon^*)$-contraction. In fact, Λ is not continuous so that Λ is not an $\mathbb{F}$-contraction also.

The following is an extension of Theorem 1.2 by $\Upsilon^* \in \Upsilon^*$, in which we used $\mathbb{F}^* \in \mathcal{G}^*$.

Theorem 2.2. Let (Ξ, ϱ) be a complete metric space. Let $\Lambda: \Xi \to \Xi$ be an $(\mathbb{F}^*, \Upsilon^*)$-weak contraction and $\mathbb{F}^*$ is continuous. Let $\curlywedge_0 \in \Xi$. We define $\{\curlywedge_n\}$ in Ξ by $\curlywedge_{n+1} = \Lambda\curlywedge_n$, for $n = 0, 1, 2, \ldots$. Then Λ has a unique fixed point $\curlywedge^* \in \Xi$. Moreover, Λ is continuous at the fixed point $\curlywedge^*$ if and only if $\lim_{n\to\infty} m(\curlywedge_n, \curlywedge^*) = 0$.

Proof. Let $⅄_0 \in \Xi$. We define the sequence $⅄_{n+1} = \Lambda ⅄_n$ for $= 0, 1, 2, \dots$.

We assume that $⅄_n \neq ⅄_{n+1}$, for every $n \in \mathbb{N}$.

By taking $⅄ = ⅄_n$ and $\wp = ⅄_{n-1}$ in (2.2), we have

$$\Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(\Lambda ⅄_n, \Lambda ⅄_{n-1}))) \leq \mathbb{F}^*(m_{\Upsilon^*}(⅄_n, ⅄_{n-1}))$$

where

$$\begin{aligned} m_{\Upsilon^*}(⅄_n, ⅄_{n-1}) &= \max\{\Upsilon^*\left(\varrho(⅄_n, ⅄_{n-1})\right), \Upsilon^*\left(\varrho(⅄_n, \Lambda ⅄_n)\right), \Upsilon^*\left(\varrho(⅄_{n-1}, \Lambda ⅄_{n-1})\right)\} \\ &= \max\{\Upsilon^*\left(\varrho(⅄_n, ⅄_{n-1})\right), \Upsilon^*\left(\varrho(⅄_n, ⅄_{n+1})\right), \Upsilon^*\left(\varrho(⅄_{n-1}, ⅄_n)\right)\}. \\ &= \max\{\Upsilon^*\left(\varrho(⅄_n, ⅄_{n-1})\right), \Upsilon^*\left(\varrho(⅄_{n+1}, ⅄_n)\right)\}. \end{aligned}$$

Let $m_{\Upsilon^*}(⅄_n, ⅄_{n-1}) = \Upsilon^*\left(\varrho(⅄_{n+1}, ⅄_n)\right)$, then we have

$$\Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄_{n+1}, ⅄_n)\right)\right) \leq \mathbb{F}^*(\Upsilon^*(\varrho(⅄_{n+1}, ⅄_n))),$$

a contradiction.

Therefore $m_{\Upsilon^*}(⅄_n, ⅄_{n-1}) = \Upsilon^*\left(\varrho(⅄_n, ⅄_{n-1})\right)$.

Thus, we have

$$\begin{aligned} \Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄_{n+1}, ⅄_n)\right)\right) &= \Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\Lambda ⅄_n, \Lambda ⅄_{n-1})\right)\right) \\ &\leq \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄_n, ⅄_{n-1})\right)\right) \end{aligned}$$

and hence

$$\begin{aligned} \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄_{n+1}, ⅄_n)\right)\right) &\leq \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄_n, ⅄_{n-1})\right)\right) - \Gamma \\ &\leq \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄_{n-1}, ⅄_{n-2})\right)\right) - 2\Gamma \\ &\vdots \\ &\leq \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄_1, ⅄_0)\right)\right) - n\Gamma. \end{aligned}$$

On letting $n \to \infty$, it follows that $\lim_{n\to\infty} \mathbb{F}^*(\Upsilon^*(\varrho(⅄_{n+1}, ⅄_n))) = -\infty$.

By using $(\mathbb{F}_2)$, we have $\lim_{n\to\infty} \Upsilon^*(\varrho(⅄_{n+1}, ⅄_n)) = 0$.

This implies that $\Upsilon^*(\lim_{n\to\infty} \varrho(⅄_{n+1}, ⅄_n)) = 0$.

Hence, by applying (Υ_3), we have $\lim_{n\to\infty} \varrho(⅄_{n+1}, ⅄_n) = 0$.

