Laplace q-Transform of Trigonometric Functions

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Abstract : In this paper, we obtain the extorial and trigonometric functions using the Laplace q-Transform, suitable examples are inserted to illustrate the main results.

Key words: Laplace -Transform, Difference operator, Inverse difference operator, Extorial functions, Trigonometric functions.

INTRODUCTION

The most crucial idea in mathematics in trigonometry, and its applications are incredibly widerspread in daily life. Trigon0metry has a wide range of immediate applications wheter it be in the domains of Aviation, Physics, Ariminology or the military among others, it is crucial. Finding the triangles angles or sides is done using trigonometry examines how a triangle’s and sides relate to one another. Trigon0metry can be used to determine the heights of enormous mountains or structures. It is commonly used in Physics, Architecture between stars and plants. In this paper, we present the extorial and trigonometric functions using the Laplace q-transform.

2. Preliminaries

This section focuses on the basic definition of the q-difference operator.

Definition 2.1. If n and q are any two positive integer, then the generalized positive polynomial factorial is defined as

$$k_q^{(n)} = k(k-q)(k-q^2)\cdots(k-q^{n-1}) \; and \; k_q^{(0)} = 1;\; k_q^{(1)} = k \quad (1)$$

Definition 2.2. If $u(k)$ is a sequence of numbers and q is any positive integer, then we define the generalized difference operator Δq as

$$\Delta q u(k) = u(kq) - u(k) \quad (2)$$

Definition 2.3. Let $q > 0$ and u(k), v(k) are real valued bounded functions. Then

$$\Delta_q^{-1}\left(u(kq) - u(k)\right) = u(k)\Delta_q^{-1}v(k) - \Delta_q^{-1}\left(\Delta_q^{-1}v(kq).\Delta q u(k)\right) \quad (3)$$

Definition 2.4. Let $q > 0, u(k)$ be a real valued functions on $[0, 1)$. Then

$$\sum_{r=0}^{m} u(kq^r) = \Delta_q^{-1}u(k)\Big|_{kq^m}^{k} \quad (4)$$

Definition 2.5. Let $k \in [0;\, 1)$ and $q > 0$ then

$$\sin k_q^{(1)} = k_q^{(1)} - \frac{k_q^{(3)}}{3!} + \frac{k_q^{(5)}}{5!} - \frac{k_q^{(7)}}{7!} + \cdots \quad (5)$$

Definition 2.6. Let $k \in [0;\, 1)$ and $q > 0$ then

$$\sin(ak)_q^{(1)} = (ak)_q^{(1)} - \frac{(ak)_q^{(3)}}{3!} + \frac{(ak)_q^{(5)}}{5!} - \frac{(ak)_q^{(7)}}{7!} + \cdots \quad (6)$$

Definition 2.7. Let $k \in [0;\, 1)$ and $q > 0$ and $k_q^{(n)}$ be a n^{th} power of polynomial factorial, then

$$\sin k_q^{(n)} = k_q^{(n)} - \frac{k_q^{(3n)}}{3!} + \frac{k_q^{(5n)}}{5!} - \frac{k_q^{(7n)}}{7!} + \cdots \quad (7)$$

3. Laplace q-Transform of sine Function

In this section, we define Laplace q-transform of sine function and results using the operator Δ_q^{-1} and present the examples to examine their results.

Lemma 3.1. *If* $e^{-skq^r} \neq 0$ *and* $q > 0$ *then the Laplace q-transform of sine function is*

$$L_q\left(sin(k)_q^{(1)}\right) = (1 - q)\left\{k\, sin(k)_q^{(1)} \sum_{r=0}^{\infty} e^{-skq^r} + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\left(kq^{t+1} sin\left(kq^{t+1}\right)_q^{(1)} - kq^t sin\left(kq^t\right)_q^{(1)}\right)\right)\right\} \quad (8)$$

Proof. Let $u(k) = sin(k)_q^{(1)}$ in difference equation we obtain,

$$L_q\, sin(k)_q^{(1)} = (q-1)\Delta_q^{-1} k\, sin(k)_q^{(1)} e^{-sk}\Big|_0^{\infty}$$

$$= (q-1)\Delta_q^{-1}\left(k_q^{(1)} - \frac{k_q^{(3)}}{3!} + \frac{k_q^{(5)}}{5!} - \frac{k_q^{(7)}}{7!} + \cdots\right) e^{-sk}\Big|_0^{\infty} \quad (9)$$

