Contractive mappings on B_n - metric spaces

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Abstract: The notion of B_n - metric spaces is introduced as a generalization of B_4 - metric spaces. In this paper, we consider the relationships between a B_n - metric space, B_{n-1} metric space, and a metric space. We show that a B_n - metric space can be generated by B_{n-1} - metric space. We also show that a B_{n-1} - metric space, gives rise to a B_n - metric space and give some examples. We also study the relationship of contractions of self-maps on B_n - metric space and B_{n-1} - metric spaces.

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I. INTRODUCTION

In 1922, [5] Banach introduced the famous Banach contraction principle. Since then, generalizations of the contraction principle in different directions as well as many new fixed point results with applications have been established by different researchers ([1] – [4], [7] – [10], [18]).

It is a well-known fact that every Banach contraction is continuous. In 1968, [16] Kannan proved the following result for not necessarily continuous mappings.

Theorem 1.1. ([15] Kannan) Let (Y_0, d_0) be a complete metric space. Suppose $T_0 : Y_0 \to Y_0$ is a mapping such that

$$\mathrm{d}(T_0x_0, T_0y_0) \leqslant \lambda_0\{d_0(x_0, T_0x_0) + d_0(y_0, T_0y_0)\}\ \forall\ x_0, y_0 \in Y_0$$

Since, then much work has been done on the contraction mappings and their extensions ([6], [11] – [13], [16], [17], [20], [24]).

We investigate relationships between B_n - metric spaces and B_{n-1}- metric spaces. It is known that every B_{n-1} - metric generates an B_n - metric, and in [21], it was given an example of an B_4 - metric which is generated by a $\mathbb{S}$ - metric. Here, we give a new example of an B_n - metric which is generated by a B_{n-1} metric, and use this new B_n - metric in the next sections. However, we give a counterexample of this result.

II. Preliminaries

We begin by defining some of the terms used in this paper. Sedghi, Shobe, and Aliouche defined an $\mathbb{S}$ - metric space as a generalization of a metric space in [23] in 2014 as following criteria:

Definition 2.1. [23] Let $Y_0 \neq \emptyset$, and $\mathbb{S}: {Y_0}^3 \to [0, \infty)$ be a function satisfying the following criteria: $\forall\ x_0, y_0, z_0, a_0 \in Y_0$

(1) $\mathbb{S}(x_0, y_0, z_0) = 0 \Leftrightarrow x_0 = y_0 = z_0$,

(2) $\mathbb{S}(x_0, y_0, z_0) \leqslant \mathbb{S}(x_0, x_0, a_0) + \mathbb{S}(y_0, y_0, a_0) + \mathbb{S}(z_0, z_0, a_0)$

Then, $\mathbb{S}$ is called an $\mathbb{S}$ - metric on Y_0 and the pair $(Y_0, \mathbb{S})$ is called an $\mathbb{S}$ - metric space.

Recently, K.K.M. Sarma, Ch. Srinivasa Rao and S. Ravi Kumar [21] have defined the concept of a

B_4 - metric space as a generalization of a metric space as follows:

Definition 2.3. [**21**] Let $Y_0 \neq \phi$ and $B_4: {Y_0}^4 \to \Re$ be a function that meets the following criteria: $\forall\, x_1, x_2, x_3, x_4, a_0 \in Y_0$

(i) $B_4(x_1, x_2, x_3, x_4) = 0 \Leftrightarrow x_1 = x_2 = x_3 = x_4$

(ii)
$$B_4(x_1, x_2, x_3, x_4) \leqslant \begin{Bmatrix} B_4(x_1, x_1, x_1, a_0) + B_4(x_2, x_2, x_2, a_0) \\ +B_4(x_3, x_3, x_3, a_0) + B_4(x_4, x_4, x_4, a_0) \end{Bmatrix}$$

Then, B_4 is called a B_4 - metric on Y_0 and the pair (Y_0, B_4) is called a B_4 - metric space. Using this definition, Convergence, Cauchy sequence, and Completeness are successfully introduced in B_4 - metric spaces.

Example 2.2. [21] Let $\Re = Y_0$ and define the function $B_4: {Y_0}^4 \to \Re$ by

$$B_4(x_1, x_2, x_3, x_4)$$

$$= |x_1 - x_3| + |x_2 - x_3| + |x_1 + x_2 + x_3 - 3x_4|$$

$\forall\, x_1, x_2, x_3, x_4 \in \Re$.

Then, B_4 is a B_4 - metric on Y_0.

Example 2.3. [21] Let $Y_0 \neq \phi$ and define the function $B_4: {Y_0}^4 \to \Re$ as

$$B_4(x_1, x_2, x_3, x_4) = \begin{cases} 0, & \text{if } x_1 = x_2 = x_3 = x_4 \\ 1, & \text{otherwise} \end{cases}$$

Then, B_4 is a B_4 - metric on Y_0.

Now we introduce the notions of limits, convergence, Cauchy sequence, and completeness in a B_4 - metric space.

Definition 2.5. [21] Let (Y_0, B_4) be a B_4 - metric space.

(1) Convergence: A sequence $\{x_0\}$ in Y_0 Converges to x_0

if $B_4(x_{0_\mathrm{m}}, x_{0_\mathrm{m}}, x_{0_\mathrm{m}}, x_0) \to 0$ as $m \to \infty$.

That is given $\varepsilon > 0 \;\exists\; n_0 \in \mathbb{N} \ni$,

$\forall\, m \geqslant n_0,\; B_4(x_{0_\mathrm{m}}, x_{0_\mathrm{m}}, x_{0_\mathrm{m}}, x_0) < \varepsilon$.

We denote this by $\lim_{n\to\infty} x_{0_\mathrm{m}} = x_0$.

(2) Cauchy Sequence: A sequence $\{x_{0_\mathrm{m}}\}$ in Y_0 is called a Cauchy Sequence if $B_4\left(x_{0_{m_1}}, x_{0_{m_1}}, x_{0_{m_1}}, x_{0_{m_2}}\right) \to 0$ as $m_1, m_2 \to \infty$. That is, given $\varepsilon > 0$, $\exists\; n_0 \in \mathbb{N} \ni$, $\forall\; m_1, m_2 \geqslant n_0$, $B_4\left(x_{0_{m_1}}, x_{0_{m_1}}, x_{0_{m_1}}, x_{0_{m_2}}\right) < \varepsilon$

(3) Completeness: A B_4 - metric space (Y_0, B_4) is called Complete if every Cauchy Sequence in Y_0 is convergent to a point $x_0 \in Y_0$.

We now prove a few lemmas, which we use in our further development.

Lemma 2.3. [21] Let (Y_0, B_4) be a B_4-metric space, $\forall\, x_0, y_0 \in Y_0$. Then, $B_4(x_0, x_0, x_0, y_0) = B_4(y_0, y_0, y_0, x_0)$

Lemma 2.4. [21] $x_{0_\mathrm{m}} \to x_0 \Leftrightarrow B_4(x_0, x_0, x_0, x_{0_\mathrm{m}}) \to 0$ as $m \to \infty$

The following Lemma shows that a convergent sequence has only one limit.

