A NEW FOUR STEP ITERATION SCHEME TO APPROXIMATE FIXED POINTS FOR CONTRACTIVE-LIKE MAPPINGS AND ITS APPLICATION

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Abstract
The focus of this paper is to present results on stability and data dependency for a new four steps iteration scheme when dealing with contractive-like mappings. We also proffer some weak and strong convergence results for generalized nonexpansive mappings endowed with the property (E) in the setting of uniformly convex Banach spaces. We provide a numerical example supported by graphs and tables to validate our proofs. We used this new iteration method to find the solution of a Volterra-Fredholm integral equation.

Keywords: Contractive-like mappings, generalized nonexpansive mappings, stability, data dependence, fixed points, iterative algorithm, weak and strong convergence, Volterra-Fredholm integral equation.
MSC(2010) : 47H09, 47H10, 54H25.

1 Introduction

Iterative methods have become one of the most interesting topics for numerical analysis to approximate fixed points and study their convergence results. The iterative method has two main aspects, the number of iterations and stability. When the number of iterations is small and stable, the method is considered successful, effective, and better than its counterpart in approximation. Moreover, we use the concept of fixed points to find the solutions of integral and differential equations.

The data-dependence result concerning Mann-Ishikawa iteration is in [30] where the data dependence of Ishikawa iteration was proved for contractions mappings. Soltuz et al. in [31] proved data-dependence results for Ishikawa iteration for the contractive-like operators.

In the last two decades, iteration schemes have been considered an easy way to calculate the desired fixed points and, due to this, several interesting iterative processes have been introduced to obtain the fixed point of various kinds of mappings in different types of domains. Some well-known-rations are Mann iteration [14], Ishikawa iteration [12], Noor [15], S-iteration [2], Abbas et al. [1], Thakur et al. ([32], [33]) K iteration [11], M *iteration [34], M iteration [35], K^* iteration [36], Picard-S iteration process [9].

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Let *UCBS,* N, R, and F (T) denote uniformly convex Banach space, the set of positive integers, the set of real numbers, and the set of fixed points of any mapping T respectively, and T be any self-mapping defined on a subset C of a Banach space B.

In 1967, Ostrowski [19] obtained the following classical stability result on metric spaces.

Theorem 1.1. *Let* T: $X \to X$ *be a Banach contraction with contraction constant* $\mu \in [0,1)$*, where* (X, ρ) *is a complete metric space. Let* p^* *be the fixed point of T. Let* $p_0 \in X$ *and* $p_{n+1} = \mathrm{T}\, p_n$ *for* $n = 0, 1, 2\ldots.$

Suppose that $\{q_n\}$ *is another sequence in* X *such that* $\epsilon_n = \rho(\mathrm{T}\, q_n, q_{n+1})$. *Then*

$$\rho(p^*, q_{n+1}) \le \rho(p^*, p_{n+1}) + \mu^{n+1}\rho(p_0, q_0) + \sum_{i=0}^{n} \mu^{n-i}\epsilon_i .$$

In addition, $\lim_{n\to\infty} q_n = p^*$ *if and only if* $\lim_{n\to\infty} \epsilon_n = 0$.

Later, Harder and Hicks [10], Osilike [18], Rhoades [22] and Zhou [37] extended above important result.

The convergence and stability of the iteration scheme are studied by Hussain et al. [[11] and Ullah et al. [35]] for K iteration and K^* iteration respectively. Hussain et al. [11] discussed data dependency and stability results for their newly defined iteration.

In 2020 Shatanvi et al. [26] introduced the following four-step iteration process, which was faster than iteration schemes given by Sintunavarat et al. [27], Agrawal et al. [2], Mann [14], Ishikawa [12], Abbas et al. [1]. for $x_1 \in C$

$$\begin{aligned} x_{n+1} &= (1-a_n)Tz_n + a_nTy_n \\ y_n &= (1-b_n)T\,z_n + b_n\,Tu_n \\ z_n &= (1-c_n)x_n + c_nu_n \\ u_n &= (1-d_n)x_n + d_nTx_n \qquad n \in N \end{aligned} \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \in (0,1)$.

In the same year, Ofem et al. [16] introduced the following AI iteration scheme and proved that the AI iteration method converges at a rate faster than normal S-iteration [24], M iterative process [34], Garodia et al. iteration [8] and Picard S-iteration process [9]. For $x_1 \in C$

$$\begin{aligned} x_{n+1} &= Ty_n, \\ y_n &= Tz_n, \\ z_n &= T\,u_n, \\ u_n &= (1-a_n)x_n + a_n\,Tx_n \qquad n \in N \end{aligned} \tag{1.2}$$

where $\{a_n\} \in (0,1)$.

Motivated by the work done in this direction, we introduced a new iteration method to prove stability, data dependency and approximate the fixed point of generalized nonexpansive mappings. For $x_1 \in C$

$$
\begin{aligned}
x_{n+1} &= T\big((1-a_n)y_n + a_n Ty_n\big)\\
y_n &= Tz_n\\
z_n &= Tu_n\\
u_n &= T\big((1-b_n)x_n + b_nTx_n\big) \qquad n\in N \qquad (1.3)
\end{aligned}
$$

where $\{a_n\}$, $\{b_n\} \in (0,1)$.

Definition 1.1. A mapping T: C → C is a contraction, if ∃ $k \in (0,1)$, such that $\|Ta - Tb\| \le k\|a - b\|$ for all $a, b \in$ C.

Definition 1.2. T is quasi-nonexpansive, if F (T) $\neq \phi$ and $\|Ta - p\| \le \|a - p\|$ $\forall\, a \in$ C and $p \in F$(T).

In the last few years, several generalizations, and extensions of nonexpansive mappings have been done by researchers. Suzuki [28] introduced condition (*C*).

A self-mapping T: C → C is said to satisfy the condition (*C*) if

$$\frac{1}{2}\|u - \mathcal{T}u\| \le \|u - v\| \Rightarrow \|\mathcal{T}u - \mathcal{T}v\| \le \|u - v\| \quad \forall u, v \in \mathcal{C}.$$

It is obvious that every mapping which satisfies condition (*C*) with at least one fixed point is quasi nonexpansive mapping (see [28]).

In 2017, generalized α-nonexpansive mapping, larger than the mappings satisfying in condition (*C*) was introduced by Pant and Shukla [20].

A self-mapping T: C → C is known as generalized α-nonexpansive if there exists $\alpha \in [0,1)$ such that

$$\frac{1}{2}\|u - \mathcal{T}u\| \le \|u - v\| \Rightarrow \|\mathcal{T}u - \mathcal{T}v\| \le \alpha\|\mathcal{T}u - v\| + \alpha\|\mathcal{T}v - u\| + (1-2\alpha)\|u - v\| \quad \forall u, v \in \mathcal{C}. \qquad (1.4)$$

Remark : Suzuki-type generalized nonexpansive mappings, α-nonexpansive mappings and generalized α-nonexpansive mappings are not necessarily continuous in their domains but nonexpansive mappings are continuous (see [20], [21], [28]).

Definition 1.3 ([7])**.** A self-mapping T defined on a nonempty subset C of a Banach space *B* satisfies the (E_μ) condition on the set C if there can be found a real number $\mu \ge 1$ so that

$$\|u - Tv\| \le \mu\|u - Tu\| + \|u - v\|,$$

for all $u, v \in$ C.

Moreover, it is said that T accomplishes the condition (*E*) if ∃ $\mu \ge 1$ such that T fulfils the condition (E_μ).

1. If T is nonexpansive, then it satisfies the condition (E_1). The converse need not to be true.
2. From Lemma 7 to [28], if T: C → C satisfies the condition (*C*) on C, then it satisfies the condition (E_3).
3. From proposition 2.4 in [21] it is obvious that if T is a generalized α-nonexpansive mapping, then T satisfies condition E_μ for $\mu = \frac{(3+\alpha)}{(1-\alpha)}$, where $\alpha \in (0,1)$.

In [7], it is proved that a mapping endowed with the (E) property with a fixed point is quasi-nonexpansive.

2 Preliminaries

A Banach space B is uniformly convex if for each $\epsilon \in$ R+, there is a $\delta \in$ R+ such that $\|a\| \leq 1$, $\|b\| \leq 1$ and $\|a - b\| > \epsilon$ implies $\frac{\|a+b\|}{2} \leq (1-\delta)$ for $a, b \in B$.

A Banach space B is said to have the Opial property [17] if for each weakly convergent sequence $\{p_n\}$ in B, converging weakly to $p \in B$, we have

$\limsup_{n\to\infty} \|p_n - p\| < \limsup_{n\to\infty} \|p_n - q\|$, for all $q \in B$ such that $p \neq q$.

Assume that $\{p_n\}$ is a bounded sequence in Banach space B. For $p \in B$, we set

$$r(p, \{p_n\}) = \limsup_{n\to\infty} \|p - p_n\|.$$

The asymptotic radius of $\{p_n\}$ relative to a nonempty closed and convex subset C of Banach space B is given by

$$r(\mathrm{C}, \{p_n\}) = \inf\{r(p, \{p_n\}) : p \in \mathrm{C}\}.$$

The asymptotic canter of $\{p_n\}$ relative to C is the set

$$A(\mathrm{C}, \{p_n\}) = \{p \in \mathrm{C} : r(p, \{p_n\}) = r(\mathrm{C}, \{p_n\}\}.$$

It is noteworthy that $A(\mathrm{C}, \{p_n\})$ has exactly one point if B is uniformly convex. Also, $A(\mathrm{C}, \{p_n\})$ is nonempty and convex when C is weakly compact and convex (for more details, see [29]).

Definition 2.1. [6] Let T: $B \to B$ be any mapping. Suppose $p_0 \in B$ and $p_{n+1} = f(\mathrm{T}, pn)$ defines an iterative scheme which produces a sequence of points $p_n \in B$. Suppose p_n converges to the fixed point p^* of T. Assume that $\{u_n\}$ be a sequence in B and $\epsilon \in [0, \infty)$ given by $\epsilon = \|u_{n+1} - u_n\|$. Then the iterative scheme defined by $p_{n+1} = f(\mathrm{T}, p_n)$ is called stable with respect to T if $\lim_{n\to\infty} \in_n = 0$ iff $\lim_{n\to\infty} u_n = 0$.

Definition 2.2. [21] Let $\{x_n\}$ and $\{y_n\}$ be two iterative schemes such that both converging to the same fixed point x^* and $\|x_n - x^*\| \leq t_n$ and $\|y_n - x^*\| \leq w_n \, \forall n \in$ N.

If $\{t_n\}$ and $\{w_n\}$ be two real number sequences converging to t and w, respectively and $\frac{|t_n - t}{|w_n - w|} = 0.$

This implies that $\{x_n\}$ converges faster than $\{y_n\}$.

