# Discussion On Some $w^*s$-$T_k$ Spaces In Topological Spaces

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## Abstract

In this article, we introduce $w^*s$-$T_o$, $w^*s$-$T_1$, $w^*s$-$T_2$ spaces by using $w^*s$-open sets and $w^*s$-symmetric spaces using $w^*s$-closure. Moreover, we investigate various characterizations and properties of these spaces.

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## 1. Introduction

The concept of generalized closed (briefly $g$-closed) sets was introduced by N. Levine[5] in 1970. In 2000, the concept of weakly closed (briefly *w*-closed) sets was introduced by M. Sheik John[6]. In 2017, Veeresha A. Sajjanar[1] introduced the concept of weakly semi closed (briefly *ws*-closed) sets in topological spaces. D. Dhana Lekshmi and T. Shyla Isac Mary introduced weakly star semi closed (briefly *w*s*-closed) sets and some of its properties are examined. Preliminaries needed to introduce this new class of closed sets are given in section 2. In section 3, some $w^*s$-$T_k$ spaces were introduced and studied. Section 4 contains the conclusion and at the end references were included.

## 2. Preliminaries

**Definition 2.1** A space $X$ is a $T_1$-space or Frechet space iff it satisfies the $T_1$ axiom, that is, for each $x, y \in X$ such that $x \neq y$, there is an open set $U \subset X$ so that $x \in U$ but $y \notin U$.

**Definition 2.2** A space $X$ is a $T_2$-space or Hausdorff space iff it satisfies the $T_2$ axiom, that is, for each $x, y \in X$ such that $x \neq y$, there are open sets $U, V \subset X$ so that $x \in U, y \in V$ and $U \cap V \neq \varphi$.

**Theorem 2.3 [3]** Let us assume $f: (X, \tau) \to (Y, \sigma)$ be a function. Then the following three statements are equivalent.

i. $f$ is a $w^*s$-continuous function.
ii. The inverse image of each open set in $(Y, \sigma)$ is a $w^*s$-open set in $(X, \tau)$.
iii. The inverse image of each closed set in (Y,σ) is a $w^*s$-closed set in (X,τ).

**Definition 2.4 [4]** For a subset $A$ of a space $X$, $w^*s$-closure is defined as follows:

$w^*s - cl(A) = \cap \{F: A \subseteq F \text{ and } F \text{ is } w^*s - closed \text{ in } X\}$.

**Remark 2.5** A function $f: (X, \tau) \to (Y, \sigma)$ is a $w^*s$-irresolute function if and only if the inverse image of every $w^*s$-open set in $(Y, \sigma)$ is $w^*s$-open in $(X, \tau)$.

**Definition 2.6 [2]**Let us assume $X$ be a topological space and let $x \in X$. A subset $N$ of $X$ is said to be a $w^*s$-neighborhood (briefly $w^*s$-nbhd) of the point $x$ if there exists a $w^*s$-open set $G$ such that $x \in G \subseteq N$.

**Definition 2.7 [4]** A subset $A$ of a topological space $(X, \tau)$ is called weakly star semi closed (briefly $w^*s$-closed) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $ws$-open.

**Remark 2.8 [4]** $A$ is a $w^*s$-closed set if and only if $w^*s - cl(A) = A$.

## 3. $w^*s$-$T_k$ spaces, where $k \in \{0, 1, 2\}$

**Definition 3.1** If for every pair of disinct points $x$ and $y$ of a topological space $X$, there exists a $w^*s$-open set $G$ such that $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$, then $X$ is said to be $w^*s$-$T_o$.

**Definition 3.2** A space $X$ is said to be $w^*s$-$T_1$ if for every pair of distinct points $x$ and $y$ of $X$, there exist $w^*s$-open sets $U$ and $V$ of $X$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

**Definition 3.3** A topological space $X$ is said to be $w^*s$-$T_2$ if for every pair of distinct points $x$ and $y$ of $X$, there exist disjoint $w^*s$-open sets $U$ and $V$ in $X$ containing $x$ and $y$ respectively.

