HOMEOMORPHISMS BETWEEN LETTERS OF ALPHABET: TOPOLOGICAL INVARIANTS OF CLASSIFICATION OF LETTERS

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ABSTRACT: There is a general agreement that problems which are highly complex in any naive sense are also difficult from the computational point of view. It is therefore of great interest to find invariants and invariant structures which measure in some respect the complexity of the given problem. One of the basic problems of Topology is to determine when two given geometric objects are homeomorphic. This can be quite difficult in general. Our first goal will be to define exactly what the geometric objects are that one studies in Topology. These are called topological spaces. The set of invariants is proven to be necessary and sufficient to characterize topological equivalence classes of binary relations between simple letters.

Keywords: Topological Space, Metric Space, Open sets, Closed sets.

1. INTRODUCTION

Topology (from Greek topos [place/location] and logos [discourse/reason/logic]) can be viewed as the study of continuous functions, also known as maps. Let X and Y are sets, and $f : X \to Y$ a function from X to Y. In order to make sense of the assertion that f is a continuous function; we need to specify some extra data. After all, continuity roughly asserts that if x and y are elements of X that are "close together" or "nearby", then the function values f(x) and f(y) are elements of Y that are also close together. Hence we need to give some sense to a notion of closeness for elements in X, and similarly for elements in Y. In many cases this can be done by specifying a real number d(x,y) for each pair of elements $x, y \varepsilon X$, called the distance between x and y, and saying at x and y are close together if d(x,y) is sufficiently small. This leads to the notion of a metric space (X,d), when the distance function (or metric) d satisfies some reasonable properties. The only information available about two elements x and y of a general set X is whether they are equal or not. Thus a set X appears as an unorganized collection of its elements, with no further structure. When (X,d) is equipped with a metric, however, it acquires a shape or form, which is why we call it a space, rather than just a set. Similarly, when (X,d) is a metric space we refer to the $x \varepsilon X$ as points, rather than just as elements.

However, metric spaces are somewhat special among all shapes that appear in Mathematics, and there are cases where one can usefully make sense of a notion of closeness, even if there does not exists a metric function that expresses this notion. An example of this is given by the notion of point wise convergence for real functions. Recall that a sequence of functions f_n for n = 1, 2, … converges point wise to a function g if for each point t in the domain the sequence $f_n(t)$ of real numbers converges to the number g(t). There is no metric d on the set of real functions that expresses this notion of convergence. To handle this, and many other more general examples, one can use a more general concept than that of metric spaces, namely topological spaces. Rather than specifying the distance between any two elements x and y of a set X, we shall instead give a meaning to which subsets $U \subset X$ are "open". Open sets will encode closeness as follows:

If U is open and $x \varepsilon U$, then all $y \varepsilon X$ that are "sufficiently close" to x also satisfy $y \varepsilon U$.

The shape of X is thus defined not by a notion of distance, but by the specification of which subsets U of X is open. When this specification satisfies some reasonable conditions, we call X together with the collection of all its open subsets a "topological space". The collection of all open subsets will be called the topology on X, and is usually denoted T. Once we have established the working definitions of topological spaces and continuous functions, or maps, we shall turn to some of the most useful properties that such topological spaces may satisfy, including being connected (not being a disjoint union of open subsets), compact (not having too many open subsets globally) or Hausdorff (having enough open subsets locally). Then we discuss consequences of these properties, such as general forms of the intermediate value theorem, existence of maximal values, or uniqueness of limits, and many more. In the case of real functions of one real variable, these are familiar from first-year Calculus, but the generalized results apply to a vastly wider range of shapes, including the plane and higher-dimensional Euclidean spaces, infinite-dimensional function spaces, and finite partially ordered sets. A well-known example of a topological result is the classification of surfaces, or more precisely, of connected compact 2-dimensional manifolds. The answer is that two facts about a surface suffice to determine it up to topological equivalence, namely, whether the surface "can be oriented", and "how many handles it has". The number of handles is also known as the genus. A sphere has genus 0, while a torus has genus 1, and the surface of a mug with two handles has genus 2. For a surface F equipped with a so-called Riemannian metric, this is a formula

$$\int_F K\ dA = 2\pi.\chi$$

expressing the integral of the locally defined curvature K of the surface in terms of the globally defined genus g, or more precisely in terms of the Euler characteristic $\chi = 2 - 2g$.