If $\{⅄_n\}$ is not Cauchy, then by Lemma 1.4 of [2], there exist $\varsigma > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $\varrho(⅄_{m_k}, ⅄_{n_k}) \geq \varsigma$ and $\varrho\left(⅄_{m_k-1}, ⅄_{n_k}\right) < \varsigma$ and $\lim_{k\to\infty} \varrho(⅄_{m_k}, ⅄_{n_k}) = \varsigma$, $\lim_{k\to\infty} \varrho(⅄_{m_k-1}, ⅄_{n_k-1}) = \varsigma$ and $\lim_{k\to\infty} \varrho\left(⅄_{m_k-1}, ⅄_{n_k}\right) = \varsigma$.

By taking $⅄ = ⅄_{m_k}$ and $\wp = ⅄_{n_k}$ in (2.2), we have

$$\begin{aligned} \Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(⅄_{m_k}, ⅄_{n_k}))) &= \Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(\Lambda ⅄_{m_k-1}, \Lambda ⅄_{n_k-1}))) \\ &\leq \mathbb{F}^*\left(m_{\Upsilon^*}\left(⅄_{m_k-1}, ⅄_{n_k-1}\right)\right) \end{aligned}$$

where

$$\begin{aligned} m_{\Upsilon^*}\left(⅄_{m_k-1}, ⅄_{n_k-1}\right) = \max\{&\Upsilon^*(\varrho(⅄_{m_k-1}, ⅄_{n_k-1})), \\ &\Upsilon^*(\varrho(⅄_{m_k-1}, \Lambda ⅄_{m_k-1})), \\ &\Upsilon^*\left(\varrho\left(⅄_{n_k-1}, \Lambda ⅄_{n_k-1}\right)\right)\}. \end{aligned}$$

On letting $k \to \infty$, we have $\lim_{k\to\infty} m_{\Upsilon^*}\left(⅄_{m_k-1}, ⅄_{n_k-1}\right) = \Upsilon^*(\varsigma)$.

Since $\mathbb{F}^*$ and Υ^* are continuous and on letting $k \to \infty$, we have

$$\Gamma + \mathbb{F}^*(\Upsilon^*(\varsigma)) \leq \mathbb{F}^*(\Upsilon^*(\varsigma)),$$

a contradiction.

Therefore $\{⅄_n\}$ is Cauchy.

Since Ξ is complete, $\lim_{n\to\infty} ⅄_n = ⅄^*$, for some $⅄^*$ in Ξ.

We now show that $\Lambda ⅄^* = ⅄^*$.

If $⅄_n = \Lambda ⅄^*$ for infinitely many n, then there exists a subsequence $\{⅄_{n_k}\}$ of $\{⅄_n\}$ which converges to $\Lambda ⅄^*$.

Therefore $\lim_{k\to\infty} ⅄_{n_k} = \Lambda ⅄^*$.

That is, $⅄* = \Lambda ⅄*$.

If $⅄_n = \Lambda ⅄^*$ for finitely many n, then $\varrho(⅄_n, \Lambda ⅄^*) > 0$ for infinitely many n.

Hence, there exists a subsequence $\left\{⅄_{n_k}\right\} \subseteq \{⅄_n\}$ such that $\varrho\left(⅄_{n_k}, \Lambda ⅄^*\right) > 0$ for all $= 1, 2, \dots$.

Now, using the inequality (2.2), we have, for all k,

$$\Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho\left(⅄_{n_k}, \Lambda ⅄^*\right)\right)\right) \leq \mathbb{F}^*\left(m_{\Upsilon^*}\left(⅄_{n_k-1}, ⅄^*\right)\right),$$

where

$$\begin{aligned} m_{\Upsilon^*}\left(⅄_{n_k-1}, ⅄^*\right) = \max\{&\Upsilon^*\left(\varrho\left(⅄_{n_k-1}, ⅄^*\right)\right), \\ &\Upsilon^*\left(\varrho\left(⅄_{n_k-1}, ⅄_{n_k}\right)\right), \\ &\Upsilon^*\left(\varrho(⅄^*, \Lambda ⅄^*)\right)\}. \end{aligned}$$

If $\varrho(\Lambda ⅄^*, ⅄^*) > 0$ then $\lim_{k\to\infty} m_{\Upsilon^*}\left(⅄_{n_k-1}, ⅄^*\right) = \Upsilon^*\left(\varrho(\Lambda ⅄^*, ⅄^*)\right)$.