**Theorem 3.4** A space $X$ is $w^*s$-$T_0$ if and only if $w^*s$-closures of any two distinct points are distinct.

**Proof:** Let us assume $X$ be a $w^*s$-$T_0$space and $x, y \in X$ with $x \neq y$. Since $X$ is a $w^*s$-$T_0$ space, by Definition 3.1, there exist a $w^*s$-open set $G$ such that $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$.

Considering the first case.

Now, $x \notin X\backslash G$ and $y \in X\backslash G$, because $x \in G$ and $y \notin G$.

Also, $X\backslash G$ is a $w^*s$-closed set in $X$.

Since $w^*s$-$cl(\{y\})$ is the intersection of all $w^*s$-closed sets containing $y$, we have $w^*s$-$cl(\{y\}) \subseteq X\backslash G$\$.

Clearly $y \in w^*s - cl(\{y\})$.

Thus $w^*s$-$cl(\{x\}) \neq w^*s - cl(\{y\})$.

Similarly, we prove the second case.

Conversely, assume that $w^*s$-$cl(\{x\}) \neq w^*s - cl(\{y\})$ if $x \neq y$ and $x, y \in X$.

Then there exists atleast one point $z$ of $X$ such that $z \in w^*s - cl(\{x\})$ and $z \notin w^*s - cl(\{y\}$ or $z \in w^*s - cl(\{y\})$ and $z \notin w^*s - cl(\{x\})$.

Considering the first case.

Suppose that $x \in w^*s - cl(\{y\})$.

Thus $w^*s - cl(\{x\}) \subseteq w^*s - cl(\{y\})$.

Therfore $z \in w^*s - cl(\{y\})$, which is a contradiction.

Therefore, $x \notin w^*s - cl(\{y\})$, which implies that $x \in X\backslash w^*s - cl(\{y\}$, which is a $w^*s$ - open set in $X$ containing $x$ but not $y$.

Then by using Definition 3.1, $X$ is a $w^*s$-$T_0$ space.

**Theorem 3.5** Let us consider $f: X \to Y$ be a $w^*s$-irresolute, injective map. If Y is $w^*s$-$T_1$, then $X$ is a $w^*s$-$T_1$ space.

**Proof:** Let us assume that Y is $w^*s$-$T_1$ and let $x, y \in X$ with $x \neq y$. Thus $f(x), f(y) \in Y$ with $f(x) \neq f(y)$. Since $Y$ is $w^*s$-$T_1$, by Definition 3.2, there exist $w^*s$-open sets $U$ and $V$ in $Y$ such that $f(x) \in U$ and $f(y) \notin U$ or $f(y) \in V$ and $f(x) \notin V$. Since $f$ is an injective function, $x \in f^{-1}(U)$ and $y \notin f^{-1}(U)$ or $y \in f^{-1}(V)$ and \$$x \notin f^{-1}(V)$. Since $f$ is a $w^*s$-irresolute function, $f^{-1}(U)$ and $f^{-1}(V)$ are $w^*s$-open in $X$. Then by Definition 3.2, X is $w^*s$-$T_1$.

**Theorem 3.6** Let us consider $f: (X, \tau) \to (Y, \sigma)$ be bijective. If $f$ is a $w^*s$-continuous function and $(Y, \sigma)$ is $T_1$, then $(X, \tau)$ is $w^*s$-$T_1$.

**Proof:** Let us assume $f: (X, \tau) \to (Y, \sigma)$ be bijective.

Suppose $f$ is a $w^*s$-continuous function and $(Y, \sigma)$ is $T_1$.

Let us aasume $x_1, x_2 \in X$ with $x_1 \neq x_2$.

Since $f$ is a bijective function, there exist $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

Since $(Y, \sigma)$ is a $T_1$-space, by Definition 2.1, there exist open sets $U$ and $V$ such that $y_1 \in U$ but $y_1 \notin V$ and $y_2 \in V$ but $y_2 \notin U$.

Since $f$ is a bijective function, $x_1 \in f^{-1}$(U) but $x_1 \notin f^{-1}$(V) and $x_2 \in f^{-1}$(V) but $x_2 \notin f^{-1}$(U).

Since $f$ is $w^*s$-continuous, by using Theorem 2.3, $f^{-1}(U)$ and $f^{-1}(V)$ are $w^*s$-open sets in $(X, \tau)$.