Lemma 2.5. [21] $x_{0_\mathrm{m}} \to x_0,\; x_{0_\mathrm{m}} \to y_0$

$\Rightarrow x_0 = y_0$.

Lemma 2.6. [21] $x_{0_\mathrm{m}} \to x_0$

$\Rightarrow \{x_{0_\mathrm{m}}\}$ is a Cauchy Sequence.

In the following Lemma, we show that an $\mathbb{S}$ - metric gives rise to a B_4-metric.

Lemma 2.7. [21] Let $(Y_0, \mathbb{S})$ be an $\mathbb{S}$ - metric space and define the function $B_{\mathcal{S}}: {Y_0}^4 \to \Re$ by $B_{\mathcal{S}}(x_0, y_0, z_0, t_0) = \mathbb{S}(y_0, z_0, t_0) + \mathbb{S}(x_0, z_0, t_0) + \mathbb{S}(x_0, y_0, t_0)\; \mathbb{S}(x_0, y_0, z_0),\; \forall\, x_0, y_0, z_0, t_0 \in Y_0$.

Then $B_{\mathcal{S}}$ is a B_4-metric on Y_0.

We call $B_{\mathcal{S}}$ as the B_4 - metric generated by $\mathbb{S}$.

Definition 2.6. [21] Let $Y_0 \neq \phi$, and $\lambda_0 \geqslant 1$. Suppose $\mathbb{S}: {Y_0}^3 \to \mathfrak{R}$ be a function meeting the criteria below: $\forall\, x_0, y_0, z_0, a_0 \in\ Y_0$

(1) $\mathbb{S}(x_0, y_0, z_0) = 0 \Leftrightarrow\ x_0 = y_0 = z_0$

(2) $\mathbb{S}(x_0, y_0, z_0) \leqslant \lambda_0\big(\mathbb{S}(x_0, x_0, a) +$
$\mathbb{S}y0,y0,a+\mathbb{S}z0,z0,a.$

Then, $\mathbb{S}$ is called a $\mathbb{S}$ - metric on Y_0 and the pair $(Y_0, \mathbb{S})$ is called a $\mathbb{S}$ - metric space with index λ_0

Note: If $\lambda_0 = 1$ we get the usual $\mathbb{S}$ - metric space (by definition 1)

Lemma 2.8. [21] Let (Y_0, B_4) be any B_4 - metric space. Define $\mathbb{S}_b: {Y_0}^3 \to \mathfrak{R}$ as follows:
$\mathbb{S}_b(x_0, y_0, z_0) =$
$B_4(x_0, x_0, y_0, z_0) + B_4(x_0, g_0, g_0, z_0) +$
$B_4(x_0, g_0, z_0, z_0) \qquad \forall\, x_0, y_0, z_0 \in Y_0$

Then $(Y_0, \mathbb{S}_b)$ is a $\mathbb{S}$ - metric space with index 2.

($\mathbb{S}_b$ is called $\mathbb{S}$ - metric space with index 2)

III. Main Result

Notation: $\mathfrak{R}$ stands for the set of real numbers, and $\mathbb{N}$ stands for a set of positive integers. Now we introduce the notion of B_4 - metric spaces. Suppose n is a positive integer and $n \geqslant 4$.

Definition 3.1. Let $Y_0 \neq \phi$ and $B_n: {Y_0}^n \to \mathfrak{R}$ be a function satisfying the following conditions:
$\forall x_1, x_2, \dots, x_n, a_0 \in Y_0$

(i) $B_n(x_1, x_2, \dots, x_n) = 0 \Leftrightarrow x_1 = x_2 = \cdots = x_n$

(ii) $B_n(x_1, x_2, \dots, x_n) \leqslant B_n(x_1, x_1, \dots, x_1, a_0) + B_n(x_2, x_2, \dots, x_2, a_0) + \cdots + B_n(x_n, x_n, \dots, x_n, a_0)$

Then, B_n is called a B_n - metric on Y_0 and the pair (Y_0, B_n) is called a B_n - metric space. Clearly, this definition extends a B_4 - metric space to B_n - metric space for $n \geqslant 4$.

The following two examples give an insight into the concept of B_n - metric spaces.

Example 3.1. Let $Y_0 = \mathfrak{R}$ and n be a positive integer, $n \geqslant 4$. Define the function $B_n: Y^n \to \mathfrak{R}$ by

$$B_n(x_1, x_2, \dots, x_n) = \left\{\begin{matrix} |x_1 - x_{n-1}| + |x_2 - x_{n-1}| + \cdots + \\ |x_{n-2} - x_{n-1}| + |x_1 + x_2 + \cdots + x_{n-1} - (n-1)x_n| \end{matrix}\right\}$$

$\forall\, x_1, x_2, \dots, x_n \in \mathfrak{R}$.Then, B_n is a B_n - metric on Y_0

Example 3.2. Let $Y_0 \neq \phi, n \geqslant 4$ and define the function $B_n: {Y_0}^n \to \mathfrak{R}$. Then, B_n is a B_n - metric on Y_0.

$$B_n(x_1, x_2, \dots, x_n) = \begin{cases} 0, \text{ if } x_1 = x_2 = \cdots = x_n \\ 1, \text{ otherwise} \end{cases}$$

Observation: The $\mathbb{S}$ - metric space mentioned in [2] is nothing but a B_3 - metric space $(n = 3)$.

Definition 3.2. Let (Y_0, B_n) be a B_n - metric space.

Now, we introduce the notions of Convergence, Cauchy sequence, and completeness.

Convergence: A sequence $\{x_{0_m}\}$ in Y_0 converges to x_0

If $B_n(\underbrace{x_{0_m}, x_{0_m}, \dots, x_{0_m}}_{(n-1)\text{ times}}, x_0) \to 0$ as $m \to \infty$

That is, given $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N} \ni \forall\ \ m \geqslant n_0$,

$$B_n(\underbrace{x_{0_m}, x_{0_m}, \dots, x_{0_m}}_{(n-1)\text{ times}}, x_0) < \varepsilon.$$

We denote this by $\lim_{m\to\infty} x_{0_m} = x_0$.

Cauchy sequence: A sequence $\{x_{0_m}\}$ in Y_0 is called a Cauchy sequence if
$B_n(\underbrace{x_{0_k}, x_{0_k}, \dots, x_{0_k}}_{(n-1)\text{ times}}, x_{0_m}) \to 0$ as $k, m \to \infty$

That is, given $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N} \ni \forall\ \ k, m \geqslant n_0$,

$$B_n(\underbrace{x_{0_k}, x_{0_k}, \dots, x_{0_k}}_{(n-1)\text{ times}}, x_{0_m}) < \varepsilon$$

Completeness: A B_n - metric space (Y_0, B_n) is called complete if every Cauchy sequence in Y_0 is convergent to a point $x_0 \in Y_0$.

We now prove a few lemmas, which we use in our further development.

Lemma 3.1. Let (Y_0, B_n) be a B_n - metric space.