Definition 2.3. [18] A mapping T defined on a Banach space B is known as contractive mapping on B if there exist constants $L, b \in [0,1)$ such that for all $x, y \in B$

$$\|\mathrm{T}x - \mathrm{T}y\| \leq L\|x - \mathrm{T}x\| + b\|x - y\| \quad (2.1)$$

Definition 2.4. [6] The operator T is called contractive-like operator if there exists a constant $b \in (0,1)$ a continuous and strictly increasing function h: $[0, \infty) \to [0, \infty)$ with $h(0) = 0$ such that for each $x, y \in B$,

$$\|\mathrm{T}x - \mathrm{T}y\| \leq b\|x - y\| + g(\|x - \mathrm{T}x\|) \quad (2.2)$$

Osilike [18] worked on many stability results as a generalization of the works done by Rhoades [22] and Harder et al [10].

Proposition 2.1. *[7] Let* T *be an arbitrary self-mapping defined on bounded subset* C *of a Banach space B.*

If

1. *There exists an almost fixed-point sequence* $\{x_n\}$ *for* T *in* C *such that* x_n *weakly converges to* p^*,
2. T *satisfies condition* (E) *on* C, *and*
3. (B, ‖‖) *satisfies the Opial condition.*

Then, T $p^* = p^*$.

Proposition 2.2. *[7] Let* T *be an arbitrary self-mapping defined on bounded subset* C *of a Banach space B. Then at least one of the following statements hold.*

1. *There exists a sequence* $\{x_n\}$ *in* C *such that* $\|Tx_n - x_n\| \to 0$. *Such a sequence is called almost fixed point sequence for* T.
2. T *satisfies condition* (E) *on* C.

Proposition 2.3. *[7] Let* T: C → C *be a mapping which satisfies condition* (E) *on* C . *If* T *has some fixed point, then* T *is quasi-nonexpansive. The converse is not necessarily true.*

Proposition 2.4. *[20] Let* T *be a generalized α-nonexpansive self-mapping defined on nonempty subset* T *of a Banach space B. Then, for all u, v* ∈ C,

$$\|u - \mathcal{T}(v)\| \leq \frac{(3+\alpha)}{(1-\alpha)}\|u - \mathcal{T}(u)\| + \|u - v\|.$$

Therefore, if T is a generalized α-nonexpansive mapping, then T satisfies condition E_μ for $\mu = \frac{(3+\alpha)}{(1-\alpha)}$.

Theorem 2.5. *[28] Let* C *be a convex and weakly compact subset of a uniformly convex Banach space B and* T : C → C *satisfies condition* (C). *Then,* T *has a fixed point.*

Theorem 2.6. *[29] Let* $0 < \alpha \leq \omega_n \leq \beta < 1$ *for all positive integers n. Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in a uniformly convex Banach space B such that* $\lim_{n\to\infty} sup \parallel x_n \parallel$, $\lim_{n\to\infty} sup \parallel y_n \parallel \leq \eta$ *and* $\lim_{n\to\infty} \parallel \omega_n x_n + (1-\omega_n) y_n \parallel = \eta$ *hold for some* $\eta \geq 0$. *Then* $\lim_{n\to\infty} \parallel x_n - y_n \parallel = 0$.

3 Stability result

Remark : Since $(1-\eta) \leq e^{-\eta}$, for $\eta \in [0,1]$. If $\{b_n\}$ be a sequence of nonnegative real numbers such that

$b_n \in (0,1]$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n = \infty$, then $\prod_{n=1}^{\infty}(1-b_n) = 0$

Theorem 3.1. *Let* T *be a self-contractive-like mappings defined on a nonempty convex and closed subset* C *of a Banach space B with* F (T) ≠ ∅ *and* $\{x_n\}$ *be the sequence defined by the new iteration scheme (1.3), where* $\{a_n\}$ *and* $\{b_n\}$ *are a sequence in* [0,1] *for all n* ∈ N *such that* $\lim_{n\to\infty} \sum_{i=0}^{n} a_i = \infty$. . *Then* $\{x_n\}$ *converges strongly to a unique fixed point of T.*

Proof. Assume that p^* is a fixed point of T. By the definition of new iteration (1.3) and contradictive-like mapping

$$\|u_n - p^*\| = \|T((1-b_n)x_n + b_n T x_n) - T p^*\|$$

$$= \gamma[\|(1-b_n)x_n + b_n T x_n - p^*\|] + g(\|T p^* - p^*\|)$$
$$\leq \gamma[(1-b_n)\|x_n - p^*\| + b_n\|T x_n - p^*\|]$$
$$\leq \gamma[(1-b_n)\|x_n - p^*\| + \gamma b_n\|x_n - p^*\| + b_n g(\|T p^* - p^*\|)]$$
$$= \gamma[1 - b_n(1-\gamma)]\|x_n - p^*\|. \quad (3.1)$$

Similarly

$$\|z_n - p^*\| = \|T u_n - T p^*\|$$
$$\leq \gamma\|u_n - p^*\| + g(\|T p^* - p^*\|)$$
$$= \gamma\|u_n - p^*\| \quad (3.2)$$

By equations (3.1)

$$\|z_n - p^*\| \leq \gamma^2[1 - b_n(1-\gamma)]\|x_n - p^*\|. \quad (3.3)$$

And

$$\|y_n - p^*\| = \|T z_n - T p^*\|$$
$$\leq \gamma\|z_n - p^*\| + g(\|T p^* - p^*\|)$$
$$= \gamma\|z_n - p^*\|. \quad (3.4)$$

Therefore, by equation (3.3) we have

$$\|y_n - p^*\| =\leq \gamma^3[1 - b_n(1-\gamma)]\|x_n - p^*\|. \quad (3.5)$$

Again

$$\|x_{n+1} - p^*\| = \|T((1-a_n)y_n + a_n T y_n) - T p^*\|$$
$$= \gamma[\|(1-a_n)y_n + a_n T y_n - p\|] + g(\|T p^* - p^*\|)$$
$$\leq \gamma[(1-a_n)\|y_n - p^*\| + a_n\|T y_n - p^*\|]$$
$$\leq \gamma[(1-a_n)\|y_n - p^*\| + \gamma a_n\|y_n - p^*\| + a_n g(\|T p^* - p^*\|)]$$
$$= \gamma[1 - a_n(1-\gamma)]\|y_n - p^*\|. \quad (3.6)$$

From equation (3.5)

$$\|x_{n+1} - p^*\| \leq \gamma^4[1 - a_n(1-\gamma)][1 - b_n(1-\gamma)]\|x_n - p^*\|. \quad (3.7)$$

Since $\gamma < 1$ and $b_n \in (0,1]$ so for all $i \in N$, we obtain $[1 - b_i(1-\gamma)] < 1$. Hence

$$\|x_{n+1} - p^*\| \leq \gamma^4[1 - a_n(1-\gamma)]\|x_n - p^*\|. \quad (3.8)$$

Inductively we have

$$\|x_{n+1} - p\| \leq \gamma^{4(n+1)} \prod_{i=0}^{n} [(1 - a_i(1-\gamma)] \| x_0 - p \|. \quad (3.9)$$

It is evident that $a_i(1-\gamma)] < 1$ and $(1-\eta) \leq e^{-\eta}$ for all $\eta \in [0,1]$. Thus, we see that $1 - a_i(1-\gamma) \leq e^{-a(1-\gamma)}$.

Therefore,

$$||x_{n+1} - p|| \leq \gamma^{4(n+1)} e^{-(1-\gamma)\sum_{i=0}^{n} a_i ||x_0 - p||}$$
$$\lim_{n\to\infty} ||x_{n+1} - p|| \leq \lim_{n\to\infty} \gamma^{4(n+1)} e^{-(1-\gamma)\sum_{i=0}^{n} a_i ||x_0 - p||}$$

Since $\lim_{n\to\infty}\sum_{i=0}^{n} a_i = \infty$. These yields $\lim_{n\to\infty} \| x_n - p^* \| = 0$ i.e. $\{x_n\}$ converges strongly to p^*.

For the uniqueness of fixed points, suppose that p^* and q^* are any two fixed points of T. Then

$$\|p - q\| = \|T p^* - T q^*\| \le \gamma\|p^* - q^*\| + g(\|T p^* - p^*\|) = \gamma\|p^* - q^*\|.$$

Thus, we have $(1 - \gamma)\|p^* - q^*\| = 0$. This implies that $p^* = q^*$. □

Lemma 3.2. *[3] For a real number $\eta \in [0,1)$ and a sequence of positive numbers $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ as $n \to \infty$, then for any sequence of positive numbers x_n satisfying*

$$xn+1 = \eta\, xn + \epsilon n$$

for $n = 1, 2, 3, \ldots$, we have

$$\lim_{n\to\infty} x_n = 0.$$

Theorem 3.3. *Let* T *be a self-contractive-like mappings defined on nonempty closed convex subset* C *of Banach space B with $F(T) \neq \emptyset$ and $\{x_n\}$ be the sequence satisfying (1.3), where $\{a_n\}, \{b_n\} \in (0,1]$* for *all positive integers n. Then the new iteration scheme (1.3) is* T *-stable.*

Proof. Let $\{p_n\}$ be an arbitrary sequence in B and $x_{n+1} = g(T, x_n)$ be defined by (1.3) which converges to the unique fixed point p^* of T and $\epsilon_n = \| p_{n+1} - g(T, p_n) \|$. We need to show that lim $\epsilon_n = 0$ if and only if $\lim_{n\to\infty} p_{n=p^*}$.

First, we assume that $\lim_{n\to\infty} \epsilon_n = 0$

and

$$\begin{aligned}
\|p_{n+1} - p^*\| &= \|p_{n+1} - g(T, p_n) + g(T, p_n) - p^*\| \\
&\le \|p_{n+1} - g(T, p_n)\| + \|g(T, p_n) - p^*\| \\
&= \epsilon_n + \|T((1 - a_n)q_n + T q_n) - p^*\| \\
&= \epsilon_n + \gamma\|(1 - a_n)q_n + a_n T q_n - p^*\| \\
&= \epsilon_n + \gamma[(1 - a_n)\|q_n - p^*\| + a_n\|T q_n - p^*\|] \\
&= \epsilon_n + \gamma[(1 - a_n)\|q_n - p^*\| + \gamma a_n\|q_n - p^*\| + g(T p^* - p^*)] \\
&= \epsilon_n + \gamma[1 - a_n(1 - \gamma)]\|q_n - p^*\| \\
&= \epsilon_n + \gamma[1 - a_n(1 - \gamma)]\|T r_n - p^*\| \\
&= \epsilon_n + \gamma[1 - a_n(1 - \gamma)][\gamma\|r_n - p^*\| + g(T p^* - p^*)] \\
&= \epsilon_n + \gamma^2[1 - a_n(1 - \gamma)]\|T s_n - p^*\| \\
&= \epsilon_n + \gamma^2[1 - a_n(1 - \gamma)][\gamma\|s_n - p^*\| + g(T p^* - p^*)] \\
&= \epsilon_n + \gamma^3[1 - a_n(1 - \gamma)]\|T((1 - b_n)p_n + b_n T p_n) - p^*\| \\
&= \epsilon_n + \gamma^4[1 - a_n(1 - \gamma)][1 - b_n(1 - \gamma)]\|p_n - p^*\|
\end{aligned}$$

Since $(1-\gamma) \le 1$ and $a_n, b_n \in (0,1]$, so $\gamma^4[1-a_n(1-\gamma)][1-b_n(1-\gamma)] \le 1$. By the virtue of Lemma 3.2, $\lim_{n\to\infty} \| pn - p^* \| = 0$ i.e. $\lim p_n = p^*$.