Separate the terms and using the equations (1), (2), (3) and (4)

$$\Delta_q^{-1} k\, k_q^{(1)} e^{-sk}|_0^{\infty} = \left[\Delta_q^{-1} k\, k_q^{(1)} \Delta_q^{-1} e^{-sk} - \Delta_q^{-1}\left(\Delta_q^{-1} e^{-skq} \Delta_q k\, k_q^{(1)}\right)\right]\Big|_0^{\infty}$$

$$= k\, k_q^{(1)} \sum_{r=0}^{\infty} e^{-skq^r} - \Delta_q^{-1}\left[-\sum_{r=0}^{\infty} e^{-skq^{r+1}}\left(kq(kq)_q^{(1)} - k(k)_q^{(1)}\right)\right]\Big|_0^{\infty}$$

$$= -k\, k_q^{(1)} \sum_{r=0}^{\infty} e^{-skq^r} + \sum_{t=0}^{\infty}\left[-\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\left(kq^{t+1}(kq^{t+1})_q^{(1)} - kq^t (kq^t)_q^{(1)}\right)\right] \quad (10)$$

Also,

$$\Delta_q^{-1} \frac{k\, k_q^{(3)}}{3!} e^{-sk}\Big|_0^{\infty} = \frac{1}{3!}\left[k\, k_q^{(3)} \Delta_q^{-1} e^{-sk} - \Delta_q^{-1}\left(\Delta_q^{-1 e^{-skq}} \Delta_q k\, k_q^{(3)}\right)\right]\Big|_0^{\infty}$$

we get the equation,

$$\Delta_q^{-1} \frac{k\, k_q^{(3)}}{3!} e^{-sk}\Big|_0^{\infty} = \frac{1}{3!}\left[-k\, k_q^{(3)} \sum_{r=0}^{\infty} e^{-skq^r} - \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(3)} - kq^t (kq^t)_q^{(3)}\right)\right)\right] \quad (11)$$

Replace $\frac{k_q^{(3)}}{3!}$ by $\frac{k_q^{(5)}}{5!}$ in equation (11) we get,

$$\Delta_q^{-1} \frac{k\, k_q^{(5)}}{5!} e^{-sk}\Big|_0^{\infty} = \frac{1}{5!}\left[-k\, k_q^{(5)} \sum_{r=0}^{\infty} e^{-skq^r}\right.$$

$$-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(5)} - kq^t(kq^t)_q^{(5)}\right)\right)\Bigg] \qquad (12)$$

Applying the process mentioned above, we get

$$\Delta_q^{-1}\frac{k\,k_q^{(7)}}{7!}e^{-sk}\Bigg|_0^{\infty} = \frac{1}{7!}\Bigg[-k\,k_q^{(7)}\sum_{r=0}^{\infty} e^{-skq^r}$$
$$-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(7)} - kq^t(kq^t)_q^{(7)}\right)\right)\Bigg] \qquad (13)$$

Substituting equation (10) to (13) in (9) we get,

$$L_q\left(sin(k)_q^{(1)}\right) = (1-q)\ \Bigg\{\Bigg[-k\,k_q^{(1)}\sum_{r=0}^{\infty} e^{-skq^r}$$
$$-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(1)} - kq^t(kq^t)_q^{(1)}\right)\right)\Bigg]$$
$$-\frac{1}{3!}\Bigg[-k\,k_q^{(3)}\sum_{r=0}^{\infty} e^{-skq^r}$$
$$-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(3)} - kq^t(kq^t)_q^{(3)}\right)\right)\Bigg]$$
$$+\frac{1}{5!}\Bigg[-k\,k_q^{(5)}\sum_{r=0}^{\infty} e^{-skq^r}$$
$$-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(5)} - kq^t(kq^t)_q^{(5)}\right)\right)\Bigg]$$
$$-\frac{1}{7!}\Bigg[-k\,k_q^{(3)}\sum_{r=0}^{\infty} e^{-skq^r}$$
$$-\sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\left(kq^{t+1}(kq^{t+1})_q^{(7)} - kq^t(kq^t)_q^{(7)}\right)\right)\Bigg] + \cdots\Bigg\}$$

$$= (1-q)\left\{\left(kk_q^{(1)} - \frac{kk_q^{(3)}}{3!} + \frac{kk_q^{(5)}}{5!} - \frac{kk_q^{(7)}}{7!} + \cdots\right)\sum_{r=0}^{\infty} e^{-skq^r}\right.$$

$$-\sum_{t=0}^{\infty}\left[\sum_{r=0}^{\infty} e^{-skq^{(t+r+1)}}\, kq^{t+1}\left[\frac{(kq^{t+1})_q^{(1)}}{1!} - \frac{(kq^{t+1})_q^{(3)}}{3!} + \frac{(kq^{t+1})_q^{(5)}}{5!}\right.\right.$$

$$\left. - \frac{(kq^{t+1})_q^{(7)}}{7!} + \cdots\right] - kq^t\left[\frac{(kq^{t+1})_q^{(1)}}{1!} - \frac{(kq^{t+1})_q^{(3)}}{3!} + \frac{(kq^{t+1})_q^{(5)}}{5!}\right.$$

$$\left.\left.\left. - \frac{(kq^{t+1})_q^{(7)}}{7!} + \cdots\right]\right]\right\}$$

which yields the proof.