Then $B_n(x_1, x_1, \dots, x_1, x_2) = B_n(x_2, x_2, \dots, x_2, x_1)$, $\forall\ x_1, x_2 \in Y_0$.

Proof. Suppose (Y_0, B_n) is a B_n - metric space.

By definition 2, replacing a by x, we get,

$B_n(x_1, x_1, \dots, x_1, x_2)$

$\leqslant B_n(x_1, x_1, \dots, x_1, x_1) + B_n(x_1, x_1, \dots, x_1, x_1)$

$+B_n(x_1, x_1, \dots, x_1, x_1) + B_n(x_2, x_2, \dots, x_2, x_1)$

$\Rightarrow B_n(x_1, x_1, \dots, x_1, x_2) \leqslant 0 + 0 + 0 + B_4(x_2, x_2, \dots, x_2, x_1)$

$\Rightarrow B_n(x_1, x_1, \dots, x_1, x_2) \leqslant B_n(x_2, x_2, \dots, x_2, x_1)$

Similarly,

$B_n(x_2, x_2, \dots, x_2, x_1) \leqslant B_n(x_1, x_1, \dots, x_1, x_2)$

From (1.1) and (1.2),
$B_n(x_1, x_1, \dots, x_1, x_2) = B_n(x_2, x_2, \dots, x_2, x_1)$.

Lemma 3.2. $x_{0_m} \to x_0 \Leftrightarrow B_n(x_0, x_0, \dots, x_0, x_{0_m}) \to 0$ as $m \to \infty$.

Lemma 3.3. $x_{0_m} \to x_0,\ x_{0_m} \to y_0 \Rightarrow x_0 = y_0$.

Lemma 3.4. $x_{0_m} \to x_0 \Longrightarrow \{x_{0_m}\}$ is a Cauchy sequence.

Notation: Suppose $n \geqslant 5$ and B_{n-1} is a B_{n-1} - metric on Y_0.

That is $B_{n-1}: {Y_0}^{n-1} \to R$ is a B_{n-1} - metric. We denote B_{n-1} - metric by K - metric.

If $n = 4, B_{n-1} = B_3$ is a $\mathbb{S}$ - metric.

In the following Lemma, we show that a K - metric gives rise to a B_n - metric.

Lemma 3.5. Suppose $n \geqslant 4$. Let K be a B_{n-1} - metric on Y_0.

Define the function $B_n: {Y_0}^n \to \Re$ by

$B_n(x_1, x_2, \dots, x_n) = K(x_2, x_3, \dots, x_n) + K(x_1, x_3 \dots, x_n) + \cdots + K(x_1, x_2 \dots, x_{n-1})$,
$\forall\ x_1, x_2, \dots, x_n \in Y_0$.
Then B_n is a B_n - metric on Y_0.

Definition 3.3. Let $Y_0 \neq \emptyset, \lambda_0 \geqslant 1$ and let $B_n: {Y_0}^n \to \Re$ be a function satisfying the following conditions: $\forall x_1, x_2, \dots, x_n, a_0 \in Y_0$

(i) $B_n(x_1, x_2, \dots, x_n) = 0 \Leftrightarrow x_1 = x_2 = \cdots = x_n$,

(ii)
$B_n(x_1, x_2, \dots, x_n) \leqslant$

$$\lambda_0 \left\{ \begin{array}{c} B_n\left(\underbrace{x_1, x_1, \dots, x_n}_{(n-1)\text{ times}}, a_0\right) + B_n\left(\underbrace{x_2, x_2, \dots, x_2}_{(n-1)\text{ times}}, a_0\right) \\ + \cdots + B_n\left(\underbrace{x_n, x_n \dots, x_n}_{(n-1)\text{ times}}, a_0\right) \end{array} \right\}$$

Then, B_n is called B_n - metric with index λ_0 on Y_0 and the pair (Y_0, B_n) is called a B_n - metric space with index λ_0.

Note: If $\lambda_0 = 1$ we get the usual B_n - metric space (by definition 2).

Now we show that every B_n - metric gives rise to a B_{n-1} - metric with index $(n-2)$.

Lemma 3.6. Let (Y_0, B_n) be any B_n - metric space.

Define $K_b: {Y_0}^{n-1} \to \Re$ as follows:

$$K_b(x_1, x_2 \dots, x_{n-1}) = \left\{ \begin{array}{c} B_n(x_1, x_1, x_2, \dots, , x_{n-1}) + B_n(x_1, x_2, x_2 \dots, x_{n-1}) \\ + \cdots + \\ B_n(x_1, x_2, \dots, x_{n-1}, x_{n-1}) \end{array} \right\}$$

$\forall x_1, x_2 \dots, x_{n-1} \in Y_0$. Then (Y_0, K_b) is a B_{n-1} - metric space with index $(n-2)$. (K_b is called K_b - metric space with index $(n-2)$ induced by B_n).
Proof. Suppose (Y_0, B_n) is a B_n - metric space.

Now $K(x_1, x_2 \dots, x_{n-1}) = 0$

$\Leftrightarrow B_n(x_1, x_1, x_2, \dots, x_{n-1})$

$+B_n(x_1, x_2, x_2 \dots, x_{n-1}) + \cdots +$

$B_n(x_1, x_2, \dots, x_{n-1}, x_{n-1}) = 0$

$\Leftrightarrow B_n(x_1, x_1, x_2, \dots, x_{n-1}) = 0$

$$B_n(x_1, x_2, x_2 \dots, x_{n-1}) = 0, \dots, B_n(x_1, x_2, \dots, x_{n-1}, x_{n-1}) = 0$$

$\Leftrightarrow x_1 = x_2 = \cdots = x_{n-1}.$

Now we show that $K_b(x_1, x_2, \dots, x_{n-1})$

$$\leq \left\{ (n-2) \begin{pmatrix} K_b\left(\underbrace{x_1, x_1, \dots, x_1}_{(n-2)\text{ times}}, a_0\right) \\ +K_b\left(\underbrace{x_2, x_2, \dots, x_2}_{(n-2)\text{ times}}, a_0\right) \\ +\cdots+ \\ K_b(\underbrace{x_{n-1}, x_{n-1}, \dots, x_{n-1}}_{(n-2)\text{ times}}, a) \end{pmatrix} \right\}$$