Conversely, suppose that $\lim p_n = p^*$. Then

$$\begin{aligned}
\epsilon_n &= \|p_{n+1} - g(T, p_n)\| \\
&\le \|p_{n+1} - p^*\| + \|p^* - g(T, p_n)\|
\end{aligned}$$

$\leq \|p_{n+1} - p^*\| + \gamma^4[1 - a_n(1 - \gamma)][1 - b_n(1 - \gamma)]\|p_n - p^*\|$ (3.10) Taking limit as $n \to \infty$ in (3.10) we get $\lim_{n\to\infty} \epsilon_n = 0$. Hence the new iteration scheme (1.3) is T−stable.

□

4 Data Dependency Theorem

We consider the new iteration process (1.3) for the operator S as follows. For $p_1 \in C$

$$p_{\{n+1\}} = S\big((1 - a_n)q_n + a_n Sy_n\big)$$
$$q_n = Sr_n$$
$$r_n = Ss_n$$
(4.1)
$$s_n = S\big((1 - b_n)p_n + b_n Sp_n\big) \quad n \in N$$

where $\{a_n\}, \{b_n\} \in (0,1]$.

Definition 4.1. [4] Let T, S: $B \to B$ be two operators. S is said to be approximate operator for T if for some $\epsilon > 0$ we have $\|T u - S u\| \leq \epsilon$ for all $u \in B$.

Lemma 4.1. *[31] Let* $\{\alpha_n\}_{n=0}^{\infty}$ *be a sequence of nonnegative real number for which there exists* $n_0 \in \in$ N *such that for all* $n \geq n_0$ *satisfying the relation.*

$$\alpha_{n+1} \leq (1 - \beta_n)\alpha_n + \beta_n \sigma_n, \quad (4.2)$$

where $\beta_n \in (0,1)$ *for all* $n \in$ N, $\sum_{n=0}^{\infty} \beta_n$. *and* $\sigma_n \geq 0$ *for all* $n \in$ $0 \leq \limsup_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \sigma_n$ N, *Then*

Theorem 4.2. *Let* T *and* S *be defined on a nonempty subset* C *such that* T *be a contractive-like operator with a fixed point* p^* *and* S *is approximate operator for* T *with* $Sq^* = q^*$. *Let* $\{x_n\}_{n=0}^{\infty}$ *be an iterative sequence generated by (1.3) and iterative sequence* $\{p\}_{n=0}^{\infty}$ *is generated by (4.1) with the assumption* $(1 - a_n) < a_n$ *and* $\sum_{n=1}^{\infty} a_n = \infty$. *If* $\lim_{n\to\infty} p_n = q^*$, *then we have* $\|p^* - q^*\| \leq \frac{11\epsilon}{1-\gamma}$, *where* $\epsilon > 0$ *is a fixed number.*

Proof. By using equations (1.3) and (4.1), we obtain

$$|u_n - s_n\| = \|T((1 - b_n)x_n + b_n T x_n) - S((1 - b_n)p_n + b_n Sp_n)\|$$
$$= \|T((1 - b_n)x_n + b_n T x_n) - T((1 - b_n)p_n + b_n Sp_n) + T((1 - b_n)p_n + b_n Sp_n) - S((1 - b_n)p_n + b_n Sp_n)\|$$
$$\leq \|T((1 - b_n)x_n + b_n T x_n) - T((1 - b_n)p_n + b_n Sp_n)\| + \|T((1 - b_n)p_n + b_n Sp_n) - S((1 - b_n)p_n + b_n Sp_n)\|$$
$$\leq \gamma\|(1 - b_n)x_n + b_n T x_n - ((1 - b_n)p_n + b_n Sp_n)\| + g(T((1 - b_n)x_n + b_n T x_n) - ((1 - b_n)x_n + b_n T x_n)) + \epsilon$$
$$\leq \gamma[(1 - b_n)\|x_n - p_n\| + b_n\|T x_n - Sp_n\|] + g(T((1 - b_n)x_n + b_n T x_n) - ((1 - b_n)x_n + b_n T x_n)) + \epsilon$$
$$\leq \gamma[(1 - b_n)\|x_n - p_n\| + b_n\{\|T x_n - T p_n\| + \|T p_n - Sp_n\|\}] + g(T((1 - b_n)x_n + b_n T x_n) - ((1 - b_n)x_n + b_n T x_n)) + \epsilon$$
$$\leq \gamma[(1 - b_n)\|x_n - p_n\| + b_n\{\gamma\|x_n - p_n\| + g(T x_n - x_n) + \epsilon\}] + g(T((1 - b_n)x_n$$

$$+ b_n\mathrm{T}\, x_n) - ((1-b_n)x_n + b_n\mathrm{T}\, x_n)) + \epsilon$$
$$\leq \gamma[(1-b_n)\|x_n - p_n\| + b_n\gamma\|x_n - p_n\| + b_n g(\mathrm{T}\, x_n - x_n) + b_n\epsilon] + g(\mathrm{T}\,((1-b_n)x_n + b_n\mathrm{T}\, x_n) - ((1-b_n)x_n + b_n\mathrm{T}\, x_n)) + \epsilon$$
$$\leq \gamma[(1-b_n(1-\gamma)]\|x_n - p_n\| + \gamma b_n g(\mathrm{T}\, x_n - x_n) + \gamma b_n\epsilon + g(\mathrm{T}\,((1-b_n)x_n + b_n\mathrm{T}\, x_n) - ((1-b_n)x_n + b_n\mathrm{T}\, x_n)) + \epsilon \tag{4.3}$$

Now

$$\|z_n - r_n\| = \|\mathrm{T}\, u_n - \mathrm{S}s_n\| \leq \|\mathrm{T}\, u_n - \mathrm{T}\, s_n\| + \|\mathrm{T}\, s_n - \mathrm{S}s_n\| \leq \gamma\|u_n - s_n\| + g\,(\|\mathrm{T}\, u_n - u_n\|) + \epsilon. \tag{4.4}$$

From equations (4.3) and (4.4)

$$\|z_n - r_n\| = \gamma^2[(1-b_n(1-\gamma)]\|x_n - p_n\| + \gamma^2 b_n g(\mathrm{T}\, x_n - x_n) + \gamma^2 b_n\epsilon + \gamma g(\mathrm{T}\,((1-b_n)x_n + b_n\mathrm{T}\, x_n) - ((1-b_n)x_n + b_n\mathrm{T}\, x_n)) + \gamma\epsilon + g(\|\mathrm{T}\, u_n - u_n\|) + \epsilon \tag{4.5}$$

Similarly,

$$\|y_n - q_n\| = \|\mathrm{T}\, z_n - \mathrm{S}r_n\| \leq \|\mathrm{T}\, z_n - \mathrm{T}\, r_n\| + \|\mathrm{T}\, r_n - \mathrm{S}r_n\| \leq \gamma\|z_n - r_n\| + g\,(\|\mathrm{T}\, z_n - z_n\|) + \epsilon. \tag{4.6}$$

!

From equations (4.5) and (4.6)

$$\|y_n - q_n\| = \gamma^3[(1-b_n(1-\gamma)]\|x_n - p_n\| + \gamma^3 b_n g(\mathrm{T}\, x_n - x_n) + \gamma^3 b_n\epsilon + \gamma^2 g(\mathrm{T}\,((1-b_n)x_n + b_n\mathrm{T}\, x_n) - ((1-b_n)x_n + b_n\mathrm{T}\, x_n)) + \gamma^2\epsilon + \gamma g(\|\mathrm{T}\, u_n - u_n\|) + \gamma\epsilon + g\,(\|\mathrm{T}\, z_n - z_n\|) + \epsilon. \tag{4.7}$$

and

$$\|x_{n+1} - p_{n+1}\| = \|\mathrm{T}\,((1-a_n)y_n + a_n\mathrm{T}\, y_n) - \mathrm{S}((1-a_n)q_n + a_n\mathrm{S}q_n)\|$$
$$\leq \|\mathrm{T}\,((1-a_n)y_n + a_n\mathrm{T}\, y_n) - \mathrm{T}\,((1-a_n)q_n + a_n\mathrm{S}q_n)\| + \|\mathrm{T}\,((1-a_n)y_n + a_n\mathrm{S}q_n) - \mathrm{S}((1-a_n)q_n + a_n\mathrm{S}q_n)\|$$
$$\leq \gamma[(1-a_n)\|y_n - q_n\| + a_n\|\mathrm{T}\, y_n - \mathrm{S}q_n\|] + g(\mathrm{T}\,((1-a_n)y_n + a_n\mathrm{T}\, y_n) - ((1-a_n)y_n + a_n\mathrm{T}\, y_n)) + \epsilon$$
$$\leq \gamma[(1-a_n)\|y_n - q_n\| + a_n\{\|\mathrm{T}\, y_n - \mathrm{T}\, q_n\| + \|\mathrm{T}\, q_n - \mathrm{S}q_n\|\}] + g(\mathrm{T}\,((1-a_n)y_n + a_n\mathrm{T}\, y_n) - ((1-a_n)y_n + a_n\mathrm{T}\, y_n)) + \epsilon$$
$$\leq \gamma[(1-a_n)\|y_n - q_n\| + a_n\{\gamma\|y_n - q_n\| + g(\mathrm{T}\, y_n - y_n) + \epsilon\}] + g(\mathrm{T}\,((1-a_n)y_n + a_n\mathrm{T}\, y_n) - ((1-a_n)y_n + a_n\mathrm{T}\, y_n)) + \epsilon$$
$$\leq \gamma[(1-a_n(1-\gamma)]\|y_n - q_n\| + \gamma a_n g(\mathrm{T}\, y_n - y_n) + \gamma a_n\epsilon + g(\mathrm{T}\,((1-a_n)y_n + a_n\mathrm{T}\, y_n) - ((1-a_n)y_n + a_n\mathrm{T}\, y_n)) + \epsilon \tag{4.8}$$