On letting $k \to \infty$,

$$\Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄^*, \Lambda ⅄^*)\right)\right) \leq \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄^*, \Lambda ⅄^*)\right)\right),$$

a contradiction.
Therefore $\Lambda\, ⅄^*=⅄^*$.
Suppose that $\Lambda\wp^* = \wp^*$ and $⅄^* \neq \wp^*$.
We now consider

$$\mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄^*,\wp^*)\right)\right) = \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\Lambda\, ⅄^*,\Lambda\wp^*)\right)\right) < \Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\Lambda\, ⅄^*,\Lambda\wp^*)\right)\right) \leq \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄^*,\wp^*)\right)\right),$$

a contradiction.
Therefore $⅄^*= \wp^*$.
Hence $⅄^*$ is the unique fixed point of Λ.
First we assume that Λ is continuous at its fixed point $⅄^*$.
Let $\{⅄_n\} \subset \Xi$ such that $⅄_n \to ⅄^*$ as $n \to \infty$.
Then we have $\Lambda\, ⅄_n \to \Lambda\, ⅄^*=⅄^*$.
Therefore

(2.3) $\quad \lim_{n\to\infty} \varrho(⅄_n,\Lambda\, ⅄_n) = 0.$

We have

$$m_{\Upsilon^*}(⅄_n, ⅄^*) = \max\{\Upsilon^*\left(\varrho(⅄_n, ⅄^*)\right), \Upsilon^*\left(\varrho(⅄_n,\Lambda\, ⅄_n)\right), \Upsilon^*\left(\varrho(⅄^*,\Lambda\, ⅄^*)\right)\}.$$

On letting $n \to \infty$, we have

$$\lim_{n\to\infty} m_{\Upsilon^*}(⅄_n, ⅄^*) = \max\{\lim_{n\to\infty}\Upsilon^*\left(\varrho(⅄_n, ⅄^*)\right), \lim_{n\to\infty}\Upsilon^*\left(\varrho(⅄_n,\Lambda\, ⅄_n)\right), \Upsilon^*\left(\varrho(⅄^*,\Lambda\, ⅄^*)\right)\}.$$

Therefore, by using (2.3), we have

$$\lim_{n\to\infty} m_{\Upsilon^*}(⅄_n, ⅄^*) = \max\{\Upsilon^*\left(\varrho(⅄^*, ⅄^*)\right), 0, \Upsilon^*\left(\varrho(⅄^*, ⅄^*)\right)\} = 0.$$

Hence $\lim_{n\to\infty} m_{\Upsilon^*}(⅄_n, ⅄^*) = 0$.
Conversely, suppose that
$\lim_{n\to\infty} m_{\Upsilon^*}(⅄_n, ⅄^*) = 0$.
Then, we have
$\lim_{n\to\infty}\Upsilon^*\left(\varrho(⅄_n, ⅄^*)\right) = 0$,
$\lim_{n\to\infty}\Upsilon^*\left(\varrho(⅄_n,\Lambda\, ⅄_n)\right) = 0$ and
$\Upsilon^*\left(\varrho(⅄^*,\Lambda\, ⅄^*)\right) = 0$.
Since $\Upsilon^* \in \Upsilon^*$, by (Υ_3), it follows that $\lim_{n\to\infty} \varrho(⅄_n, ⅄^*) = 0$, $\lim_{n\to\infty} \varrho(⅄_n,\Lambda\, ⅄_n) = 0$ and $\Upsilon^*\left(\varrho(⅄^*,\Lambda\, ⅄^*)\right) = 0$ which implies that $\lim_{n\to\infty} \Lambda\, ⅄_n = \lim_{n\to\infty}⅄_n = ⅄^* = \Lambda\, ⅄^*$, so that Λ is continuous at $⅄^*$.
Hence the theorem follows.

Remark 2.1. The selfmap Λ defined on [0,1] in Example 2.3 satisfies all the hypotheses of Theorem 2.2 and $\frac{1}{2}$ is the unique fixed point of Λ.

Corollary 2.1. Let (Ξ,ϱ) be a complete metric space. Let $\Lambda: \Xi \to \Xi$ be an $\mathbb{F}^*$-weak contraction and $\mathbb{F}^*$ is continuous. Let $⅄_0 \in \Xi$. We define $\{⅄_n\}$ in Ξ by $⅄_{n+1} = \Lambda\, ⅄_n$, for $n = 0, 1, 2, \dots$. Then Λ has a unique fixed point $⅄^* \in \Xi$. Moreover, Λ is continuous at the fixed point $⅄^*$ if and only if $\lim_{n\to\infty} m(⅄_n, ⅄^*) = 0$.
Proof. By choosing $\Upsilon^*(\jmath) = \jmath, \jmath \geq 0$ in (2.2), the conclusion of this corollary follows from of Theorem 2.2.