$\forall\, x_1, x_2, \dots, x_{n-1}, a_0 \in Y_0.$

$K_b(x_1, x_2 \dots, x_{n-1})$

$$= \left\{ \begin{matrix} B_n(x_1, x_1, x_2, \dots, x_{n-1}) \\ +B_n(x_1, x_2, x_2 \dots, x_{n-1}) \\ +\cdots+ \\ B_n(x_1, x_2, \dots, x_{n-1}, x_{n-1}) \end{matrix} \right\}$$

$$\leq \left\{ \begin{matrix} B_n(x_1, x_1, x_1, \dots, x_1, a_0) \\ +B_n(x_1, x_1, x_1, \dots, x_1, a_0) \\ +B_n(x_2, x_2, x_2, \dots, x_2, a_0) \\ +B_n(x_{n-1}, x_{n-1}, x_{n-1}, \dots, x_{n-1}, a_0) \\ +B_n(x_1, x_1, x_1, \dots, x_1, a_0) \\ +B_n(x_2, x_2, x_2, \dots, x_2, a) \\ +B_n(x_2, x_2, x_2, \dots, x_2, a_0) \\ +B_n(x_{n-1}, x_{n-1}, x_{n-1}, \dots, x_{n-1}, a_0) \\ +B_n(x_1, x_1, x_1, \dots, x_1, a_0) \\ +B_n(x_2, x_2, x_2, \dots, x_2, a) \\ +\cdots+ \\ B_n(x_{n-1}, x_{n-1}, x_{n-1}, \dots, x_{n-1}, a_0) \\ +B_n(x_{n-1}, x_{n-1}, x_{n-1}, \dots, x_{n-1}, a_0) \end{matrix} \right\}$$

$$\leq \left\{ n \begin{pmatrix} B_n(x_1, x_1, x_2, \dots, x_{n-1}) \\ +B_n(x_1, x_2, x_2 \dots, x_{n-1}) \\ +\cdots+ \\ B_n(x_{n-1}, x_{n-1}, x_{n-1} \dots, x_{n-1}) \end{pmatrix} \right\}$$

It can be easily shown that

$K_b(x_1, x_1, \dots, x_1, a) + K_b(x_2, x_2, \dots, x_2, a) + \cdots + K_b(x_{n-1}, x_{n-1} \dots, x_{n-1}, a)$

$$= (\mathrm{n}-2) \left\{ \begin{matrix} B_n\left(\underbrace{x_1, x_1, \dots, x_1}_{(n-1)\text{ times}}, a_0\right) \\ +B_\mathrm{n}\left(\underbrace{x_2, x_2, \dots, x_2}_{(n-1)\text{ times}}, a_0\right) \\ +\cdots+ \\ B_\mathrm{n}\left(\underbrace{x_{n-1}, x_{n-1}, \dots, x_{n-1}, a_0}_{(n-1)\text{ times}}\right) \\ +B_n\left(\underbrace{x_1, x_1, \dots, x_1}_{(n-1)\text{ times}}, a_0\right) \\ +B_\mathrm{n}\left(\underbrace{x_2, x_2, \dots, x_2}_{(n-1)\text{ times}}, a_0\right) \\ +\cdots+ \\ B_\mathrm{n}(\underbrace{x_{n-1}, x_{n-1}, \dots, x_{n-1}}_{(n-1)\text{ times}}, a_0) \end{matrix} \right\}.$$

From (6.2) and (6.3) we get,

$$K_b(x_1, x_2, \dots, x_{\mathrm{n}-1}) = \left\{ \begin{matrix} n(B_n(x_1, x_1, \dots, x_1, a_0) \\ +B_n(x_2, x_2, \dots, x_2, a_0) \\ +\cdots+ \\ B_n(x_{n-1}, x_{n-1} \dots, x_{n-1}, a_0)) \end{matrix} \right\}$$

$$\leqslant \left\{ n \begin{pmatrix} B_n(x_1, x_1, \dots, x_1, a) \\ +B_n(x_2, x_2, \dots, x_2, a_0) \\ +\cdots+ \\ B_n(x_{n-1}, x_{n-1} \dots, x_{n-1}, a_0) \end{pmatrix} \right\}$$

$$\leqslant (\mathrm{n}-2) \left\{ (\mathrm{n}-2) \begin{pmatrix} B_n(x_1, x_1, \dots, x_1, a) \\ +B_n(x_2, x_2, \dots, x_2, a_0) \\ +\cdots+ \\ B_n(x_{n-1}, x_{n-1} \dots, x_{n-1}, a_0) \\ +B_n(x_1, x_1, \dots, x_1, a) \\ +B_n(x_2, x_2, \dots, x_2, a_0) \\ +\cdots+ \\ B_n(x_{n-1}, x_{n-1} \dots, x_{n-1}, a_0) \end{pmatrix} \right\}$$

$$= (n-2)\begin{Bmatrix} (n-2)\begin{pmatrix} B_n(x_1,x_1,\dots,x_1,a) \\ +B_n(x_1,x_1,\dots,x_1,a_0) \end{pmatrix} \\ +(n-2)\begin{pmatrix} B_n(x_2,x_2,\dots,x_2,a_0) \\ +B_n(x_2,x_2,\dots,x_2,a_0) \end{pmatrix} \\ (n-2)\begin{pmatrix} B_n(x_{n-1},x_{n-1},\dots,x_{n-1},a_0) \\ +B_n(x_{n-1},x_{n-1}\dots,x_{n-1},a_0) \end{pmatrix} \end{Bmatrix}$$

$$= (n-2)\begin{pmatrix} K_b(x_1,x_1,\dots,x_1,a) \\ +K_b(x_2,x_2,\dots,x_2,a_0) \\ +\cdots+ \\ K_b(x_{n-1},x_{n-1}\dots,x_{n-1},a_0) \end{pmatrix}$$

Hence K_b is a B_{n-1} - metric on Y_0 with index $(n-2)$.

Definition 3.4. This K_b is called the B_{n-1} - metric on Y_0 induced by B_n - metric B_n and K_b has index $(n-2)$

Example 3.3. Let $Y_0 \neq \phi, \mu > 0$ and define the function $B_n: {Y_0}^n \to \Re$ as

$$B_n(x_1,x_2,\dots,x_n) = \begin{cases} 0, \text{ if } x_1 = x_2 = \cdots = x_n \\ \mu, \text{ otherwise} \end{cases}$$

Then (Y_0, B_n) is a B_n - metric space with index $(n-1)$.

Definition 3.5: Suppose $\mu > 0$ and (Y_0, B_n) is a B_n - metric space. $T_0: Y_0 \to Y_0$ is called a contraction with contraction constant μ, if $B_n(T_0x_1, T_0x_2, \dots, T_0x_n) \leqslant \mu\, B_n(x_1,x_2,\dots,x_n)$, $\forall x_1,x_2,\dots,x_n \in Y_0$.

IV. Relation between B_n Contractions and $K = B_{n-1}$ Contractions

Theorem 4.1. Suppose $T_0: Y_0 \to Y_0$ is a contraction with contraction constant μ on (Y_0, B_n), and K_b is the induced B_{n-1} - metric on Y_0 induced by B_n. Then T_0 is a K_b - contraction on Y_0 with contraction constant μ.

Proof. Suppose $x_1, x_2, \dots, x_{n-1} \in Y_0$.
Then, by definition,

$$K_b(T_0x_1, T_0x_2, \dots, T_0x_{n-1}) = \begin{Bmatrix} B_n(T_0x_1,T_0x_1,T_0x_2,\dots,T_0x_{n-1}) \\ +B_n(T_0x_1,T_0x_2,T_0x_2,\dots,T_0x_{n-1}) \\ +\cdots+ \\ B_n(T_0x_1,T_0x_2,\dots,T_0x_{n-1},T_0x_{n-1}) \end{Bmatrix}$$

$$\leqslant \mu \begin{Bmatrix} B_n(x_1,x_1,x_2,\dots,x_{n-1}) \\ +B_n(x_1,x_2,x_2,\dots,x_{n-1}) \\ +\cdots+ \\ B_n(x_1,x_2,\dots,x_{n-1},x_{n-1}) \end{Bmatrix}$$

$$= \mu\, K_b(x_1,x_2,\dots,x_{n-1})$$

Therefore T_0 is a contraction with respect to K_b with contraction constant μ.