From equations (4.7) and (4.8)

$$\|x_{n+1} - p_{n+1}\| \leq \gamma[(1-a_n(1-\gamma)]\{\gamma^3[(1-b_n(1-\gamma)]\|x_n - p_n\| + \gamma^3 b_n g(\mathrm{T}\, x_n - x_n) + \gamma^3 b_n\epsilon + \gamma^2 g(\mathrm{T}\,((1-b_n)x_n + b_n\mathrm{T}\, x_n) - ((1-b_n)x_n + b_n\mathrm{T}\, x_n)) + \gamma^2\epsilon + \gamma g(\|\mathrm{T}\, u_n - u_n\|) + \gamma\epsilon + g(\|\mathrm{T}\, z_n - z_n\|) + \epsilon\} + \gamma a_n g(\mathrm{T}\, y_n - y_n) + \gamma a_n\epsilon + g(\mathrm{T}\,((1-a_n)y_n + a_n\mathrm{T}\, y_n) - ((1-a_n)y_n + a_n\mathrm{T}\, y_n)) + \epsilon$$
$$= \gamma^4[(1-a_n(1-\gamma)][(1-b_n(1-\gamma)]\|x_n - p_n\| + \gamma^4[(1-a_n(1-\gamma)]b_n g(\mathrm{T}\, x_n - x_n) + \gamma^4[(1-a_n(1-\gamma)]b_n\epsilon + \gamma^4[(1-a_n(1-\gamma)]g(\mathrm{T}\,((1-b_n)x_n + b_n\mathrm{T}\, x_n) - ((1-b_n)x_n + b_n\mathrm{T}\, x_n)) + \gamma^3[(1-a_n(1-\gamma)]\epsilon$$

$$+ \gamma^2[(1 - a_n(1 - \gamma)]g(\|T u_n - u_n\|) + \gamma^2[(1 - a_n(1 - \gamma)]\epsilon$$
$$+ \gamma[(1 - a_n(1 - \gamma)]g(\|T z_n - z_n\|) + \gamma[(1 - a_n(1 - \gamma)]\epsilon + \gamma a_n g(T y_n - y_n)$$
$$+ \gamma a_n\epsilon + g(T((1 - a_n)y_n + a_n T y_n) - ((1 - a_n)y_n + a_n T y_n)) + \epsilon \quad (4.9)$$

Since a_n, $b_n \in (0,1]$ and $\gamma \in (0,1)$ it implies $(1 - a_n(1 - \gamma)$, $(1 - b_n(1 - \gamma)$, $\gamma^4 b_n$, γ^2, γ^3, $\gamma^4 < 1$. Thus,

$$\|x_{n+1} - p_{n+1}\| \leq [(1 - a_n(1 - \gamma)] \|x_n - p_n\| + g(T x_n - x_n) + \epsilon$$
$$+ g(T((1 - b_n)x_n + b_n T x_n) - ((1 - b_n)x_n + b_n T x_n)) + \epsilon + g(\|T u_n - u_n\|)$$
$$+ \epsilon + g(\|T z_n - z_n\|) + \epsilon + a_n g(T y_n - y_n) + a_n\epsilon$$
$$+ g(T((1 - a_n) y_n + a_n T y_n) - ((1 - a_n)y_n + a_n T y_n)) + \epsilon$$
$$= [(1 - a_n(1 - \gamma)]\|x_n - p_n\| + g(T x_n - x_n) + g(T((1 - b_n)x_n + b_n T x_n)$$
$$- ((1 - b_n)x_n + b_n T x_n)) + g(\|T u_n - u_n\|) + g(\|T z_n - z_n\|) + a_n g(T y_n - y_n)$$
$$+ g(T((1 - a_n)y_n + a_n T y_n) - ((1 - a_n)y_n + a_n T y_n)) + a_n\epsilon + 5\epsilon$$

By the assumption $(1-a_n) < a_n$, i.e. $1 < 2a_n$ and taking $(1-a_n) y_n + a_n T y_n = l_n$, $(1-b_n)x_n + b_n T x_n = t_n$, we obtain

$$\|x_{n+1} - p_{n+1}\| \leq [(1 - a_n(1 - \gamma)]\|x_n - p_n\| + g(T x_n - x_n) + g(T((1 - b_n)x_n + b_n T x_n)$$
$$- ((1 - b_n)x_n + b_n T x_n)) + g(\|T u_n - u_n\|) + g(\|T z_n - z_n\|)$$
$$+ a_n g(T y_n - y_n) + g(T((1 - a_n)y_n + a_n T y_n) - ((1 - a_n)y_n + a_n T y_n)) \quad (4.10)$$
$$+ a_n\epsilon + 5\epsilon$$
$$\leq [(1 - a_n(1 - \gamma)]\|x_n - p_n\| + 2a_n g(\mathcal{T}x_n - x_n) + 2a_n g(\mathcal{T}t_n - t_n)$$
$$+ 2a_n g(\|\mathcal{T}u_n - u_n\|) + 2a_n g(\|\mathcal{T}z_n - z_n\|) + a_n g(\mathcal{T}y_n - y_n)$$
$$+ 2a_n g(\mathcal{T}l_n - l_n) + a_n\epsilon + 5.2a_n\epsilon$$
$$= [(1 - a_n(1 - \gamma)]\|x_n - p_n\| + a_n(1 - \gamma)\Bigg\{\frac{2g(\mathcal{T}x_n - x_n) + 2g(\mathcal{T}t_n - t_n)}{(1 - \gamma)}$$
$$+ \frac{2g(\|\mathcal{T}u_n - u_n\|) + 2g(\|\mathcal{T}z_n - z_n\|) + g(\mathcal{T}y_n - y_n)}{(1 - \gamma)}$$
$$+ \frac{2g(\mathcal{T}l_n - l_n) + 11\epsilon}{(1 - \gamma)}\Bigg\} \quad (4.11)$$

Take $\alpha_n = \| x_n - p_n \|$, $\beta_n = a_n(1 - \gamma)$, and

$$\sigma_n = \frac{2g(\mathcal{T}x_n - x_n) + 2g(\mathcal{T}t_n - t_n)}{(1 - \gamma)} + \frac{2g(\|\mathcal{T}u_n - u_n\|) + 2g(\|\mathcal{T}z_n - z_n\|) + g(\mathcal{T}y_n - y_n)}{(1 - \gamma)}$$
$$+ \frac{2g(\mathcal{T}l_n - l_n) + 11\epsilon}{(1 - \gamma)}.$$

Since f is a continuous strictly increasing mapping and $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$ are sequences converge to the fixed point of T, then

$$\lim_{n\to\infty} f(\| T x_n - x_n \|) \lim_{n\to\infty} f(\| T t_n - t_n \|) = \lim_{n\to\infty} f(\| T y_n - y_n \|) \lim_{n\to\infty} f(\| T z_n - z_n \|)$$
$$\lim_{n\to\infty} f(\| T u_n - u_n \|) \lim_{n\to\infty} f(\| T l_n - l_n \|) = 0.$$

By using Lemma 4.1, we have

$$\limsup_{n\to\infty} \|x_n - p_n\| \leq \frac{2g(\mathcal{T}x_n - x_n) + 2g(\mathcal{T}t_n - t_n)}{(1-\gamma)} + \frac{2g(\|\mathcal{T}u_n - u_n\|) + 2g(\|\mathcal{T}z_n - z_n\|) + g(\mathcal{T}y_n - y_n)}{(1-\gamma)} + \frac{2g(\mathcal{T}l_n - l_n) + 11\epsilon}{(1-\gamma)}. \quad (4.12)$$

By using Theorem 3.1 and the assumption $\lim_{n\to\infty} p_n = q*$, we have

$$\|p^{\wedge} - q^*\| \leq \frac{11\epsilon}{(1-\gamma)}$$

This completes the proof. □

5 Rate of Convergence

In this section, we will prove that our new iteration process (1.3) converges faster than the iteration processes (1.1) and (1.2) under contractive-like mappings.

Theorem 5.1. *Let* C*, B,* T *and* $\{x_n\}$ *be as in Theorem 3.3. If* $p^* \in F$ (T) *then the iteration process (1.3) converges faster than iterations (1.1) and (1.2).*

Proof. From equation (3.9) of Theorem 3.1 and the assumption $a \leq a_n \leq 1$ for some $a > 0$ and $n \in$ N, we have

$$\|xn{+}1 - p*\| \leq \gamma^{4(n+1)} \prod_{i=0}^{n}[(1 - a_i(1-\gamma)] \parallel x_0 - p^* \parallel$$

$$\leq \gamma^{4(n+1)} [(1 - a(1-\gamma)](n+1)\|x0 - p*\|. \quad (5.1)$$

Set $hn = \gamma^{4(n+1)} [(1 - a(1-\gamma)](n+1)\|x0 - p\|$.

From equation (1.1) with assumption a_n, b_n, c_n, d_n are sequences in $[a,1-a],[b,1-b],[c,1-c]$ and $[d,1-d]$

Where $a, b, c, d \in (0, \frac{1}{2})$, we get

$$\|x_{n+1} - p^*\| \leq [1 - (1-\gamma)\, d_n(c_n + a_nb_n(1-c_n))]\|x_n - p^*\|$$
$$\leq [1 - (1-\gamma)\, d(c + ab(1-c))]\|x_n - p^*\|$$

Inductively, we find that

$$\|x_{n+1} - p^*\| \leq [1 - (1-\gamma)\, d(c + ab(1-c))]^{(n+1)}\|x_0 - p^*\|$$

Set $\mu_n = [1 - (1-\gamma)d(c + ab(1-c))]^{n+1}\|x_0 - p^*\|$. Then from equation (1.2), we obtain

$$\|xn{+}1 - p*\| \leq \gamma^{3(n+1)} [(1 - a(1-\gamma)] \|^{(n+1)}\|x0 - p*\|. \quad (5.2)$$

Set $v_n = \gamma^{3(n+1)} [(1 - a(1-\gamma)]^{(n+1)}\|x_0 - p^*\|$. Now we observe that

$$\frac{h_n}{\mu_n} = \frac{\gamma^{4(n+1)}[(1-a(1-\gamma)]^{(n+1)}\|x_0 - p\|}{[1-(1-\gamma)d(c+ab(1-c))]^{n+1}\|x_0 - p^*\|} \to 0 \ as \ n \to \infty.$$

This implies that iteration (1.3) converges faster than (1.1). Also

$$\frac{h_n}{\nu_n} = \frac{\gamma^{4(n+1)}[(1-a(1-\gamma)]^{(n+1)}\|x_0 - p\|}{\gamma^{3(n+1)}[(1-a(1-\gamma)]^{(n+1)}\|x_0 - p^*\|} \to 0 \ as \ n \to \infty$$

This shows iteration (1.3) converges faster than iteration (1.2). □

Remark 5.1. Since Shatanvi et al. iteration 1.1 is faster than Sintunavarat et al. [27], Agrawal et al. [2], Mann [14], Ishikawa [12], Abbas et al. [1] and our new iteration (1.3) is faster than Shatanvi et al. iteration (1.1). Therefore, our new defined iteration is faster than Sintunavarat et al. [27], Agrawal et al. [2], Mann [14], Ishikawa [12], Abbas et al. [1].