2. TOPOLOGICAL SPACES AND CONTINUOUS FUNCTIONS

2.1 Metric spaces

A metric space is a set X where we have a notion of distance. That is, if $x, y \varepsilon X$, then d(x,y) is the "distance" between x and y. The particular distance function must satisfy the following conditions:

- d(x, y) ≥0 for all $x, y \varepsilon X$
- d(x, y) = 0 if and only if x = y
- d(x, y)=d(y, x)
- d(x, z) ≤ d(x, y) + d(y, z)

To understand this concept, it is helpful to consider a few examples of what does and does not constitute a distance function for a metric space. For any space X, let d(x, y) = 0 if x = y and d(x, y) = 1 otherwise. This metric, called the discrete metric, satisfies the conditions one through four. The Pythagorean Theorem gives the most familiar notion of distance for points in R_n. In particular, when given x = (x_1, x_2, ..., x_n) and y = (y_1, y_2, ..., y_n), the distance f as

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

2.2 Open Sets (in a metric space)

Now that we have a notion of distance, we can define what it means to be an open set in a metric space. Let X be a metric space. A ball B of radius r around a point $x \varepsilon X$ is

$$B = \{ y \varepsilon X \mid d(x, y) < r \}.$$

We recognize that this ball encompasses all points whose distance is less than r from x. A subset $O \subseteq U$ is open if for every point $x \varepsilon O$, there is a ball around x entirely contained in O. Let X = [0, 1]. The interval (0, 1/2) is open in X. Let X = R. The interval [0, 1/2) is not open in X. With an open set, we should be able to pick any point within the set, take an infinitesimal step in any direction within our given space, and find another point within the open set. In the first example, we can take any point 0 < x < 1/2 and find a point to the left or right of it, within the space [0, 1], that also is in the open set [0, 1). However, this cannot be done with the second example. For instance, if we take the point within the set [0, 1), say 0, and take an infinitesimal step to the left while staying within our given space X, we are no longer within the set [0, 1). Therefore, this would not be an open set within R.

If a set is not open, this does not imply that it is closed. Furthermore, there exists sets that are neither open, nor closed, and sets that are open and closed. Open sets in spaces X have the following properties:

- The empty set is open
- The whole space X is open
- The union of any collection of open sets is open
- The intersection of any finite number of open sets is open.

2.3 Closed Sets (in a metric space)

While we can and will define closed sets by using the definition of open sets, we first define it using the notion of a limit point. A point z is a limit point for a set A if every open set U containing z intersects A in a point other than z. Notice, the point z could be in A or it might not be in A. The following example will help make this clear. Consider the open unit disk

$$D=\left\{(x,y):x^2+y^2<1\right\}$$

Any point in D is a limit point of D. Take a point (0, 0) in D. Any open set U about this point will contain other points in D. Now consider (1, 0), which is not in D. This is still a limit point because any open set about (1, 0) will intersect the disk D. A set C is a closed set if and only if it contains all of its limit points. The unit disk in the previous example is not closed because it does not contain all of its limit points; namely, (1, 0). Let A = Z, a subset of R. This is a closed set because it does contain all of its limit points; no point is a limit point! A set that has no limit points is closed, by default, because it contains all of its limit points. Every intersection of closed sets is closed, and every finite union of closed sets is closed.

2.4 Topological Spaces

We now consider a more general case of spaces without metrics, where we can still make sense of (or rather define appropriately) the notions of open and closed sets. These spaces are called topological spaces. A topological space is a pair (X, T) where X is a set and T is a set of subsets of X satisfying certain axioms. T is called a topology. A topology T on a set X consists of subsets of X satisfying the following properties:

- The empty set ϕ and the space X are both sets in the topology.
- The union of any collection of sets in T is contained in T.
- The intersection of any finitely many sets in T is also contained in T.

Consider the following set consisting of 3 points; X = {a, b, c} and determine if the set $T=\left\{\phi,X,\{a\},\{b\}\right\}$ satisfies the requirements for a topology. This is, in fact, not a topology because the union of the two sets {a} and {b} is the set {a, b}, which is not in the set T.

2.5 Closed Sets (Revisited)

As promised, we can now generalize our definition for a closed set to one in terms of open sets alone which removes the need for limit points and metrics. A set C is closed if $X-C$ is open.