Now we establish some relations between B_n contractions and K contractions

Theorem 4.2. Let (Y_0, B_n) be a B_n - metric space. Define $K: {Y_0}^{n-1} \to \Re$ by

$$K(x_1,x_2,\dots,x_{n-1}) = \begin{Bmatrix} B_n(x_1,x_1,x_2,\dots,x_{n-1}) \\ +B_n(x_1,x_2,x_2,\dots,x_{n-1}) \\ +\cdots+ \\ B_n(x_1,x_2,\dots,x_{n-1},x_{n-1}) \end{Bmatrix}$$

$\forall x_1, x_2, \dots, x_{n-1} \in Y_0$, (by definition 05)

(i.e., K is the B_{n-1} - metric on Y_0 induced by B_n)

Suppose

$$B_n(T_0x_1,T_0x_2,\dots,T_0x_n) < \max\begin{Bmatrix} B_n\left(\underbrace{T_0x_1,T_0x_1,\dots,T_0x_1}_{(n-1)\text{ times}}, x_1\right), \\ B_n\left(\underbrace{T_0x_2,T_0x_2,\dots,T_0x_2}_{(n-1)\text{ times}}, x_2\right), \\ \dots, B_4(\underbrace{T_0x_n,T_0x_n,\dots,T_0x_n}_{(n-1)t\text{ times}}, x_n) \end{Bmatrix}$$

$\forall x_1, x_2, \dots, x_n \in Y_0$

(T_0 is called a generalized contraction on (Y_0, B_n) if it satisfies (2.1)) Further suppose, $B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$

$$\leqslant B_n(\underbrace{T_0x_1,T_0x_1,\dots,T_0x_1}_{(n-2)\text{ times}}, x_1, x_1), \forall x_1 \in Y_0.$$

Then

$K(T_0x_1, T_0x_2, \dots, T_0x_{n-1}) <$

$$\max\begin{Bmatrix} K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots , \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{Bmatrix}$$

$\forall x_1, x_2, \dots, x_{n-1} \in Y_0$

(i.e., T_0 is a generalized contractions w.r.t K)

Proof. Suppose $B_n(T_0x_1, T_0x_2, \dots, T_0x_n)$

$$< \max\begin{Bmatrix} B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots , \\ B_n(T_0x_n, T_0x_n, \dots, T_0x_n) \end{Bmatrix}$$

$\forall x_1, x_2, \dots, x_n \in Y_0$

Then we show that

$K(T_0x_1, T_0x_2, \dots, T_0x_{n-1})$

$$< \max\begin{Bmatrix} K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2), \\ \dots, K\left(T_0x_{x_{n-1}}, T_0x_{x_{n-1}}, \dots, T_0x_{x_{n-1}}\right) \end{Bmatrix}$$

$\forall\, x_1, x_2, \dots, x_{n-1} \in Y_0$.

L.H.S: $K(T_0x_1, T_0x_2, \dots, T_0x_{n-1})$

$$= \begin{Bmatrix} B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1}) \\ +B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1}) \\ +B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1}) \end{Bmatrix}$$

Now, $B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1}) <$

$$\max\begin{Bmatrix} B_n(T_0x_1, T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_1, T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_2, T_0x_2, T_0x_2, \dots, T_0x_2, x_2), \\ , \dots , \\ B_n(T_0x_{n-1}, T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{Bmatrix}$$

$$= B_n(T_0x_1, T_0x_1, T_0x_1, \dots, T_0x_1, x_1) \text{ (say)}$$

$B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1}) <$

$$\max\begin{Bmatrix} B_n(T_0x_1, T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_2, T_0x_2, T_0x_2, \dots, T_0x_2, x_2), \\ B_n(T_0x_2, T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots , \\ B_n(T_0x_{n-1}, T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{Bmatrix}$$

$= B_n(T_0x_1, T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$ (say)

$B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1}) <$

$$\max\begin{Bmatrix} B_n(T_0x_1, T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_2, T_0x_2, T_0x_2, \dots, T_0x_2, x_2), \\ , \dots , \\ B_n(T_0x_{n-1}, T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}), \\ B_n(T_0x_{n-1}, T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{Bmatrix}$$

$= B_n(T_0x_1, T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$ (say)

Also, (a_1)

$K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$

$$= \begin{Bmatrix} B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) \\ +B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) \\ + \cdots + \\ B_n(T_0x_1, T_0x_1, \dots, x_1, x_1) \end{Bmatrix}$$

(a_2) $K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2)$

$$= \begin{Bmatrix} B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ +B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ + \cdots + \\ B_n(T_0x_2, T_0x_2, \dots, x_2, x_2) \end{Bmatrix}$$

…

…

$$(a_{n-1})K\left(\underbrace{T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}}_{(n-2)\text{ times}}, x_{n-1}\right)$$

$$= \begin{Bmatrix} B_n\left(\underbrace{T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}}_{(n-1)\text{ times}}, x_{n-1}\right) \\ +B_n\left(\underbrace{T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}}_{(n-1)\text{ times}}, x_{n-1}\right) \\ + \cdots + \\ B_n(\underbrace{T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}}_{(n-1)\text{ times}}, x_{n-1}) \end{Bmatrix}$$

And
$B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) <$
$K(T_0x_1, T_0x_1, \dots, x_1)$,

L.H.S: $K(T_0x_1, T_0x_1, \dots, x_{n-1})$

$$< \left\{\begin{matrix} B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) \\ +B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) \\ + \cdots + \\ B_n\left(\underbrace{T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}}_{(n-1)\text{ times}}, x_{n-1}\right) \\ +B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) \end{matrix}\right\}$$

$$\leqslant B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) + B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$$

$$= K(\underbrace{T_0x_1, T_0x_1, \dots, T_0x_1}_{(n-2)\text{ times}}, x_1)$$

$$\leqslant \max\left\{\begin{matrix} K(T_0x_1, T_0x_1, \dots, x_1), \\ K(T_0x_2, T_0x_2, \dots, x_2), \\ , \dots, \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, x_{n-1}) \end{matrix}\right\}$$

Therefore, $K(T_0x_1, T_0x_2, \dots, T_0x_{n-1})$

$$< \max\left\{\begin{matrix} K(T_0x_1, T_0x_1, \dots, x_1), \\ K(T_0x_2, T_0x_2, \dots, x_2), \\ , \dots, \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, x_{n-1}) \end{matrix}\right\}$$

Thus T_0 is a generalized contraction with respect to the B_{n-1} - metric K.