Remark 5.2. Similarly, the AI iteration method (1.2) converges at a rate faster than normal S-iteration [24], M iterative process [34], Garodia et al. iteration [8] and Picard S-iteration process [9] and our new iteration (1.3) is faster than the AI iteration method (1.2). Therefore, the new iteration (1.3) converges at a rate faster than the normal S-iteration [24], M iterative process [34], Garodia et al. iteration [8] and Picard S-iteration process [9].

To illustrate the efficiency of new iteration process (1.3), we consider the following numerical example. *Example* 5.1. Let $B = \mathbb{R}$, $C = [0,4]$, and let $T: C \rightarrow C$ be defined by

$$\mathcal{T}(u) = \begin{cases} \frac{u}{3}, & \text{if } u \in [0,2) \\ \frac{u}{6}, & \text{if } u \in [2,4]. \end{cases}$$

Clearly $0 \in T(F)$. Since T is not continuous at 2, it implies that T is neither contraction mapping nor nonexpansive mapping. Next, we show that T is contractive-like mapping. For this, consider a strictly increasing continuous function $f: [0, \infty) \rightarrow [0, \infty)$ defined by

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in (0, \infty) \\ 0, & \text{if } x = 0. \end{cases}$$

When $u \in [0, 2)$, we have

$$\|u - \mathcal{T}u\| = \|u - \frac{u}{3}\| = \frac{2u}{3} \text{ and } f(\frac{2u}{3}) = \frac{u}{3}.$$

When $u \in [2, 4]$, we have

$$\|u - \mathcal{T}u\| = \|u - \frac{u}{6}\| = \frac{5u}{6} \text{ and } f(\frac{5u}{6}) = \frac{5u}{12}.$$

Now we consider the following cases.

Case 1: If $u, v \in [0, 2)$, then

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}v\| &= \|\frac{u}{3} - \frac{v}{3}\| = \frac{1}{3}\|u - v\| \leq \frac{1}{3}\|u - v\| + \|\frac{u}{3}\| \\ &= \frac{1}{3}\|u - v\| + f(\|\frac{2u}{3}\|) = \frac{1}{3}\|u - v\| + f(\|u - \mathcal{T}u\|). \end{aligned}$$

Case 2: If $u \in [0, 2)$, $v \in [2, 4]$, then

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}v\| &= \|\frac{u}{3} - \frac{v}{6}\| = \|\frac{u}{6} + \frac{u}{6} - \frac{v}{6}\| \leq \frac{1}{6}\|u - v\| + \|\frac{u}{6}\| \\ &\leq \frac{1}{3}\|u - v\| + \|\frac{u}{3}\| \leq \frac{1}{3}\|u - v\| + f(\|\frac{2u}{3}\|) \\ &= \frac{1}{3}\|u - v\| + f(\|u - \mathcal{T}u\|). \end{aligned}$$

Case 3: If $u \in [2, 4]$, $v \in [0, 2)$, then

$$\|\mathcal{T}u-\mathcal{T}v\| = \|\frac{u}{6}-\frac{v}{3}\| = \|\frac{u}{3}-\frac{u}{6}-\frac{v}{3}\| \le \frac{1}{3}\|u-v\| + \|\frac{u}{6}\|$$
$$\le \frac{1}{3}\|u-v\| + \|\frac{5u}{12}\| \le \frac{1}{3}\|u-v\| + f(\|\frac{5u}{6}\|)$$
$$= \frac{1}{3}\|u-v\| + f(\|u-\mathcal{T}u\|).$$

Case 4: If u, $v \in [2, 4]$, then

$$\|\mathcal{T}u-\mathcal{T}v\| = \|\frac{u}{6}-\frac{v}{6}\| \le \frac{1}{6}\|u-v\| + \|\frac{5u}{12}\|$$
$$\le \frac{1}{3}\|u-v\| + f(\|\frac{5u}{6}\|) = \frac{1}{3}\|u-v\| + f(\|u-\mathcal{T}u\|).$$

Thus, **Case 1, 2, 3** and **Case 4** are satisfied for $b = \frac{1}{3}$. Hence, T is a contractive-like mapping for for $b = \frac{1}{3}$.

By using example 5.1, we tried to show that the rate of convergence of the iteration process (1.3) is better than some known iteration processes for contractive-like mapping. Parameters are

$$a_n = \frac{29n}{30n+7}, b_n = \frac{19n}{20n+3}, c_n = \frac{n}{(5n+7)^2}, d_n = \frac{n}{(5n^2+7)^2},$$ for all $n \in$ N.

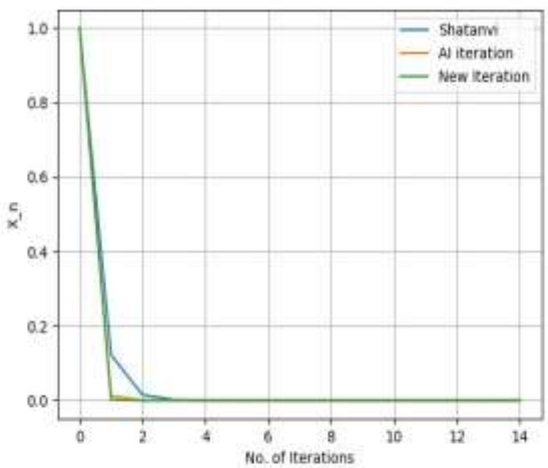
Clearly $p^* = 0$ is a fixed point of contractive-like mapping T. Table 1 shows the behaviour of some iteration processes to fixed point of T for initial value $x_0 = 1.5$.

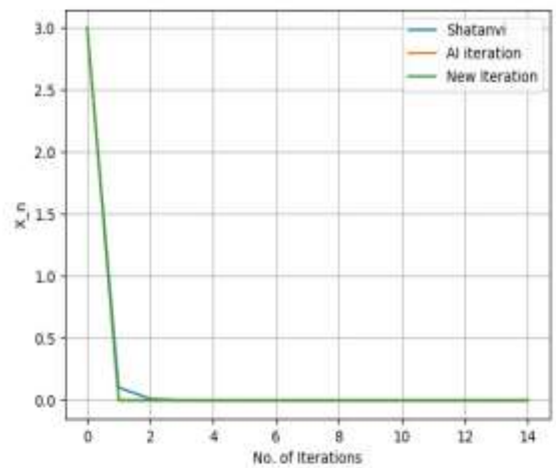
Table 1: Comparison Table

Convergence behaviour of the iterative schemes of Shatanvi et al., AI iteration and the new iteration (1.3) for the function given in Example 5.1 when the initial guesses are $x_0 = 1$ *and* 3.

Steps	Shatanvi	AI Iteration	New Iteration	Shatanvi	AI iteration	New Iteration
1	1.000000000000000	1.000000000000000	1.000000000000000	3.000000000000000	3.000000000000000	3.000000000000000
2	0.121808121906936	0.012345679012346	0.001682484481530	0.103390956476623	0.002314814814815	0.000103473832066
3	0.014837218562495	0.000152415790276	0.000002830754031	0.012593858230579	0.000028577960677	0.000000174093117
4	0.001807293727420	0.000001881676423	0.000000004762700	0.001534034218629	0.000000352814329	0.000000000292909
5	0.000220143054671	0.000000023230573	0.000000000008013	0.000186857827112	0.000000004355732	0.000000000000493
6	0.000026815212040	0.000000000286797	0.000000000000013	0.000022760800984	0.000000000053774	0.000000000000001
7	0.000003266310617	0.000000000003541	**0.000000000000000**	0.000002772450421	0.000000000000664	**0.000000000000000**

8	0.000000397863162	0.000000000000044	0.000000000000000	0.000000337706979	0.000000000000008	0.000000000000000
9	0.000000048462965	0.000000000000001	0.000000000000000	0.000000041135453	0.000181882440	0.000000000000000
10	0.000000005903183	**0.000000000000000**	0.000000000000000	0.000000000000000	**0.000000000000000**	0.000000000000000





(a) Graph corresponding to Table 1 for initial value 1**(b)** Graph corresponding to Table 1 for initial value 3

Figure 1: Convergence behaviour of the iterative schemes of Shatanvi et al., AI iteration and new iteration (1.3)

6 Convergence Theorem

Now, we introduce our convergence theorem.

Theorem 6.1. *Let* T: C → C *be a self-mapping endowed with the property (E) defined on a nonempty convex and closed subset* C *of a UCBS B with* F(T) $\neq \emptyset$ *and* $\{x_n\}$ *be the sequence defined by iteration scheme (1.3), where sequences* $\{a_n\}$, $\{b_n\} \in (0,1]$ *for all* $n \in \mathbb{N}$. *Then* F(T) $\neq \emptyset$ if *and only if* $\{x_n\}$ *is bounded and* llim$_{n\to\infty}$ $\|$ T $x_n - x_n$ $\|$ = 0.