Theorem 2.3. Let (Ξ,ϱ) be a metric space. Let $\Lambda: \Xi \to \Xi$ be an $(\mathbb{F}^*,\Upsilon^*)$-weak contraction and $\mathbb{F}^*$ is continuous. Suppose that for some $⅄_0$ in Ξ, the sequence $\{\Lambda^n ⅄_0\}$ has a cluster point æ in Ξ. If Λ is continuous then æ is the unique fixed point of Λ and the sequence $\{\Lambda^n ⅄_0\}$ converges to æ.
Proof. Let $⅄_0 \in \Xi$. We define $⅄_n = \Lambda^n ⅄_0$, $n = 1, 2, \dots$.
If $⅄_{n+1} = ⅄_n$ for some n, then $\Lambda\, ⅄_n = ⅄_n$, and we are through.
Hence, we suppose that $\Lambda\, ⅄_{n+1} \neq \Lambda\, ⅄_n$.
Then, by applying the inequality (2.2), we have

$$\Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄_{n+1}, ⅄_{n+2})\right)\right) = \Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(\Lambda\, ⅄_n,\Lambda\, ⅄_{n+1}))) \leq \mathbb{F}^*(m_{\Upsilon^*}(⅄_n, ⅄_{n+1}))$$

where

$$m_{\Upsilon^*}(⅄_n,⅄_{n+1}) = \max\{\Upsilon^*\left(\varrho(⅄_n, ⅄_{n+1})\right), \Upsilon^*\left(\varrho(⅄_n,\Lambda\, ⅄_n)\right), \Upsilon^*(\varrho(⅄_{n+1},\Lambda\, ⅄_{n+1}))\} = \max\{\Upsilon^*\left(\varrho(⅄_n, ⅄_{n+1})\right), \Upsilon^*\left(\varrho(⅄_n, ⅄_{n+1})\right), \Upsilon^*\left(\varrho(⅄_{n+1}, ⅄_{n+2})\right)\}. = \Upsilon^*\left(\varrho(⅄_{n+1}, ⅄_{n+2})\right)$$

and hence

$$\Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄_{n+1}, ⅄_{n+2})\right)\right) \leq \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄_{n+1}, ⅄_{n+2})\right)\right),$$

a contradiction.
Therefore, $⅄_{n+1} = ⅄_{n+2}$ and hence $⅄_{n+2} = ⅄_{n+1} = ⅄_n$.
In general, it follows that $⅄_m = ⅄_n$ for all $m \geq n$.
Therefore, $\lim_{m\to\infty}⅄_m = ⅄_n =$ æ, and the conclusion of the theorem holds.

Hence, we suppose that $\lambda_{n+1} \neq \lambda_n$ for all n.
Then, by taking $\lambda = \lambda_{n+1}$ and $\wp = \lambda_n$ in the inequality (2.2), we have

$$\Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(\Lambda \lambda_{n+1}, \Lambda \lambda_n))) \leq \mathbb{F}^*(m_{\Upsilon^*}(\lambda_{n+1}, \lambda_n))$$

where

$$\begin{aligned} m_{\Upsilon^*}(\lambda_{n+1}, \lambda_n) &= \max\{\Upsilon^*\left(\varrho(\lambda_n, \lambda_{n+1})\right), \Upsilon^*\left(\varrho(\lambda_n, \Lambda \lambda_n)\right), \Upsilon^*(\varrho(\lambda_{n+1}, \Lambda \lambda_{n+1}))\} \\ &= \max\{\Upsilon^*\left(\varrho(\lambda_n, \lambda_{n+1})\right), \Upsilon^*\left(\varrho(\lambda_{n+1}, \lambda_{n+2})\right), \Upsilon^*\left(\varrho(\lambda_n, \lambda_{n+1})\right)\}. \\ &= \max\{\Upsilon^*\left(\varrho(\lambda_{n+1}, \lambda_n)\right) \Upsilon^*\left(\varrho(\lambda_{n+1}, \lambda_{n+2})\right)\}. \end{aligned}$$

If $m_{\Upsilon^*}(\lambda_{n+1}, \lambda_n) = \Upsilon^*\left(\varrho(\lambda_{n+2}, \lambda_{n+1})\right)$, then

$$\Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(\lambda_{n+2}, \lambda_{n+1}))) \leq \mathbb{F}^*(\Upsilon^*(\varrho(\lambda_{n+2}, \lambda_{n+1}))),$$

a contradiction.
Therefore, $m_{\Upsilon^*}(\lambda_{n+1}, \lambda_n) = \Upsilon^*\left(\varrho(\lambda_{n+1}, \lambda_n)\right)$ and hence

$$\Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(\lambda_{n+2}, \lambda_{n+1}))) \leq \mathbb{F}^*(\Upsilon^*(\varrho(\lambda_{n+1}, \lambda_n))) \text{ for all } n.$$