Theorem-1: A set C is a closed set if and only if it contains all of its limit points.

Proof: Suppose a set A is closed. If it has no limit points, there is nothing to check as it trivially contains its limit points. Now suppose z is a limit point of A. Then if $z\varepsilon A$, it contains this limit point. So suppose for the sake of contradiction that z is a limit point and z is not in A. Now we have assumed A was closed, so its complement is open. Since z is not in A, it is in the complement of A, which is open; which means there is an open set U containing z contained in the complement of A. This contradicts that z is a limit point because a limit point is, by definition, a point such that every open set about z meets A. Conversely, if A contains all its limit points, then its complement is open. Suppose x is in the complement of A. Then it cannot be a limit point (by the assumption that A contains all of its limit points). So x is not a limit point which means we can find some open set around x that does not meet A. This proves the complement is open, i.e. every point in the complement has an open set around it that avoids A.

3. CONTINUITY

In topology a continuous function is often called a map. There are two different ideas we can use on the idea of continuous functions.

3.1 Calculus Style

$f:\mathbb{R}^n \to \mathbb{R}^m$ is continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that when $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. The map is continuous if for any small distance in the pre-image an equally small distance is apart in the image. That is to say the image does not “jump”.

3.2 Topology Style

In topology it is necessary to generalize down the definition of continuity, because the notion of distance does not always exist or is different than our intuitive idea of distance. A function $f: X \to Y$ is continuous if and only if the pre-image of any open set in Y is open in X. If for whatever reason you prefer closed sets to open sets, you can use the following equivalent definition: A function $f: X \to Y$ is continuous if and only if the pre-image of any closed set in Y is closed in X. Given a point x of X, we call a subset N of X a neighborhood of X if we can find an open set O such that $x \varepsilon O \subseteq N$.

- A function $f: X \to Y$ is continuous if for any neighborhood V of Y there is a neighborhood U of X such that f(U) $\subseteq$ V .
- A composition of 2 continuous functions is continuous.

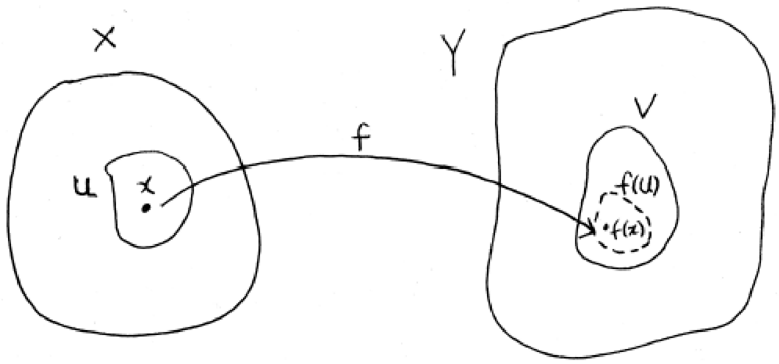

Figure-1: Continuity of f with neighborhoods.

3.3 Homeomorphisms

Homeomorphism is the notion of equality in topology and it is a somewhat relaxed notion of equality. For example, a classic example in topology suggests that a doughnut and coffee cup are indistinguishable to a topologist. This is because one of the geometric objects can be stretched and bent continuously from the other. A homeomorphism is a function $f: X \to Y$ between two topological spaces X and Y that is a continuous bijection, and has a continuous inverse function f^{-1}.

Figure-2: Homeomorphism between a doughnut and a coffee cup.

Two topological spaces X and Y are said to be homeomorphic if there are continuous map $f: X \to Y$ and $g: Y \to Z$ such that $f \circ g = I_Y$ and $g \circ f = I_X$. Moreover, the maps f and g are homeomorphisms and are inverses of each other, so we may write f^{-1} in place of g and g^{-1} in place of f. Here, I_X and I_Y denote the identity maps.