Theorem 4.3. Let (Y_0, B_n) be a B_n-metric space. Define $K: {Y_0}^{n-1} \to \mathfrak{R}$ by

$$K(x_1, x_2 \dots, x_{n-1}) = \left\{\begin{matrix} B_n(x_1, x_1, x_2, \dots, , x_{n-1}) \\ +B_n(x_1, x_2, x_2 \dots, x_{n-1}) \\ + \cdots + \\ B_n(x_1, x_2, \dots, x_{n-1}, x_{n-1}) \end{matrix}\right\}$$

$\forall\, x_1, x_2 \dots, x_{n-1} \in Y_0$. Suppose

$$B_n(x_1, x_1, x_2, \dots, x_{n-1}) = B_n(x_1, x_2, x_2, \dots, x_{n-1})$$

$= B_n(x_1, x_2, \dots, x_{n-1}, x_{n-1}) = (C)$ (say),
$\forall\, x_1, x_2 \dots, x_{n-1} \in Y$.

Suppose $T_0: Y_0 \to Y_0$ satisfies that

$$B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_n) < \max\left\{\begin{matrix} B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_2, T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ B_n(T_0x_{n-1}, T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}), \\ B_n(T_0x_n, T_0x_n, T_0x_n, \dots, T_0x_n, x_n) \end{matrix}\right\}$$

$\forall\, x_1, x_2 \dots, x_{n-1}, x_n \in Y_0$.

Further, suppose

$$B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) \leqslant B_n(T_0x_1, T_0x_1, \dots, x_1, x_1),$$

$\forall\, x_1 \in Y_0$

Then
$K(T_0x_1, T_0x_2, \dots, T_0x_{n-1}) <$

$$\max\left\{\begin{matrix} K(x_1, x_2 \dots, x_{n-1}), \\ K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{matrix}\right\}$$

$\forall\, x_1, x_2 \dots, x_{n-1} \in Y_0$.

(In other words, K is also a generalized contraction that satisfies a condition similar to (3.1))

Proof. L.H.S: $K(T_0x_1, T_0x_2, \dots, T_0x_{n-1}) =$

$$\left\{\begin{matrix} B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1}), \\ +B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1}) \\ + \cdots + \\ B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1}) \end{matrix}\right\}$$

Now
(a_1) $K(x_1, x_2, \dots, x_{n-1}) =$

$$\left\{\begin{matrix} B_n(x_1, x_1, x_2, \dots, x_{n-1}) \\ +B_n(x_1, x_2, x_2, \dots, x_{n-1}) \\ + \cdots + \\ B_n(x_1, x_2, \dots, x_{n-1}, x_{n-1}) \end{matrix}\right\}$$

$= (n-1)(C)$

$(a_2) K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$

$$= \begin{Bmatrix} B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) \\ +B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) \\ +B_n(T_0x_1, T_0x_1, \dots, x_1, x_1) \end{Bmatrix}$$

$> (n-1)B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$

(a_3) $K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2)$

$$= \begin{Bmatrix} B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ +B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ +B_n(T_0x_2, T_0x_2, \dots, x_2, x_2) \end{Bmatrix}$$

$> (n-1)B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2)$

…

…

$(a_{n-1})K(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1})$

$$= \begin{Bmatrix} B_n(Tx_{n-1}, Tx_{n-1}, \dots, Tx_{n-1}, x_{n-1}) \\ +\ B_n(Tx_{n-1}, Tx_{n-1}, \dots, Tx_{n-1}, x_{n-1}) \\ + \cdots + \\ B_n(Tx_{n-1}, Tx_{n-1}, \dots, Tx_{n-1}, x_{n-1}) \end{Bmatrix}$$

$> (n-1)\ B_n(Tx_{n-1}, Tx_{n-1}, \dots, Tx_{n-1}, x_{n-1})$ $(by (4.2))$

Also $B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1}) <$

$$\max \begin{Bmatrix} B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{Bmatrix}$$

$B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1})$

$$< \max \begin{Bmatrix} B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2), \\ B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2), \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{Bmatrix}$$

$$< \max \begin{Bmatrix} B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1}), \\ B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2), \\ , \dots, \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}), \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{Bmatrix}$$

By hypothesis, $B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1})$

$= B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1})$
$= B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1})$.

Without loss of generality, we may assume that

$$\max \begin{Bmatrix} B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{Bmatrix}$$

$= B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$.

Then $B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1})$
$< max\{(C), B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)\}$

$B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1})$
$< max\{(C), B_n(T_0x_2, T_0x_2, \dots, T_0x_2, x_2)\}$

$B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1})$
$< max\{(C), B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1})\}$

L.H.S $= K(T_0x_1, T_0x_2, \dots, T_0x_{n-1})$

$$= \begin{Bmatrix} B_n\left(\underbrace{T_0x_1, T_0x_1, \dots, T_0x_1}_{(n-1)\text{ times}}, x_1\right) \\ +B_n\left(\underbrace{T_0x_2, T_0x_2, \dots, T_0x_2}_{(n-1)\text{ times}}, x_2\right) \\ + \cdots + \\ B_n\left(\underbrace{T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}}_{(n-1)\text{ times}}, x_{n-1}\right) \end{Bmatrix}$$

$< (n-1)\max\{(C), B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)\}$

$= \max\{(n-1)(C), (n-1)B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)\}$

$$< \max \begin{Bmatrix} K(Tx_1, Tx_2, \dots, Tx_{n-1}), \\ K(Tx_1, Tx_1, \dots, Tx_1, x_1), \\ K(Tx_2, Tx_2, \dots, Tx_2, x_2) \\ , \dots, \\ K(Tx_{n-1}, Tx_{n-1}, \dots, Tx_{n-1}, x_{n-1}) \end{Bmatrix}$$

$=$ R.H.S

Therefore, $K(T_0x_1, T_0x_2, \dots, T_0x_{n-1}) <$

$$\max\left\{\begin{array}{c} K(x_1,x_2 \ldots,x_{n-1}), \\ K\left(\underbrace{T_0x_1,T_0x_1,\ldots,T_0x_1}_{(n-2)\text{ times}},x_1\right), \\ K\left(\underbrace{T_0x_2,T_0x_2,\ldots,T_0x_2}_{(n-2)\text{ times}},x_2\right) \\ ,\ldots, \\ K(\underbrace{T_0x_{n-1},T_0x_{n-1},\ldots,T_0x_{n-1}}_{(n-2)\text{ times}},x_{n-1}) \end{array}\right\}$$

$\forall\ x_1,x_2 \ldots,x_{n-1} \in Y_0$

Theorem 4.4. Let (Y_0,B_n) be a B_n-metric space. Define $K\colon {Y_0}^{n-1} \to \Re$ by

$$K(x_1,x_2 \ldots,x_{n-1}) = \left\{\begin{array}{c} B_n(x_1,x_1,x_2,\ldots,,x_{n-1}) \\ +B_n(x_1,x_2,x_2 \ldots,x_{n-1}) \\ +B_n(x_1,x_2,\ldots,x_{n-1},x_{n-1}) \end{array}\right\}$$

$\forall\ x_1,x_2 \ldots,x_{n-1} \in Y_0$.