Now

$$\begin{aligned} \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\lambda_{n+1}, \lambda_n)\right)\right) &\leq \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\lambda_n, \lambda_{n-1})\right)\right) - \Gamma \\ &\leq \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\lambda_{n-1}, \lambda_{n-2})\right)\right) - 2\Gamma \\ &\vdots \\ &\leq \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\lambda_1, \lambda_0)\right)\right) - n\Gamma. \end{aligned}$$

On letting $n \to \infty$, it follows that
$\lim_{n\to\infty} \mathbb{F}^*(\Upsilon^*(\varrho(\lambda_{n+1}, \lambda_n))) = -\infty$.
By using $(\mathbb{F}_2)$, we have
$\lim_{n\to\infty} \Upsilon^*(\varrho(\lambda_{n+1}, \lambda_n)) = 0$.
This implies that $\Upsilon^*(\lim_{n\to\infty} \varrho(\lambda_{n+1}, \lambda_n)) = 0$ and hence, by (Υ_3), we have

(2.4) $\quad \lim_{n\to\infty} \varrho(\lambda_{n+1}, \lambda_n) = 0$.

Since $\{\Lambda^n \lambda_0\}$ has a cluster point æ in Ξ, there exists a subsequence $\{\Lambda^{n_k} \lambda_0\}$ of $\{\Lambda^n \lambda_0\}$ such that the sequence $\{\Lambda^{n_k} \lambda_0\}$ converges to æ (say) in Ξ.
Now, from (2.4), $\lim_{k\to\infty} \varrho(\lambda_{n_{k+1}}, \lambda_{n_k}) = 0$.
Therefore

(2.5) $\quad \lim_{k\to\infty} \lambda_{n_{k+1}} = \lim_{k\to\infty} \lambda_{n_k} = \text{æ}$.

Now, by the continuity of Λ, it follows that

$$\lim_{k\to\infty} \lambda_{n_{k+1}} = \lim_{k\to\infty} \Lambda(\lambda_{n_k}) = \Lambda\left(\lim_{k\to\infty} \lambda_{n_k}\right) = \Lambda\text{æ}$$

, and hence by (2.5), we have $\Lambda\text{æ} = \text{æ}$.
As in the proof of Theorem 2.2, it is easy to see that the sequence $\{\lambda_n\}$ is Cauchy.
Since this sequence $\{\lambda_n\}$ has a subsequence that converges to æ, it follows that the sequence $\{\lambda_n\}$ converges to æ.

Hence the theorem follows.

III. A fixed point theorem in orbits by using generalized altering distance function

Let Λ be a selfmap on a nonempty set Ξ. For $\lambda_0 \in \Xi$, $O(\lambda_0) = \{\Lambda^n \lambda_0; n = 0, 1, 2, \ldots\}$ is called the *orbit* of λ_0, where $\Lambda^0 = I$, I the identity map of Ξ.

Definition 3.1. [9] A metric space Ξ is said to be Λ-*orbitally complete* if every Cauchy sequence which is contained in $O(\lambda)$ for all λ in Ξ converges to a point of Ξ.

Definition 3.2. [9] A selfmap Λ of a metric space Ξ is said to be *orbitally continuous* at a point $\imath$ in Ξ if for any sequence $\{\lambda_n\} \subseteq O(\lambda)$, $\lambda \in \Xi$, $\lambda_n \to \imath$ as $n \to \infty$ implies $\Lambda \lambda_n \to \Lambda\imath$ as $n \to \infty$.

Motivated by the works of Ćirić **[3]**, Sastry and Babu **[8]** on the existence of fixed points in orbits, we prove the following.

Theorem 3.1. Let Λ be a selfmap of a metric space (Ξ, ϱ). Suppose that there exists a point λ_0 in Ξ such that the orbit $O(\lambda_0)$ has a cluster point $\imath$ in Ξ. If there exist $\Upsilon^* \in \Upsilon^*$, $\mathbb{F}^* \in \mathcal{G}^*$ and $\Gamma > 0$ such that $\varrho(\Lambda\lambda, \Lambda\wp) > 0$ implies that

(3.1) $$\Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\Lambda\lambda, \Lambda\wp)\right)\right) \leq \mathbb{F}^*(\Upsilon^*(\varrho(\lambda, \wp))$$

for each $\lambda, \wp \in \overline{O(\lambda_0)}$ and if Λ is orbitally continuous at $\imath$ then $\imath$ is a fixed point of Λ in Ξ.