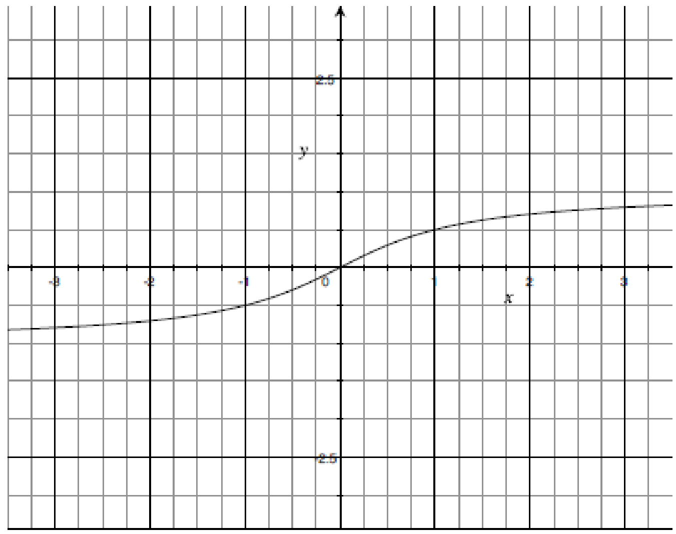

Figure-3: The graph of $f^{-1}(x)$.

4. HOMEOMORPHISMS BETWEEN LETTERS OF ALPHABET

Letters of the alphabet are treated as one-dimensional objects. This means that the lines and curves making up the letters have zero thickness. Thus, we cannot deform or break a single line into multiple lines. Homeomorphisms between the letters in most cases can be extended to homeomorphisms between $\mathbb{R}^2$ and $\mathbb{R}^2$ that carry one letter into another, although this is not a requirement - they can simply be a homeomorphism between points on the letters.

4.1 Topological Invariants

Before classifying the letters of the alphabet, it is helpful to get a feel for the topological invariants of one-dimensional objects in $\mathbb{R}^2$. The number of 3-vertices, 4-vertices, (n-vertices, in fact, for n ≥3), and the number of holes in the object are the topological invariants which we have identified.

4.2 Vertices

The first topological invariant is the number and type of vertices in an object. We think of a vertex as a point where multiple curves intersect or join together. The number of intersecting curves determines the vertex type. An n-vertex in a subset L of a topological space S is an element v ε L such that there exists some neighborhood $N_0 \subseteq S$ of v where all neighborhoods $N \subseteq N_0$ of v satisfy the following properties:

- $N \cap L$ is connected.

- The set formed by removing v from N $\cap$ L, i.e., $\{a \varepsilon N \cap L \mid a \neq v\}$, is not connected, and is composed of exactly n disjoint sets, each of which is connected.

The number of 2-vertices is not a useful topological invariant. This is true because every curve has an infinite number of 2-vertices (every point on the curve not intersecting another curve is a 2-vertex). This does not help us to distinguish between classes of letters.

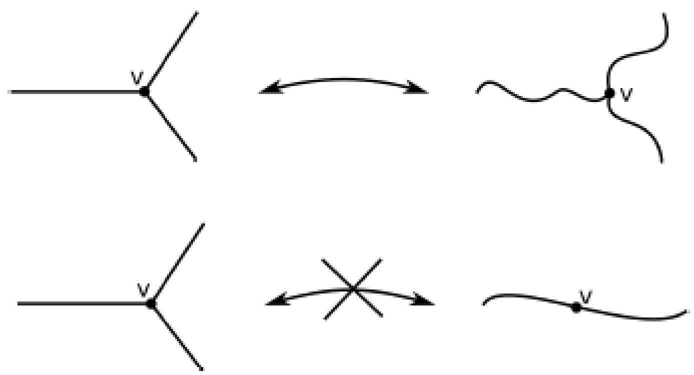

Figure-4: 3-vertex homeomorphism

Three curves intersecting in a 3-vertex are homeomorphic to any other three curves intersecting in a 3-vertex. However, they are not homeomorphic to a single curve. This point is illustrated in Figure-4. Similarly, any set close to a 4-vertex is homeomorphic to any other set close to a 4-vertex. In this case, “close” means a small neighborhood in which there are no other n-vertices (n $\geq$3). Figure-5 shows an example of this.