Proof. Since F (T) is nonempty, by Proposition 2.3 T is quasi-nonexpansive. If $p^* \in F$(T), then from the
iteration (1.3)

$$\begin{aligned}\|u_n - p^*\| &= \|T((1-b_n)x_n + b_n T x_n) - p^*\| \\ &\leq \|(1-b_n)x_n + b_n T x_n - p^*\| \\ &\leq (1-b_n)\|x_n - p^*\| + b_n\|T x_n - p^*\|\end{aligned}$$

$$\leq (1-b_n)\|x_n-p^*\| + b_n\|x_n-p^*\| \leq \|x_n-p^*\|. \tag{6.1}$$

So, we have

$$\|z_n-p^*\| = \|T\,u_n-p^*\| \leq \|u_n-p^*\| \leq \|x_n-p^*\|. \tag{6.2}$$

Similarly, we have

$$\|y_n-p^*\| = \|T\,z_n-p^*\| \leq \|z_n-p^*\| \leq \|x_n-p^*\|. \tag{6.3}$$

From equations (6.1) and (6.2), we have

$$\begin{aligned}\|x_{n+1}-p^*\| &= \|T\,((1-a_n)y_n + a_nT\,y_n) - p^*\|\\ &\leq \|(1-a_n)y_n + a_nT\,y_n - p^*\|\\ &\leq (1-a_n)\|y_n-p^*\| + a_n\|T\,y_n-p^*\|\\ &\leq (1-a_n)\|y_n-p^*\| + a_n\|y_n-p^*\| \leq \|y_n-p^*\| \leq \|x_n-p^*\|. \end{aligned}\tag{6.4}$$

Thus, $\|x_n-p^*\|$ is bounded below and nonincreasing. Hence $\lim \|x_n-p^*\|$ exists. Assume that

$$\lim_{n\to\infty} \| x_n - p^* \| = \varepsilon. \tag{6.5}$$

Again by (6.1), we have $limsup \| u_n - p^* \| \leq limsup \| x_n - p^* \| = \varepsilon.$

Therefore,

$$limsup \| u_n - p^* \| \leq \varepsilon. \tag{6.6}$$

Since T is a generalized nonexpansive mapping endowed with the property (E) with a fixed point, it implies that T is quasi-nonexpansive mapping. Hence,

$$\limsup \|T\,x_n-p^*\| \leq \limsup\|x_n-p^*\| = \varepsilon \tag{6.7}$$

and (6.8)

$$\begin{aligned}\|x_{n+1}-p^*\| &= \|T\,((1-a_n)y_n + a_nT\,y_n) - p^*\|\\ &\leq \|(1-a_n)y_n + a_nT\,y_n - p^*\|\\ &\leq (1-a_n)\|y_n-p^*\| + a_n\|T\,y_n-p^*\|\\ &\leq (1-a_n)\|T\,z_n-p^*\| + a_n\|y_n-p^*\|\\ &\leq (1-a_n)\|z_n-p^*\| + a_n\|T\,z_n-p^*\|\\ &\leq (1-a_n)\|T\,u_n-p^*\| + a_n\|z_n-p^*\|\\ &\leq (1-a_n)\|u_n-p^*\| + a_n\|T\,u_n-p^*\|\\ &\leq (1-a_n)\|x_n-p^*\| + a_n\|u_n-p^*\|\end{aligned}$$

which implies that

$$\|x_{n+1}-p^*\| \leq \|x_n-p^*\| + a_n(\|u_n-p^*\| - \|x_n-p^*\|)$$
$$\frac{\|x_{n+1}-p^*\| - \|x_n-p^*\|}{a_n} \leq \|u_n-p^*\| - \|x_n-p^*\|. \tag{6.9}$$

Since $a_n \in (0,1]$, then from (6.9)

$$\|x_{n+1}-p^*\| - \|x_n-p^*\| \leq \frac{\|x_{n+1}-p^*\| - \|x_n-p^*\|}{a_n} \leq \|u_n-p^*\| - \|x_n-p^*\|. \tag{6.10}$$

Thus, we have

$$\|xn{+}1 - p*\| \leq \|un - p*\|$$
$$\varepsilon \leq \liminf \|u_n-p^*\|. \tag{6.11}$$

From equations (6.6) and (6.11), we have $\|u_n - p^*\| = \varepsilon$.
Therefore,

$$\varepsilon = \lim_{n\to\infty} \| u_n - p^* \| = \lim_{n\to\infty} \| \mathrm{T}\,(\mathrm{T}\,((1 - b_n)x_n + b_n \mathrm{T}\, x_n)) - p^* \|$$
$$\leq \lim_{n\to\infty} \| (\mathrm{T}\,((1 - b_n)x_n + b_n \mathrm{T}\, x_n) - p^* \|$$
$$\leq \lim_{n\to\infty} \| (1 - b_n)x_n + b_n \mathrm{T}\, x_n - p^* \|$$
$$\leq \lim_{n\to\infty} (1 - b_n) \| x_n - p^* \| + b_n \| \mathrm{T}\, x_n - p^* \|$$

Now using equations (6.5), (6.7) and Theorem 2.5, we conclude that $\lim_{n\to\infty} \| x_n - \mathrm{T}\, x_n \| = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \| x_n - \mathrm{T}\, x_n \| = 0$. Let $p^* \in A$ (C, $\{x_n\}$). By Proposition (2.4)

$$r\,(\mathrm{T}\,p^*, \{x_n\}) = \lim_{n\to\infty} sup \| x_n - \mathrm{T}\,p * \|$$
$$\leq \text{limsup}\, [\mu_{n\to\infty} \| x_n - \mathrm{T}\, x_n \| + \| x_n - \mathrm{p}^* \|]$$
$$\leq \mu\, \text{limsup}_{n\to\infty} \| x_n - \mathrm{T}\, x_n \| + \limsup_{n\to\infty} \| x_n - \mathrm{p}^* \|].$$

Since $\lim_{n\to\infty} \| x_n - \mathrm{T}\, x_n \| = 0$, it follows that

$$r\,(\mathrm{T}\; p^*, \{x_n\}) \leq \limsup_{n\to\infty} \| x_n - \mathrm{p}^* \|]$$
$$r\,(\mathrm{T}\,p^*, \{x_n\}) = r\,(p^*, \{x_n\}).$$

Therefore, we conclude that T $p^* \in A$(C, $\{x_n\}$). Since B is uniformly convex, A (C ,$\{x_n\}$) possesses a unique element. Thus, we have T $p^* = p^*$.

□

It is obvious that if T: C → C is nonexpansive, then it satisfies condition (E_1). Thus, we get the following corollary.

Corollary 6.2. *Let B,* C *and* $\{x_n\}$ *be as in Theorem 6.1 and* T *be a nonexpansive self-mapping defined on a nonempty closed and convex subset* C *of a UCBS B. Then F* (T) ≠ ∅ *iff* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty} \| x_n - \mathrm{T}\, x_n \| = 0$.

If, in Theorem 6.1, we assume that T satisfied the condition (E_3), then we have that T satisfies condition (C) i.e., T is Suzuki's generalized nonexpansive mapping and hence we get the following corollary.

Corollary 6.3. *Let* T *be a Suzuki's generalized nonexpansive self-mapping define on a nonempty closed and convex subset* C *of a UCBS B,* $\{x_n\}$ *defined by (1.3), where* $\{a_n\}$ *and* $\{b_n\}$ *are sequences in* [0,1] *for all n* ∈ N. *Then F*(T) *is a nonempty set iff* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty} \| x_n - \mathrm{T}\, x_n \| = 0$.

In Theorem (6.1), if we assume that T is a generalized α−nonexpansive mapping, i.e., it satisfies the condition (E_μ), for $\mu = \frac{3+\alpha}{1-\alpha}$ and $3 < \mu < \infty$. Then T satisfies the condition (E) and hence we obtain the following corollary.

Corollary 6.4. *Let* T *be a generalized* α*−nonexpansive self-mapping defined on a nonempty closed and convex subset* C *of a UCBS B, sequence* $\{x_n\}$ *is defined by (1.3), where* $\{a_n\}$ *and* $\{b_n\}$ *are sequences in* [0,1] *for all* $n \in$ N. *Then* F (T) $\neq \emptyset$ *iff* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty} \| x_n - \mathrm{T}x_n \| = 0$.

Theorem 6.1 plays an important role in proving the following weak convergence theorem.

Theorem 6.5. T *be a generalized nonexpansive self-mapping defined on a nonempty closed convex subset* C *of a uniformly convex Banach space B satisfying condition* (E)*,* $\{x_n\}$ *defined by (1.3), where* $\{a_n\}$, $\{b_n\}$ *are sequences in* [0,1] *for all* $n \in$ N. *Suppose that B has Opial's property and* F(T) $\neq \phi$. *Then the sequence* $\{x_n\}$ *converges weakly to an element of* F (T).

Proof. In Theorem 6.1, it is proved that $\{x_n\}$ is a bounded sequence, $\lim_{n\to\infty} \| x_n - \mathrm{T}x_n \| = 0$ and $\lim_{n\to\infty} \| x_n - p^* \| = 0$ exists. Since B is uniformly convex, it is reflexive. Therefore, there are a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ *weakly converges to* $p_1 \in C$. By Proposition 2.1, p_1 is an element of F (T). It is sufficient to prove that $\{x_n\}$ converges weakly to p_1. If we assume that $\{x_n\}$ does not converge weakly to p_1. Then, there exists a weakly convergent subsequence $\{x_{n_l}\}$ of $\{x_n\}$ which converges weakly to $p_2 \in$ C and $p_1 \neq p_2$. Again, by Proposition 2.1, $p_2 \in F$(T). By Theorem 6.1, $\lim_{n\to\infty} \| x_n - p^* \| = 0$ exists for all fixed points $p^* \in F$(T). By Opial's property

$$
\begin{aligned}
\lim_{n\to\infty} \| x_n - p_1 \| &= \lim_{i\to\infty} \| x_{n_i} - p_1 \| \leq \lim_{i\to\infty} \| x_{n_i} - p_2 \| \\
&= \lim_{n\to\infty} \| x_n - p_2 \| = \lim_{l\to\infty} \| x_{n_l} - p_2 \| \\
&\leq \lim_{l\to\infty} \| x_{n_l} - p_1 \| = \lim_{n\to\infty} \| x_n - p_1 \|
\end{aligned}
$$

which is a contradiction. Thus, $p_1 = p_2$. This proves that $\{p_n\}$ *weakly converges to* $p_1 \in F$(T). □

The Strong convergence theorem using the condition (C) is proved as follows.

Theorem 6.6. *Let* T *be Suzuki's generalized nonexpansive mapping defined on a nonempty convex and closed subset* C *of uniformly convex Banach space B. If* $\{x_n\}$ *is generated by iteration process (1.3), then the sequence* $\{x_n\}$ *converges strongly to an element of* F(T).

Proof. By Theorem 2.5, F(T) $\neq \emptyset$. Therefore, Theorem 6.1 implies the $\lim_{n\to\infty} \| \mathrm{T}x_n - x_n \| = 0 = 0$ and the sequence $\{x_n\}$ is bounded. Because of the compactness property of C, $\{x_n\}$ have subsequence $\{x_{n_i}\}$ such that $\{x_{n_i}\} \to p \in$ C.

Since the mapping T satisfies condition (C) is a generalized nonexpansive mapping which satisfies condition (E_3), we have by Definition 1.3,

$$\|u - \mathrm{T}(v)\| \leq 3\|u - \mathrm{T}(u)\| + \|u - v\|.$$

Put $u = x_{nj}, v = p^*$, we get

$$\|x_{nj} - \mathrm{T}(p^*)\| \leq 3\|x_{nj} - \mathrm{T}x_{nj}\| + \|x_{nj} - p^*\|,$$

as $j \to \infty$, so the conclusion is $\{x_{nj}\}$ converges to T p^*, so T $p^* = p^*$. Therefore, $x_n \to p^*$.

□

We now prove a strong convergence theorem for the mapping endowed with the property (E) $\rho(x_n, F(T))$.