Proof. Let $\lambda_0 \in \Xi$. We now define the sequence $\{\lambda_n\}$ by $\lambda_{n+1} = \Lambda \lambda_n$ for $= 0, 1, 2, \ldots$.
We assume, without loss of generality, that $\lambda_{n+1} \neq \lambda_n$ for every $n \in \mathbb{N}$.
Let $\iota_n = \Upsilon^*\left(\varrho(\lambda_{n+1}, \lambda_n)\right)$. Then by taking $\lambda = \lambda_n$ and $\wp = \lambda_{n-1}$ in (3.1), we have

$$\Gamma + \mathbb{F}^*(\Upsilon^*(\varrho(\Lambda \lambda_n, \Lambda \lambda_{n-1})))$$

$$\leq \mathbb{F}^*(\Upsilon^*(\varrho(⅄_n, ⅄_{n-1})))$$

which implies that $\Gamma + \mathbb{F}^*(\iota_n) \leq \mathbb{F}^*(\iota_{n-1})$.
Therefore

(3.2) $$\mathbb{F}^*(\iota_n) \leq \mathbb{F}^*(\iota_{n-1}) - \Gamma \leq \mathbb{F}^*(\iota_{n-2}) - 2\Gamma \;\vdots\; \leq \mathbb{F}^*(\iota_0) - n\Gamma.$$

From (3.2), we obtain that

(3.3) $$\lim_{n\to\infty} \mathbb{F}^*(\iota_n) = -\infty.$$

Now by $(\mathbb{F}_2)$, we have

(3.4) $$\lim_{n\to\infty} \iota_n = 0.$$

Let $n(k)$ be a subsequence of positive integers such that $\{⅄_{n(k)}\}$ converges to ı.
Then $\{\iota_{n(k)}\}$ converges to 0.
By the continuity of Υ^* and orbital continuity of Λ at ı, we have

$$0 = \lim_{k\to\infty} \iota_{n(k)} = \lim_{k\to\infty} \Upsilon^*\left(\varrho\left(⅄_{n(k)}, ⅄_{n(k)+1}\right)\right) = \Upsilon^*(\varrho(ı, Λı)).$$

Thus, by the property (Υ_3) of Υ^*, we have Λı = ı.

Corollary 3.1. Let Λ be a selfmap of a metric space (Ξ, ϱ). Suppose that there exists a point $⅄_0 \in \Xi$ such that the orbit $O(⅄_0)$ has a cluster point ı in Ξ. If there exist $\mathbb{F}^* \in \mathcal{G}^*$ and $\Gamma > 0$ such that $\varrho(Λ⅄, Λ℘) > 0$ implies that

$$\Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(Λ⅄, Λ℘)\right)\right) \leq \mathbb{F}^*(\Upsilon^*(\varrho(⅄, ℘)))$$

for each $⅄, ℘ \in \overline{O(⅄_0)}$ and if Λ is orbitally continuous at ı then ı is a fixed point of Λ.

Proof. Follows by choosing $\Upsilon^*(\jmath) = \jmath, \jmath \geq 0$ in the inequality (3.1), the conclusion of this corollary holds from Theorem 3.1.

Example 3.1. Let $\Xi = \{0,1,2\} \cup \left\{1 + \frac{1}{2(n+1)}; n = 1, 2, \ldots\right.$ with the usual metric. Define Λ:Ξ→Ξ by

$$Λ⅄ = \begin{cases} 2 \; if \; ⅄ = 0 \\ 1 + \frac{1}{2(n+2)} \; if \; ⅄ = 1 + \frac{1}{2(n+1)}, n = 1, 2, \ldots \\ 1 \; if \; ⅄ = 1 \\ 2 \; if \; ⅄ = 2 \end{cases}.$$

We define $\Upsilon^*(\jmath) = \begin{cases} \jmath^2, & 0 \leq \jmath \leq 1 \\ \frac{1}{\jmath}, & \jmath \geq 1 \end{cases}$. Then $\Upsilon^* \in \boldsymbol{\Upsilon}^*$.

$\mathbb{F}^* \in \mathcal{G}^*$ is defined by $\mathbb{F}^*(\iota) = -\frac{1}{\sqrt{\iota}} + \ln \iota, \iota > 0$.