Then by using Definition 3.2, $(X, \tau)$ is $w^*s$-$T_1$.

**Theorem 3.7** Every $w^*s$-$T_2$ space is a $w^*s$-$T_1$ space.

**Proof:** Let us assume $X$ be a $w^*s$-$T_2$ space and let $x$ and $y$ be two distinct points in $X$. Since $X$ is $w^*s$-$T_2$, by Definition 3.3, there exist disjoint $w^*s$-open sets $U$ and $V$ of $X$ containing $x$ and $y$ respectively. Since $U$ and $V$ are disjoint, we have $y \notin U$ and $x \notin V$. Then by Definition 3.2, $X$ is $w^*s$-$T_1$ space.

**Theorem 3.8** For a topological space $X$, the following three statements are equivalent.

i. The space $X$ is a $w^*s$-$T_2$ space.
ii. Let $x$ be an element of the space $X$. Then for each distinct points $x$ and $y$, there exists a $w^*s$-open set $U$ such that $x \in U$ and $y \notin w^*s - cl(U)$.
iii. For each element $x \in X$, $\cap \{(w^*s - cl\, U: U \in W^*SO(X)\ and\ x \in U\} = \{x\}$.

**Proof:**

(i)⟹(ii) Let us assume $X$ be a $w^*s$-$T_2$ space. Then by Definition 3.3, for every pair of distinct points $x$ and $y$, there exists disjoint $w^*s$-open sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$. Since $V$ is a $w^*s$-open set, $X \backslash V$ is a $w^*s$-closed set. Since $U$ and $V$ are disjoint $w^*s$-open sets, $U \subseteq X \backslash V$. Then by Definition 2.4, $w^*s - cl(U) \subseteq X \backslash V$. Since $y \notin X \backslash V$$, we have $y \notin w^*s - cl\, U$.

(ii)⟹(iii) Let us assume $y \neq x$ in $X$. Then by (ii), there exists a $w^*s$-open set $U$ such that $x \in U$ and $y \notin w^*s - cl\, U$. Thus $y \notin \{w^*s - cl\, U: U \in W^*SO(X)\ and\ x \in U\}$. Hence $\cap\ (w^*s - cl\, U: U \in W^*SO(X)\ and\ x \in U\} = \{x\}$.

(iii)⟹(i) Let us assume $y \neq x$ in $X$. Thus $y \notin \{x\} = \cap \{w^*s - cl\, U: U \in W^*SO(X)\ and\ x \in U\}$. Then there exist a $w^*s$-open set $U$ such that $x \in U$ and $y \notin w^*s - cl\ U$. Let us assume $V = X \backslash w^*s - cl\, U$. Then V is a $w^*s$-open set and $y \in V$. Also $U \cap V = U \cap (X \backslash w^*s - cl\, U) \subseteq U \cap (X \backslash U) = \varphi$. Then by using Definition 3.3, $X$ is a $w^*s$-$T_2$ space.

**Theorem 3.9** Let us consider $f: X \to Y$ be bijective.

i. If $f$ is a $w^*s$-continuous function and $Y$ is a $T_2$-space, then $X$ is a $w^*s$-$T_2$ space.

ii. If $f$ is a $w^*s$-irresolute function and $Y$ is a $w^*s$-$T_2$ space, then $X$ is a $w^*s$-$T_2$ space.

**Proof:**

i. Suppose $f$ is a $w^*s$-continuous function and $Y$ is a $T_2$-space. Let us assume $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since $f$ is a bijective function, there exist $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since $Y$ is a $T_2$-space, by Definition 2.2 there exist disjoint open sets $U$ and $V$ containing $y_1$ and $y_2$ respectively. Since $f$ is $w^*s$-continuous, by using Theorem 2.3, $f^{-1}(U)$ and $f^{-1}(V)$ are $w^*s$-open sets containing $x_1$ and $x_2$ respectively. Since $f$ is a bijective function, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. Then by Definition 3.3, $X$ is $w^*s$-$T_2$.