Suppose $B_n(x_1,x_1,x_2,\ldots,x_{n-1})$

$= B_n(x_1,x_2,x_2,\ldots,,x_{n-1})$

$= B_n(x_1,x_1,\ldots,x_{n-1},x_{n-1})$

$= (C)(\text{ say })(4.1)$

and suppose $T_0\colon Y_0 \to Y_0$ satisfies that

$$B_n(T_0x_1,T_0x_2,\ldots,T_0x_{n-1},T_0x_n) < \max\left\{\begin{array}{c} B_n(x_1,x_2,\ldots,x_{n-1},x_n), \\ B_n(T_0x_1,T_0x_1,\ldots,x_1,x_1), \\ B_n(T_0x_2,T_0x_2,\ldots,x_2,x_2) \\ ,\ldots, \\ B_n(T_0x_{n-1},T_0x_{n-1},\ldots,x_{n-1},x_{n-1}), \\ B_n(T_0x_n,T_0x_n,\ldots,x_n,x_n) \end{array}\right\}$$

$\forall\ x_1,x_2 \ldots,x_{n-1},x_n \in Y_0$ (4.2)

Further, suppose

$B_n(Tx_1,Tx_1,\ldots,x_1,x_1) \leq B_n(Tx_1,Tx_1,\ldots,Tx_1,x_1) = (D)\ (say) \quad \forall\ x_1 \in Y$ (4.2). Then

$$K(Tx_1,Tx_2,\ldots,Tx_{n-1}) < \max\left\{\begin{array}{c} K(x_1,x_2,\ldots,x_{n-1}), \\ K(Tx_1,Tx_1,\ldots,x_1), \\ K(Tx_2,Tx_2,\ldots,x_2) \\ ,\ldots,K(Tx_{n-1},Tx_{n-1},\ldots,x_{n-1}) \end{array}\right\}$$

$\forall\ x_1,x_2,\ldots,x_{n-1} \in Y_0$.

Proof. L.H.S: $K(T_0x_1,T_0x_2,\ldots,T_0x_{n-1}) = \left\{\begin{array}{c} B_n(T_0x_1,T_0x_1,T_0x_2,\ldots,T_0x_{n-1}) \\ +B_n(T_0x_1,T_0x_2,T_0x_2,\ldots,T_0x_{n-1}) \\ +\cdots+ \\ B_n(T_0x_1,T_0x_2,\ldots,T_0x_{n-1},T_0x_{n-1}) \end{array}\right\}$

$$(a_1)K(x_1,x_2 \ldots,x_{n-1}) = \left\{\begin{array}{c} B_n(x_1,x_1,x_2,\ldots,x_{n-1}) \\ +B_n(x_1,x_2,x_2 \ldots,x_{n-1}) \\ +\cdots+B_n(x_1,x_2,\ldots,x_{n-1},x_{n-1}) \end{array}\right\}$$

$= (n-1)(C)(\text{ by }(4.1))$

$(a_2)K(T_0x_1,T_0x_1,\ldots,x_1)$

$$= \left\{\begin{array}{c} B_n(T_0x_1,T_0x_1,\ldots,T_0x_1,x_1) \\ +B_n(T_0x_1,T_0x_1,\ldots,T_0x_1,x_1) \\ +B_n(T_0x_1,T_0x_1,\ldots,x_1,x_1) \end{array}\right\}$$

$> (n-1)B_n(T_0x_1,T_0x_1,\ldots,T_0x_1,x_1)\ (by(4.2))$

$(a_3)K(T_0x_2,T_0x_2,\ldots,x_2)$

$$= \left\{\begin{array}{c} B_n(T_0x_2,T_0x_2,\ldots,T_0x_2,x_2) \\ +B_n(T_0x_2,T_0x_2,\ldots,T_0x_2,x_2) \\ +B_n(T_0x_2,T_0x_2,\ldots,x_2,x_2) \end{array}\right\}$$

$> (n-1)B_n(T_0x_2,T_0x_2,\ldots,T_0x_2,x_2)\ (by(4.2))$

$(a_{n-1})K(T_0x_{n-1},T_0x_{n-1},\ldots,x_{n-1})$

$$= \left\{\begin{array}{c} B_n(T_0x_{n-1},T_0x_{n-1},\ldots,T_0x_{n-1},x_{n-1}) \\ +B_n(T_0x_{n-1},T_0x_{n-1},\ldots,T_0x_{n-1},x_{n-1}) \\ +B_n(T_0x_{n-1},T_0x_{n-1},\ldots,T_0x_{n-1},x_{n-1}) \end{array}\right\}$$

$> (n-1)B_n(T_0x_{n-1},T_0x_{n-1},\ldots,T_0x_{n-1},x_{n-1})$ $(by(4.2))$

Also $B_n(T_0x_1,T_0x_1,T_0x_2,\ldots,T_0x_{n-1})$

$$< \max \left\{ \begin{matrix} B_n(T_0x_1, T_0x_1, \dots, x_1, x_1), \\ B_n(T_0x_1, T_0x_1, \dots, x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, x_2, x_2) \\ , \dots, \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, x_{n-1}, x_{n-1}) \end{matrix} \right\}$$

$= B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$ (say)

$B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1})$

$$< max \left\{ \begin{matrix} B_n(T_0x_1, T_0x_1, \dots, x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, x_2, x_2), \\ B_n(T_0x_2, T_0x_2, \dots, x_2, x_2), \dots, \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{matrix} \right\}$$

$= B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$(say)

$B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1})$

$$< \max \left\{ \begin{matrix} B_n(T_0x_1, T_0x_1, \dots, x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, x_2, x_2) \\ , \dots, \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, x_{n-1}, x_{n-1}), \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, x_{n-1}, x_{n-1}) \end{matrix} \right\}$$

$= B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$(say)

$B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1})$
$< max\{(C), B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)\}$

$B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1})$
$< max\{(C), B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)\}$

Therefore,

$$\left\{ \begin{matrix} B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1}) \\ +B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1}) \\ +B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1}) \end{matrix} \right\}$$

$< (n-1)(C)$

(if $\max\{(C), B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)\} = (C)$)

$< (n-1)(C)$(if $\max\{(C), B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)\} = (C)$

$= K(T_0x_1, T_0x_2, \dots, T_0x_{n-1})$

$$< \max \left\{ \begin{matrix} K(x_1, x_2, \dots, x_{n-1}), \\ K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{matrix} \right\}$$

$$< \max \left\{ \begin{matrix} K(x_1, x_2, \dots, x_{n-1}), \\ K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{matrix} \right\}$$

$$\leqslant \max \left\{ \begin{matrix} K(x_1, x_2, \dots, x_{n-1}), \\ K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{matrix} \right\}$$

$K(T_0x_1, T_0x_1, \dots, x_1)$

$> (n-1)B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$ from (D)

$= (n-1)B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$

Therefore if $\max\{(C), B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)\}$

$= B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$

Then, $\left\{ \begin{matrix} B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1}) \\ +B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1}) \\ +B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1}) \end{matrix} \right\}$

$\leqslant (n-1)B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$

$$\leqslant K(T_0x_1, T_0x_1, \dots, x_1) \leqslant \max \left\{ \begin{matrix} K(x_1, x_2, \dots, x_{n-1}), \\ K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{matrix} \right\}$$

$$\leqslant K(T_0x_1, T_0x_2, \dots, T_0x_{n-1}) < \max \left\{ \begin{matrix} K(x_1, x_2, \dots, x_{n-1}), \\ K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{matrix} \right\}$$

$\forall\, x_1, x_2, \dots, x_{n-1} \in Y_0$.