We choose $\Gamma = 2\ln 2 > 0$. Let $⅄_0 = 1 + \frac{1}{4}$,

$$O(⅄_0) = \left\{1 + \frac{1}{4}, 1 + \frac{1}{6}, 1 + \frac{1}{8}, \ldots, 1 + \frac{1}{2(n+1)}, \ldots\right\},$$

$O(⅄_0)$ has a cluster point ı, and
$\overline{O(⅄_0)} = O(⅄_0) \cup \{1\}$.
We now verify the inequality (3.1).

Case (i): Let $⅄ = 1 + \frac{1}{2(n+1)}, ℘ = 1$.

We now consider

$$\begin{aligned}
&\Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(Λ⅄, Λ℘)\right)\right) \\
&= 2\ln 2 + \mathbb{F}^*\left(\Upsilon^*\left(\varrho\left(1 + \frac{1}{2(n+2)}, 1\right)\right)\right) \\
&= 2\ln 2 + \mathbb{F}^*\left(\Upsilon^*\left(\frac{1}{2(n+2)}\right)\right) \\
&= 2\ln 2 + \mathbb{F}^*\left(\left(\frac{1}{2(n+2)}\right)^2\right) \\
&= 2\ln 2 - \frac{1}{\sqrt{\left(\frac{1}{2(n+2)}\right)^2}} + \ln\left(\frac{1}{2(n+2)}\right)^2 \\
&= 2\ln 2 - 2(\mathrm{n}+2) + \ln\left(\frac{1}{2(n+2)}\right)^2 \\
&\leq 2\ln 2 - 2(n+2) + \ln \frac{1}{2(n+2)} \\
&= 2\ln 2 - 2(n+1) - 2 + \ln \frac{1}{2(n+2)} \\
&\leq -2(n+1) + \ln \frac{1}{2(n+1)} \\
&= \mathbb{F}^*\left(\Upsilon^*\left(\varrho\left(1 + \frac{1}{2(n+1)}, 1\right)\right)\right) \\
&= \mathbb{F}^*\left(\Upsilon^*\left(\varrho(⅄, ℘)\right)\right).
\end{aligned}$$

Case (ii): Let $⅄ = 1 + \frac{1}{2(n+1)}$,

$$℘ = 1 + \frac{1}{2(m+1)}, n > m.$$

We now consider

$$\begin{aligned}
&\Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(Λ⅄, Λ℘)\right)\right) \\
&= 2\ln 2 + \mathbb{F}^*\left(\Upsilon^*\left(\varrho\left(1 + \frac{1}{2(n+2)}, 1 + \frac{1}{2(m+2)}\right)\right)\right) \\
&= 2\ln 2 + \mathbb{F}^*\left(\Upsilon^*\left(\frac{1}{2(n+2)} - \frac{1}{2(m+2)}\right)\right) \\
&= 2\ln 2 + \mathbb{F}^*\left(\Upsilon^*\left(\frac{2(m+2) - 2(n+2)}{4(n+2)(m+2)}\right)\right) \\
&= 2\ln 2 + \mathbb{F}^*\left(\Upsilon^*\left(\frac{|m-n|}{2(n+2)(m+2)}\right)\right) \\
&= 2\ln 2 + \mathbb{F}^*\left(\left(\frac{n-m}{2(n+2)(m+2)}\right)^2\right) \\
&= 2\ln 2 - \frac{1}{\sqrt{\left(\frac{n-m}{2(n+2)(m+2)}\right)^2}} + \ln\left(\frac{n-m}{2(n+2)(m+2)}\right)^2
\end{aligned}$$

$$= 2\ln 2 - \frac{2(m+2)(n+2)}{n-m} + \ln\left(\frac{n-m}{2(n+2)(m+2)}\right)^2$$
$$= 2\ln 2 - \frac{2(m+1)(n+1)}{n-m} - \frac{2(m+n+3)}{n-m} + \ln\left(\frac{n-m}{2(n+2)(m+2)}\right)^2$$
$$< -\frac{2(m+1)(n+1)}{n-m} + \ln\left(\frac{n-m}{2(n+2)(m+2)}\right)^2$$
$$\leq -\frac{2(m+1)(n+1)}{n-m} + \ln\left(\frac{n-m}{2(n+1)(m+1)}\right)^2$$
$$= \mathbb{F}^*\left(\Upsilon^*\left(\varrho\left(1+\frac{1}{2(n+1)}, 1+\frac{1}{2(m+1)}\right)\right)\right)$$
$$= \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\lambda, \wp)\right)\right).$$

Thus, from Case (i) and Case (ii), we have Λ satisfies the inequality (3.1). Also, Λ is orbitally continuous at the limit point 1. Thus, Λ satisfies all the hypotheses of Theorem 3.1 and '1' is the unique fixed point of Λ in $\overline{O(\lambda_0)}$.