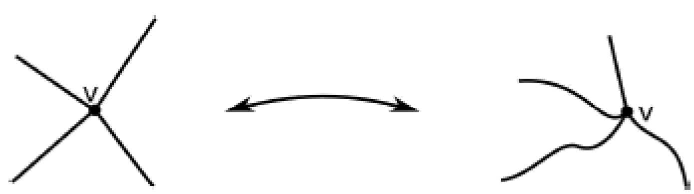

Figure-5: 4-vertex homeomorphism

4.3 Holes

It is very clear when a one dimensional shape in two dimensional spaces has a hole: a shape possessing a hole closes in on itself. As a result, a topological structure with a hole cannot be continuously shrunk to a single point. Moreover, a function mapping a space with a hole to one without a hole cannot be a homeomorphism. To see this point, we note that the tearing of the circle means those nearby points on the circle can be mapped to very distant points on the line. This point, which is illustrated by Figure-6, violates the continuity requirements of a homeomorphism. The number of holes is therefore an important topological invariant when classifying letters of the alphabet.

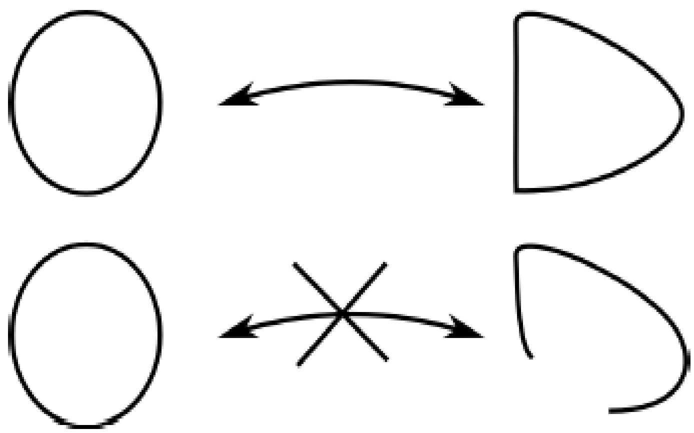

Figure-6: A loop is homeomorphic to any other loop (top), but not to a path with endpoints (bottom)

4.4 Hausdorff Property

If X is a topological space and S is a subset of X then a neighbourhood of S is a set V, which contains an open set U containing S. i.e. $S \subseteq U \subseteq V \subseteq X$. Let X be a topological space. Let x, y ε X. We say that x and y can be separated by neighbourhoods if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint i.e. $U \cap V = \phi$. A space is Hausdorff if every two distinct points have disjoint neighbourhoods.

5. CLASSIFICATION OF LETTERS: TOPOLOGICAL INVARIANTS

A topological invariant of a space X is a property that depends solely on the topology of the space X. That is, a property shared by any other space that is homeomorphic to X. Intuitively, a homeomorphism between X and Y maps points in X that are “close together” to points in Y that are “close together”, and points in X not “close together” to points in Y that are not “close together”. We now use the preceeding arguments to classify the letters of the alphabet based on their topological invariants. We must use the letters printed as in the project specifications since changing the font can change the topological invariants of the letter shows in Table-1.

0 holes, 0 three-vertices, 0 four-vertices:	C, G, I, J, L, M, N, S, U, V, W, Z
1 hole, 0 three-vertices, 0 four-vertices:	D, O
0 holes, 1 three-vertex, 0 four-vertices:	E, F, T, Y
1 hole, 1 three-vertex, 0 four-vertices:	P
0 holes, 2 three-vertices, 0 four-vertices:	H, K
1 hole, 2 three-vertices, 0 four-vertices:	A, R
2 holes, 2 three-vertices, 0 four-vertices:	B
0 holes, 0 three-vertices, 1 four-vertex:	X
1 hole, 0 three-vertices, 1 four-vertex:	Q

Table-1: Classification of the letters using the topological invariant

6. CONCLUSION

The notion of topological equivalence is based on homeomorphism classes. For each element of such classes there exists a topological transformation to elements of the same class. All these elements are said to be topologically equivalent. In other words, the elements are 'equal' with respect to topological properties, disregarding other geometric properties such as metric and shape. The topological properties are called topological invariants, since they do not change after topological transformations. The general way to assess topological equivalence would be to find a bi-continuous bijection between the two point-sets. The alternative is to show that topological invariants are preserved. The latter approach, which is adopted in the paper, implies as the basic step the definition of a set of topological invariants (the classifying invariant) that must satisfy two requirements: to be necessary and to uniquely identify a topological equivalence class. The classifying invariant is a description of all topological properties of any two-dimensional scene of objects and constitutes the basis for defining topological primitives of a spatial query language and for checking topological consistency among multiple representations of spatial data.

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