Theorem 4.5. Let (Y_0, B_n) be a B_n-metric space. Define $K: {Y_0}^{n-1} \to \Re$ by

$$K(x_1, x_2 \dots, x_{n-1}) = \left\{\begin{matrix} B_n(x_1, x_1, x_2, \dots, , x_{n-1}) \\ +B_n(x_1, x_2, x_2 \dots, x_{n-1}) \\ + \cdots + \\ B_n(x_1, x_2, \dots, x_{n-1}, x_{n-1}) \end{matrix}\right\},$$

$\forall\, x_1, x_2 \dots, x_{n-1} \in Y_0$.

Suppose
$B_n(T_0x_1, T_0x_1, x_1, x_1) \leqslant B_n(T_0x_1, T_0x_1, T_0x_1, x_1)$
$\forall\, x_1 \in Y_0$. (5.1)

and suppose $T_0: Y_0 \to Y_0$ satisfies that

$$B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_n) < \left\{\begin{matrix} B_n(T_0x_1, T_0x_1, x_1, x_1), \\ B_n(T_0x_2, T_0x_2, x_2, x_2), \\ B_n(T_0x_{n-1}, T_0x_{n-1}, x_{n-1}, x_{n-1}), \\ B_n(T_0x_n, T_0x_n, x_n, x_n) \end{matrix}\right\}$$

$\forall\, x_1, x_2 \dots, x_n \in Y_0$

Then $K(T_0x_1, T_0x_2, \dots, T_0x_{n-1})$

$$< \max\left\{\begin{matrix} K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{matrix}\right\}$$

$\forall\, x_1, x_2 \dots, x_n \in Y_0$

(That is T_0 also satisfies a condition similar to (5.2) with respect to K)

Proof: L.H.S: $K(T_0x_1, T_0x_2, \dots, T_0x_{n-1}) =$

$$\left\{\begin{matrix} B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1}) \\ +B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1}) \\ + \cdots + \\ B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1}) \end{matrix}\right\}$$

(i) $K(\underbrace{T_0x_1, T_0x_1, \dots, T_0x_1}_{(n-2)\text{ times}}, x_1)$

$$= \left\{\begin{matrix} B_n\left(\underbrace{T_0x_1, T_0x_1, \dots, T_0x_1}_{(n-1)\text{ times}}, x_1\right) \\ +B_n\left(\underbrace{T_0x_1, T_0x_1, \dots, T_0x_1, x_1}_{(n-1)\text{ times}}\right) \\ + \cdots + \\ B_n(\underbrace{T_0x_1, T_0x_1, \dots, T_0x_1, x_1}_{(n-1)\text{ times}}) \end{matrix}\right\}$$

$\leqslant (n-1)B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$

(ii) $K\left(\underbrace{T_0x_2, T_0x_2, \dots, T_0x_2}_{(n-2)\text{ times}}, x_2\right)$

$$= \left\{\begin{matrix} B_n\left(\underbrace{T_0x_2, T_0x_2, \dots, T_0x_2}_{(n-1)\text{ times}}, x_2\right) \\ +B_n\left(\underbrace{T_0x_2, T_0x_2, \dots, T_0x_2}_{(n-1)\text{ times}}, x_2\right) \\ + \cdots + \\ B_n(\underbrace{T_0x_2, T_0x_2, \dots, T_0x_2, x_2}_{(n-1)\text{ times}}) \end{matrix}\right\}$$

(iii) $K\left(\underbrace{T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}}_{(n-2)\text{ times}}, x_{n-1}\right)$

$$= \left\{\begin{matrix} B_n\left(\underbrace{T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}}_{(n-1)\text{ times}}, x_{n-1}\right) \\ +B_n\left(\underbrace{T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}}_{(n-1)\text{ times}}, x_{n-1}\right) \\ +B_n(\underbrace{T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}}_{(n-1)\text{ times}}, x_{n-1}) \end{matrix}\right\}$$

Also $B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1})$

$$< \max\left\{\begin{matrix} B_n(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ B_n(T_0x_1, T_0x_1, \dots, x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, x_2, x_2) \\ , \dots, \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, x_{n-1}, x_{n-1}) \end{matrix}\right\}$$

$= B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$ (say)

$$B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1}) < max\left\{\begin{matrix} B_n(T_0x_1, T_0x_1, \dots, x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, x_2, x_2), \\ B_n(T_0x_2, T_0x_2, \dots, x_2, x_2), \dots, \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{matrix}\right\}$$

$= B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$ (say)

$$B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1}) < \max\left\{\begin{matrix} B_n(T_0x_1, T_0x_1, \dots, x_1, x_1), \\ B_n(T_0x_2, T_0x_2, \dots, x_2, x_2) \\ , \dots, \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, x_{n-1}, x_{n-1}), \\ B_n(T_0x_{n-1}, T_0x_{n-1}, \dots, x_{n-1}, x_{n-1}) \end{matrix}\right\}$$

$= B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$ (say)

Therefore,

$$B_n(T_0x_1, T_0x_1, T_0x_2, \dots, T_0x_{n-1}) < B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$$

$$B_n(T_0x_1, T_0x_2, T_0x_2, \dots, T_0x_{n-1}) < B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$$

$$B_n(T_0x_1, T_0x_2, \dots, T_0x_{n-1}, T_0x_{n-1}) < B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$$

$$(n-1)B_n(T_0x_1, T_0x_1, \dots, x_1, x_1) < K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1)$$

Therefore from (5.4)

we have $K(T_0x_1, T_0x_2, \dots, T_0x_{n-1})$

$< (n-1)B_n(T_0x_1, T_0x_1, \dots, x_1, x_1)$ (from (5.5))

$< K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1) \leq$

$$\max\left\{\begin{matrix} K(T_0x_1, T_0x_1, \dots, T_0x_1, x_1), \\ K(T_0x_2, T_0x_2, \dots, T_0x_2, x_2) \\ , \dots, \\ K(T_0x_{n-1}, T_0x_{n-1}, \dots, T_0x_{n-1}, x_{n-1}) \end{matrix}\right\}$$

Thus (5.3) is established.

V. Conclusion and future work

We show B_{n-1} - metric space, gives rise to a B_n - metric space and B_n - metric space, gives rise to a B_{n-1} - metric space with give some examples. We also study the relationship of contractions of self-maps on B_n - metric space and B_{n-1} - metric spaces. In future we will apply a fixed point theorem on B_n- metric spaces.

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VII. References

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