Here we observe that Λ fails to satisfy the inequality (3.1) on Ξ for any $\Gamma > 0$, $\mathbb{F}^* \in \mathcal{G}^*$ and $\Upsilon^* \in \Upsilon^*$. For, by choosing $\lambda = 0, \wp = 2$ in the inequality (3.1), we have

$$\Gamma + \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\Lambda 0, \Lambda 2)\right)\right) = \Gamma + \mathbb{F}^*\left(\Upsilon^*(2)\right) \nleq \mathbb{F}^*\left(\Upsilon^*(2)\right) = \mathbb{F}^*\left(\Upsilon^*\left(\varrho(\lambda, \wp)\right)\right).$$

Thus Wardowski's theorem, Theorem 1.1, is not applicable. Here we observe that the inequality (1.1) fails to hold even though $\mathbb{F} \in \mathcal{G}$.
So Theorem 3.1 generalizes Wardowski's theorem, Theorem 1.1.

Example 3.2. Let $\Xi = [0,1]$ with the usual metric. We define $\Lambda: \Xi \to \Xi$ by $\Lambda\lambda = \begin{cases} \frac{\lambda}{2} \ if \ \lambda \in \left[0, \frac{1}{2}\right] \\ \lambda \ if \ \lambda \in \left(\frac{1}{2}, 1\right] \end{cases}$.
Let $\lambda_0 = \frac{1}{2}$, then $O(\lambda_0) = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\right\}$ and $\overline{O(\lambda_0)} = O(\lambda_0) \cup \{0\}$. Ξ is Λ-orbitally complete and satisfies the inequality (3.1) with
$\Upsilon^*(\jmath) = \begin{cases} \jmath^2, 0 \leq \jmath \leq 1 \\ \frac{1}{\jmath}, \quad \jmath \geq 1 \end{cases}$, $\Upsilon^* \in \Upsilon^*$;
$\mathbb{F}^*(\iota) = -\frac{1}{\sqrt{\iota}} + \ln\iota, \iota > 0$, $\mathbb{F}^* \in \mathcal{G}^*$ and $\Gamma = 2\ln 2$.
Let $\lambda_0 = \frac{1}{2}$, and $\Gamma = 2 \ln 2$. Also, Λ is orbitally continuous at 0. Hence, Λ satisfies the hypotheses of Theorem 3.1 and '0' is the unique fixed point of Λ on $\overline{O(\lambda_0)}$. But it is not an $\mathbb{F}$-contraction for any $\mathbb{F} \in \mathcal{G}$ and hence Theorem 1.1 is not applicable.

IV. References

[1] Alfaqih. W. M., Imdad, M., and Gubran, R., An observation on F −weak contractions and discontinuity at the fixed point with an application, J. Fixed Point Theory Appl., 22:16, (2020), 10 pages.

[2] Babu, G. V. R., and Sailaja, P. D., A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces, Thai. J. Math., 9(1), (2011), 1-10.

[3] Ćirić, Lj. B., On contraction type mappings, Math. Balkanica 1, (1971), 52-57. ´

[4] Khan, M. S., Swaleh, M., and Sessa, S., Fixed point theorems by altering distances between the points, Bull. Austral. Math. Soc., 30(1), (1984), 1-9.

[5] Naidu, S. V. R., Some fixed point theorems in metric spaces by altering distances, Czechoslovak Math. J., 53(128), (2003), 205-212.

[6] Piri, H., and Kumam, P., Some fixed point theorems concerning F −contraction in complete metric spaces, Fixed Point Theory Appl. 210(2014), (2014), 1-11.

[7] Sastry, K. P. R., and Babu, G. V. R., Fixed points theorems in metric spaces by altering distances, Bull. Cal. Math. Soc., 90, (1998), 175-182.

[8] Sastry, K. P. R., and Babu, G. V. R., Some fixed points theorems by altering distances between the points, Indian J. of Pure and Appl. Math., 30, (1999), 641-647.

[9] Turkoglu, D., Ozer, O., and Fisher, B., Fixed point theorems for T −orbitally complete metric spaces, Mathematica Nr.9, (1999), 211-218.

[10] Wardowski, D., Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 94(2012),

(2012), 6 pages.

[11] Wardowski, D., and Van Dung, N., Fixed points of F −weak contractions on complete metric spaces, Demonstr. Math., 47(1), (2014), 146-155.