Theorem 6.7. *Let* T , *B*, C *and* $\{x_n\}$ *be as in Theorem 6.1. Suppose that* $p^* \in F(T) \neq \emptyset$ *and* $\lim_{n\to\infty} \inf \rho(x_n, F(T)) = 0$ *(where* $\rho(x, F(T)) = \inf \|x - p^*\|$*). Then,* $\{x_n\}$ *converges strongly to an element of* F(T).

$p^* \in F(T)$

Proof. In Theorem 6.1, $\lim_{n\to\infty} \| x_n - p^* \|$ exists for all $p^* \in F(T)$. So $\lim_{n\to\infty} \rho(x_n, F(T))$ exists. Thus,

$$\lim_{n\to\infty} \rho(x_n, F(T)) = 0.$$

Therefore, there exists a sequence $\{q_j\}$ in F(T) and $\{x_n\}$ has a subsequence $\{x_{nj}\}$ which satisfies the following inequality $\|x_{n_j} - q_j\| \leq \frac{1}{2^j}$ for all $j \in$ N. In Theorem 6.1, it is proved that $\{x_n\}$ is nonincreasing, so

$$\|x_{n_{j+1}} - q_j\| \leq \|x_{n_j} - q_j\| \leq \frac{1}{2^j}.$$

Therefore, $\|q_{nj+1} - q_j\| \leq \|q_{nj+1} - x_{nj+1}\| + \|x_{nj+1} - q_j\|$

$$\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \leq \frac{1}{2^{j-1}} \to 0 \text{ as } j \to \infty.$$

The above argument shows that $\{q_j\}$ is a Cauchy sequence in F(T), so $\{q_j\}$ converges to some $p^* \in F$(T).

Now, apply triangle inequality

$$\|x_{nj} - p^*\| \leq \|x_{nj} - q_j\| + \|q_j - p^*\|.$$

$$\lim_{j\to\infty} \|x_{nj} - p^*\| \leq \lim_{j\to\infty} \|x_{nj} - q_j\| + \lim_{j\to\infty} \|q_j - p^*\|.$$

Above argument completes that $\{x_{nj}\}$ converges strongly to p^*. By Theorem 6.1, $\lim_{n\to\infty} \| x_n - p^* \|$ exists, hence $\{x_n\}$ converges strongly to $p^* \in F$(T).

□

In 1974, Senter et al. [25] introduced the condition (A) as follows.

A mapping T: C → C satisfies the condition (A) if there exists a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0, g(a) > 0$ for all $a \in (0,\infty)$ such that $\rho(a, Ta) \geq g(\rho(a, F(T)))$ for all $a \in$ C.

Theorem 6.8. *Let* T, *B*, C *and* $\{x_n\}$ *be as in Theorem 6.1 such that* F(T) $\neq \emptyset$. *If* $\{x_n\}$ *is a sequence defined by the iteration process (1.3) and* T *satisfies the condition* (A)*, then* $\{x_n\}$ *strongly converges to an element of* F(T).

Proof. By Theorem 6.1, $\lim_{n\to\infty} \| x_n - p^* \|$ exists for all p^* of T and

$$\|x_{n+1} - p^*\| \leq \|x_n - p^*\|$$

Taking $\inf_{p^* \in F(T)}$ in both sides

$$\inf_{p^* \in F(T)} \|x_{n+1} - p^*\| \leq \inf_{p^* \in F(T)} \|x_n - p^*\|$$

which yields

$$\|x_{n+1} - F(T)\| \leq \|x_n - F(T)\|.$$

From the above inequality it is obvious that the sequence $\{\|x_n - F(T)\|\}$ is bounded below and nonincreasing. Therefore, by Theorem 6.1 $\lim_{n\to\infty} \| x_n - F(T) \|$ exists.
Also, by Theorem 6.1, we have

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0.$$

By the condition (A),

$$\lim_{n\to\infty} g(\rho(x_n, F(T)) \leq \lim_{n\to\infty} \rho(x_n, T x_n) = 0.$$

Thus, we have

$$\lim_{n\to\infty} g(\rho(x_n, F(T)) = 0.$$

Since g is a non-decreasing function satisfying $g(0) = 0$ and $g(a) > 0$ for all points $a \in (0, \infty)$.
It compels that $\lim_{n\to\infty} \rho(x_n, F(T)) = 0$. As all the relevant conditions for Theorem 6.7 are satisfied. Thus, the sequence $\{x_n\}$ converges strongly to a fixed point of T. □

If, in Theorem 6.8, we assume that T is a generalized α-nonexpansive mapping, where $\alpha = \frac{\mu-3}{1+\mu}$ and $3 < \mu < \infty$, then we have the following corollary.

Corollary 6.9. *Let* T *be a generalized α-nonexpansive self-mapping defined on a nonempty closed and convex subset* C *of a uniformly convex Banach space B such that* $F(T) \neq \emptyset$. *If* $\{x_n\}$ *is a sequence defined by the iteration process (1.3) and* T *satisfies the condition* (A)*, then* $\{x_n\}$ *converges strongly to an element of* F(T).

If, in Theorem 6.8, T is generalized nonexpansive mapping satisfying the condition (E_3), i.e., T is Suzuki's generalized nonexpansive mapping then we obtain the following corollary.

Corollary 6.10. *Let* T: C → C *be a Suzuki's generalized nonexpansive mapping defined on a nonempty closed and convex subset* C *of a uniformly convex Banach space B such that* $F(T) \neq \emptyset$. *If* $\{x_n\}$ *is a sequence defined by iteration process (1.3) and* T *satisfy the condition* (A)*, then* $\{x_n\}$ *converges strongly to an element of* F(T).

Now, we give an example of generalized nonexpansive mapping endowed with the property (E) to illustrate the convergence of the iteration process (1.3).

Example 6.1. Let C = [1,6], which is a closed and convex subset of the Banach space B = R, endowed with the usual norm. T: C → C is defined by

$$T(u) = \begin{cases} \frac{u}{2} + 2 \\ 4, \end{cases}$$

, if $u \in [0,6)$
if $u = 6$.

It is obvious that $4 \in F(T)$ and T fulfils the Garc´ıa-Falset condition for all $\mu \geq 1$ and for $u = \frac{55}{10}$ and $v = 6$, T fails to satisfy condition (C).

Case 1: If $v \in [0,6)$ and $u = 6$, then $\|u - Tu\| = \|6 - 4\| = 2, \|u - v\| = \|6 - v\|$, and

$$\|u - \mathcal{T}v\| = \|6 - \frac{v}{2} - 2\| = \|1 + \frac{6}{2} - \frac{v}{2}\| \leq \|1\| + \|\frac{6-v}{2}\|$$
$$\leq \|2\| + \|6 - v\| \leq \mu\|2\| + \|6 - v\| = \mu\|u - \mathcal{T}u\| + \|u - v\|$$

Case 2: If $u \in [0,6)$ and $v = 6$, it is similar to **Case 1:**.

Case 3: If u and $v \in [0,6)$, then $\|u - \mathcal{T}u\| = \|u - \frac{u}{2} - 2\| = \|\frac{u}{2} - 2\|$ and

$$\|u - \mathcal{T}v\| = \|u - \frac{v}{2} - 2\| = \|\frac{u}{2} + \frac{u}{2} - \frac{v}{2} - 2\| \leq \|\frac{u}{2} - 2\| + \|\frac{u-v}{2}\|$$
$$\leq \mu\|\frac{u}{2} - 2\| + \|u - v\| = \mu\|u - \mathcal{T}u\| + \|u - v\|$$

Thus, **Case 1, Case 2, Case 3** hold true for $\mu \geq 1$. Therefore, T is a Garc´ıa-Falset mapping.

7 An Application to Volterra-Fredholm integral equation

In this section, we use iteration (1.3) to solve the following Volterra-Fredholm integral equation given by Lungu and Rus [13]:

$$f(x,y) = g(x,y,h(f(x,y))) + \int_0^x \int_0^y K(x,y,u,v,f(u,v))dudv, \tag{7.1}$$

Where $x, y \in R+$. Let $(W, |\,.\,|)$ be a Banach space. Let $\tau > 0$ and

$X_\tau = \{f \in C(R^2, W) |\ \exists\ M(f) > 0 : |f(x,y)|e^{-\tau(x+y)} \leq M(f)\}$. We now consider BieleckiâĂŹ norm on X_τ as follows:

$$\|f\|_\tau = \sup_{x, y \in R_+} (|f(x, y)|e^{-\tau(x+y)}).$$

Obviously, $(X_\tau, \|.\|_\tau)$ is a Banach space (see [5]).

The following result will play a major role in proving the main result.

Theorem 7.1. *[13] Suppose the following conditions are fulfilled*

(A) $g \in C(\mathbb{R}_+^2 \times W, W), K \in C(\mathbb{R}_+^4 \times W, W)$;

(B) *h:* $X_\tau \to X_\tau$ *such that*

$$\exists l_h > 0: |h(f_1(x, y)) - h(f_2(x, y))| \leq l_h \|f_1 - f_2\|.e^{\tau(x+y)},$$

for all $x,y \in R+$ *and* $f_1,f_2 \in X_\tau$;

(C) $\exists\ l_g > 0: |g(x, y, u_1) - g(x, y, u_2)| \leq l_g \|u_1 - u_2\|$, *for all* $x, y \in R_+$ *and* $u_1, u_2 \in W$;

(D) $\exists\ l_K(x, y, u, v) : |K(x,y,u,v,u_1) - K(x,y,u,v,u_2)| \leq l_K(x, y, u, v)|u_1 - u_2|$,

for all $x, y, u, v \in R+$ *and* $u_1, u_2 \in W$;

(E) $l_K \in C(\mathbb{R}_1^4, \mathbb{R}_+)$ *and*

$$\int_0^x \int_0^y l_K(x,y,u,v)e^{\tau(x+y)}dudv \leq le^{\tau(x+y)},$$

for all $x, y \in R+$;

(F) $(l_g l_h + l) < 1$.

Then the equation (7.1) has a unique solution $z \in X_\tau$ *and the sequence of successive approximations*

$$f_{(n+1)}(x,y) = g(x,y,h(f_n(x,y))) + \int_0^x \int_0^y K(x,y,u,v,f_n(u,v))dudv, \tag{7.2}$$

for all $n \in \mathbb{N}$ *converges uniformly to z.*

Our main result is as follows:

Theorem 7.2. *If all the conditions from* (*A*) *to* (*F*) *in theorem 7.1 are satisfied, then the equation (7.2) has a unique fixed point* $p^* \in X_\tau$ *and the iteration (1.3) with sequence* $\{a_n\}$, $\{b_n\} \in (0,1)$ *such that* $\sum_{n=0}^{\infty} a_n = \infty$ *converges strongly to* p^*.