Theorem: 3.2 *If* $e^{-skq^{t+r+1}} \neq 0$ *and* $q > 0$ *then the Laplace q-transform of sine function is*

$$L_{\boldsymbol{q}}\left(\sin(ak)_q^{(1)}\right) = (1-q)\left\{k\sin(ak)_q^{(1)}\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\right.$$

$$\left. + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, kq^{t+1}\sin(akq^{t+1})_q^{(1)} - kq^t\sin(akq^t)_q^{(1)}\right)\right\} \qquad (14)$$

Proof. *Replacing* $\sin\ k_q^{(1)} = \sin(ak)_q^{(1)}$ *in the previous Lemma 3.1, we get the proof.*

Theorem: 3.3 *If* $e^{-skq^r} \neq 0$ *and* $q > 0$ *then the Laplace q-transform of* n^{th} *power of sine function is*

$$L_{\boldsymbol{q}}\left(\sin(ak)_q^{(n)}\right) = (1-q)\left\{k\sin(ak)_q^{(n)}\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\right.$$

$$\left. + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, kq^{t+1}\sin(akq^{t+1})_q^{(n)} - kq^t\sin(akq^t)_q^{(n)}\right)\right\} \qquad (15)$$

Proof. *Replace* $\sin\ k_q^{(1)}$ *by* $\sin(ak)_q^{(2)}$ *in equation (8), and using (7)and (10) we arrive.*

$$L_{\boldsymbol{q}}\left(\sin(ak)_q^{(2)}\right) = (1-q)\left\{k\sin(ak)_q^{(2)}\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\right.$$

$$\left. + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\, kq^{t+1}\sin(akq^{t+1})_q^{(2)} - kq^t\sin(akq^t)_q^{(2)}\right)\right\} \qquad (16)$$

Replace $\sin\ k_q^{(1)}$ *by* $\sin(ak)_q^{(3)}$ *in equation (8), and using (7)and (10) we obtain*

$$L_{\boldsymbol{q}}\left(\sin(ak)_q^{(3)}\right) = (1-q)\left\{k\sin(ak)_q^{(3)}\sum_{r=0}^{\infty} e^{-skq^{t+r+1}} + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\; kq^{t+1}\sin(akq^{t+1})_q^{(3)} - kq^t\sin(akq^t)_q^{(3)}\right)\right\} \qquad (17)$$

Replace $\sin\ k_q^{(1)}$ *by* $\sin(ak)_q^{(4)}$ *in equation (8), and using (7) and (10), we arrive.*

$$L_{\boldsymbol{q}}\left(\sin(ak)_q^{(4)}\right) = (1-q)\left\{k\sin(ak)_q^{(4)}\sum_{r=0}^{\infty} e^{-skq^{t+r+1}} + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\; kq^{t+1}\sin(akq^{t+1})_q^{(4)} - kq^t\sin(akq^t)_q^{(4)}\right)\right\} \qquad (18)$$

By repeating the process n times, we get

$$L_{\boldsymbol{q}}\left(\sin(ak)_q^{(n)}\right) = (1-q)\left\{k\sin(ak)_q^{(n)}\sum_{r=0}^{\infty} e^{-skq^{t+r+1}} + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\; kq^{t+1}\sin(akq^{t+1})_q^{(n)} - kq^t\sin(akq^t)_q^{(n)}\right)\right\} \qquad (18)$$

Corollary: 3.4 *If* $e^{-skq^{t+r+1}} \neq 0$ *and* $q > 0$ *then the Laplace q-transform of sine function is*

$$L_{\boldsymbol{q}}\left(\sin(ak)_q^{(n)}\right) = (1-q)\left\{k\sin(ak)_q^{(n)}\sum_{r=0}^{\infty} e^{-skq^{r}} + \sum_{t=0}^{\infty}\left(\sum_{r=0}^{\infty} e^{-skq^{t+r+1}}\; kq^{t+1}\sin(akq^{t+1})_q^{(n)} - kq^t\sin(akq^t)_q^{(n)}\right)\right\} \qquad (19)$$

Proof. The proof follows from taking $\sin k_q^{(n)} = \sin(akq^t)_q^{(n)}$ *in equation (16), we get the proof of the corollary.*

Conclusion:

In this paper we have developed the discrete Laplace q-transform for sine function. The given example shows the values of Laplace q-transform for sine function.

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