i. Suppose $f$ is a $w^*s$-irresolute function and Y is a $w^*s$-$T_2$ space. Let us assume $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since $f$ is a bijective function, there exist $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since $Y$ is a $w^*s$-$T_2$ space, by Definition 3.3 there exist disjoint $w^*s$-open sets $U$ and $V$ containing $y_1$ and $y_2$ respectively. Since $f$ is a bijective function, $x_1 = f^{-1}(y_1) \in f^{-1}(U)$ and $x_2 = f^{-1}(y_2) \in f^{-1}(V)$. Since $f$ is $w^*s$-irresolute, by Remark 2.5, $f^{-1}(U)$ and $f^{-1}(V)$ are $w^*s$-open sets in $X$. Since $f$ is a bijective function, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. Then by Definition 3.3, $X$ is $w^*s$-$T_2$.

**Theorem 3.10** A topological space $X$ is a $w^*s$-$T_2$ space if and only if the intersection of all $w^*s$-closed $w^*s$-neighborhoods of each point of the space is reduced to that point.

**Proof:** Let us consider $X$ be a $w^*s$-$T_2$ space and $x \in X$.

Then by Definition 3.3, for each $y \neq x$ in $X$, there exist disjoint $w^*s$-open sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$. Now, $U \cap V = \varphi$ implies that $x \in U \subseteq X \backslash V$. Then by using Definition 2.6, $X \backslash V$ is a $w^*s$-nbhd of $x$. Thus $X \backslash V$ is $w^*s$-closed and $w^*s$-nbhd of $x$ which does not contain $y$. Therefore the intersection of all $w^*s$-closed $w^*s$-nbhd of $x$ does not contain any point other than $x$. That is reduced to $\{x\}$.

Conversely, Let us consider $x, y \in X$ with $x \neq y$ in $X$. By our assumption, there exists a $w^*s$-closed $w^*s$-nbhd $V$ of $x$ such that $y \notin V$. Since $V$ is a $w^*s$-nbhd of $x$, by using Definition 2.6, there exists a $w^*s$-open set $U$ such that $x \in U \subseteq V$. Then $U$ and $X \backslash V$ are disjoint $w^*s$-open sets containing the points $x$ and $y$ respectively. Then by Definition 3.3, $X$ is $w^*s$-$T_2$.

**Theorem 3.11** If $X$ is a topological space such that each one point set is $w^*s$-closed, then $X$ is a $w^*s$-$T_1$ space.

**Proof:** Let us consider $x, y \in X$ with $x \neq y$ in $X$. Now $\{x\}$ and $\{y\}$ are $w^*s$-closed sets, because each one point set is $w^*s$-closed. Then $U = X \backslash \{x\}$ is a $w^*s$-open set containing the point $y$ but not $x$ and $V = X \backslash \{y\}$ is a $w^*s$-open set containing the point $x$ but not $y$. Then by Definition 3.2, $X$ is a $w^*s$-$T_1$ space.

**Definition 3.12** A topological space $(X, \tau)$ is said to be $w^*s$-symmetric space if for $x$ and $y$ in $X$, $x \in w^*s - cl(\{y\})$ implies $y \in w^*s - cl(\{x\})$.

**Proposition 3.13** A topological space $(X, \tau)$ is a $w^*s$-symmetric space if $\{x\}$ is $w^*s$-closed, for each $x \in X$.

**Proof:** Assume $\{x\}$ be $w^*s$-closed, for every $x \in X$. Let us take $y \in X$ and $x \in w^*s - cl(\{y\})$. If $y \notin w^*s - cl(\{x\})$, then $\{y\} \subseteq X \backslash w^*s - cl(\{x\})$. By our assumption, $\{y\}$ is $w^*s$-closed. Therefore, by Remark 2.8, $\{y\} = w^*s - cl(\{y\})$. Thus, $w^*s - cl(\{y\}) \subseteq X \backslash w^*s - cl(\{x\})$. And hence, $x \in X \backslash w^*s - cl(\{x\})$, which contradicts the assumption. Therefore $y \in w^*s - cl(\{x\})$.

## 4. Conclusion

Introducing different kinds of $w^*s$-$T_k$ spaces may help us to extend our research in topological spaces and introduce the one in Bi-topological and other topological spaces. Hence we aim to extend the one in other topological spaces in our further work.

## 5. References

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