Proof. Let $\{x_n\}$ be the sequence defined by the new iteration scheme (1.3) for the operator T: $X_\tau \to X_\tau$ defined by

$$\mathcal{T}(f(x,y)) = g(x,y,h(f(x,y))) + \int_0^x \int_0^y K(x,y,u,v,f(u,v))dudv. \tag{7.3}$$

We must show that $\{x_n\} \to 0$ as $n \to \infty$. By (1.3)

$$\|x_{n+1} - p^*\|_\tau = \sup_{x,y \in R+} (|T((1-a_n)y_n + a_n T x_n)(x,y) - T(p^*(x,y))|e^{-\tau(x+y)}).$$

$$\begin{aligned}
&|\mathcal{T}((1-a_n)y_n + a_n\mathcal{T}y_n)(x,y) - \mathcal{T}(p^*(x,y))| \\
&\le |g(x,y,h((1-a_n)y_n + a_n\mathcal{T}y_n)(x,y)) - g(x,y,h(p^*(x,y))| + \\
&\Big|\int_0^x \int_0^y K(x,y,u,v,((1-a_n)y_n + a_n\mathcal{T}y_n)(u,v))dudv \\
&- \int_0^x \int_0^y K(x,y,u,v,p^*(u,v))dudv\Big| \\
&\le l_g|h((1-a_n)y_n + a_n\mathcal{T}y_n)(x,y) - h(p^*(x,y))| + \int_0^x \int_0^y |K(x,y,u,v,((1-a_n)y_n \\
&+ a_n\mathcal{T}y_n)(u,v)) - K(x,y,u,v,p^*(u,v))|dudv \\
&\le l_g l_h\|((1-a_n)y_n + a_n\mathcal{T}y_n) - p^*\|_\tau e^{\tau(x+y)} + \int_0^x \int_0^y l_K(x,y,u,v)|((1-a_n)y_n \\
&+ a_n\mathcal{T}y_n)(u,v) - p^*(u,v)|dudv \\
&\le l_g l_h\|((1-a_n)y_n + a_n\mathcal{T}y_n) - p^*\|_\tau e^{\tau(x+y)} + l\|((1-a_n)y_n + a_n\mathcal{T}y_n) - p^*\|_\tau e^{\tau(x+y)} \\
&= (l_g l_h + l)\|(1-a_n)y_n + a_n\mathcal{T}y_n - p^*\|_\tau e^{\tau(x+y)} \\
&\le (l_g l_h + l)\|(1-a_n)y_n + a_n\mathcal{T}y_n - p^*\|_\tau
\end{aligned} \tag{7.4}$$

and

$$\begin{aligned}
\|(1-a_n)y_n + a_n T y_n - p^*\|_\tau &= \|(1-a_n)(y_n - p^*) + a_n(T y_n - p^*)\|_\tau \\
&\le (1-a_n)\|(y_n - p^*)\|_\tau + a_n\|T y_n - p^*\|_\tau.
\end{aligned} \tag{7.5}$$

So

$$\|T y_n - p^*\|_\tau = \sup_{x,y \in R+} (|T y_n(x,y) - T p^*(x,y)|e^{-\tau(x+y)}), \tag{7.6}$$

Now

$$|T\, y_n(x, y) - T\, p^*(x, y)| \le |g(x, y, h(y_n(x, y))) - g(x, y, h(p^*(x, y)))| +$$
$$\left|\int_0^x \int_0^y K(x,y,u,v,y_n(u,v))dudv - \int_0^x \int_0^y K(x,y,u,v, p^*(u,v))dudv\right|$$
$$\le l_g |h(y_n(x,y)) - p^*(x,y)| + \int_0^x \int_0^y |K(x,y,u,v,y_n(u,v)) - K(x,y,u,v, p^*(u,v))| dudv$$
$$\le l_g\, l_h\, \|y_n - p^*\|_\tau e^{\tau(x+y)} + \int_0^x \int_0^y l_k(x,y,u,v)|y_n - p^8| dudv$$
$$\le lg\ lh\ \|yn - p*\|\tau\ e\tau(x+y) + l\ \|yn - p^*\|\tau$$
$$e^{\tau(x+y)}$$
$$= (lg\ \ lh + l)\ \|yn - p*\|\ \tau\ e^{\tau(x+y)}$$
$$\le (l_g\, l_h + l)\ \|y_n - p^*\|_\tau. \tag{7.7}$$

From equations (7.5) and (7.7), we get

$$\|(1 - a_n)y_n + a_n T\, y_n - p^*\|_\tau \le (1 - a_n)\|(y_n - p^*)\|_\tau + a_n(l_g\, l_h + l)\|y_n - p^*\|_\tau$$
$$= [1 - a_n\{1 - (l_g\, l_h + l)\}]\|y_n - p^*\|_\tau. \tag{7.8}$$

Thus, from equations (7.4) and (7.8) we obtain

$$|T((1 - a_n)y_n + a_n T\, y_n)(x, y) - T(p^*(x, y))| \le (l_g\, l_h + l)[1 - a_n\{1 - (l_g\, l_h + l)\}]\|y_n - p^*\|_\tau.$$

Therefore,

$$\|x_{n+1} - p^*\|_\tau \le (l_g\, l_h + l)[1 - a_n\{1 - (l_g\, l_h + l)\}]\|y_n - p^*\|_\tau. \tag{7.9}$$

Again,

$$\|y_n - p^*\|_\tau = \sup_{x,y \in R+} (|T\, z_n)(x, y) - T(p^*(x, y))|e^{-\tau(x+y)})$$

$$\begin{aligned}
&|\mathcal{T}(z_n(x,y)) - \mathcal{T}(p^*(x,y))| \\
&\le |g(x,y,h(z_n(x,y))) - g(x,y,h(p^*(x,y)))| + \\
&\quad \left|\int_0^x \int_0^y K(x,y,u,v,z_n(u,v))dudv - \int_0^x \int_0^y K(x,y,u,v,p^*(u,v))dudv\right| \\
&\le l_g|h(z_n(x,y)) - h(p^*(x,y))| + \\
&\quad \int_0^x \int_0^y |K(x,y,u,v,z_n(u,v)) - K(x,y,u,v,p^*(u,v))|dudv \\
&\le l_g l_h \|z_n - p^*\|_\tau e^{\tau(x+y)} + \int_0^x \int_0^y l_K(x,y,u,v)|z_n(u,v) - p^*(u,v)|dudv \\
&\le l_g l_h \|z_n - p^*\|_\tau e^{\tau(x+y)} + l\|q_n - p^*\|_\tau e^{\tau(x+y)} \\
&= (l_g l_h + l)\|z_n - p^*\|_\tau e^{\tau(x+y)}.
\end{aligned}$$

Hence, we get

$$\|y_n - p^*\|_\tau \le (l_g\, l_h + l)\|z_n - p^*\|_\tau. \tag{7.10}$$

From (7.9) and (7.10), we obtain

$$\|x_{n+1} - p^*\|_\tau \le (l_g\, l_h + l)^2[1 - a_n\{1 - (l_g\, l_h + l)\}]\|z_n - p^*\|_\tau. \tag{7.11}$$

Similarly, we get

$$\|z_n - p^*\|_\tau \leq (l_g\, l_h + l)\|u_n - p^*\|_\tau. \tag{7.12}$$

Putting (7.12) into (7.11)

$$\|x_{n+1} - p^*\|_\tau \leq (l_g\, l_h + l)^3[1 - a_n\{1 - (l_g\, l_h + l)\}]\|u_n - p^*\|_\tau. \tag{7.13}$$

Similarly,

$$\|u_n - p^*\|_\tau \leq (l_g\, l_h + l)[1 - b_n\{1 - (l_g\, l_h + l)\}]\|x_n - p^*\|_\tau. \tag{7.14}$$

Now from equations (7.13) and (7.14)

$$\|x_{n+1} - p^*\|_\tau \leq (l_g\, l_h + l)^4[1 - a_n\{1 - (l_g\, l_h + l)\}][1 - b_n\{1 - (l_g\, l_h + l)\}]\|x_n - p^*\|_\tau. \tag{7.15}$$

Recalling assumption $(l_g\, l_h + l) < 1$ and $b_n \in (0,1]$, it follows that $[1 - b_n\, \{1 - (l_g\, l_h + l)\}] \leq 1$. Thus, equation (7.15) becomes

$$\|x_{n+1} - p^*\|_\tau \leq [1 - a_n\{1 - (l_g\, l_h + l)\}]\|x_n - p^*\|_\tau. \tag{7.16}$$

From equation (7.16), Inductively we obtain

$$\|x_{n+1} - p^*\|_\tau \leq \|x_0 - p^*\|_\tau \prod_{i=0}^{i=n}[1 - a_i\{1 - (lg\ l_h + l)\}]. \tag{7.17}$$

Since $a_i \in (0,1]$ for all $i \in \mathbb{N}$ and from condition (F) we have $(l_g\, l_h + 1 < 1)$. Thus, $[1 - a_i\, \{1 - (l_g\, l_h + l)\}] \leq 1$. We also know that $1 - \eta \leq e^{-\eta}$ for all $\eta \in [0,1]$. From (7.17) we have

$$\|xn+1 - p*\|\tau \leq \|x0 - p*\|\tau e^{-[a_i\{1 - (l_g l_h + l)\}]\sum_{i=0}^{i=n} a_i} \tag{7.18}$$

Therefore, $\lim_{n\to\infty} \| x_n - p^* \| = 0$ as $\sum_{i=0}^{i=n} a_i = \infty$ whenever $n \to \infty$ and $e^{-\infty} = 0$.
This completes the proof. □

8 Conclusion

In Section 3 we proved stability result of the iteration scheme (1.3). Section 4 deals with the data dependence results for the iteration process (1.3) under contractive-like conditions. In Section 5 we proved that the new iteration scheme has a better rate of convergence than some well-known iteration schemes for example, Mann, Ishikawa, Abbas et al., Sintunavarat et al., normal S-iteration, M iterative process, Garodia et al. iteration and Picard S-iteration process, Shatanvi et al. and AI iteration process. Our assertion is supported by a numerical example with table and graphs. Some weak and strong convergence results for new iteration scheme for generalized nonexpansive mappings endowed with the property (E) in uniformly convex Banach spaces are discussed in section 6. More precisely, Theorem (6.1) extends and generalized the works done by Asha Rani et al. [23], Ullah et al. [35] and Thakur [32], [33] in the sense that it provides almost fixed point property results for generalized nonexpansive mappings endowed with the property (E) more general than Suzuki's generalized nonexpansive mappings and nonexpansive mappings.

In final Section 7 we used new iterative algorithm to solve the Volterra-Fredholm integral equation.

Acknowledgment: All authors contributed equally to this work. All authors have read and agreed to submit the manuscript for possible publication.

Conflict of interest: The authors declare no conflict of interest.

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