PELL GRACEFUL LABELING OF FEW SPECIFIC TREES AND BOUNDS ON PELL GRACEFUL NUMBER

S. Mangaleswaran[1*], R. Ganapathyraman[2]

Abstract
Pell graceful labeling was introduced in [1]. In this paper, pell graceful labeling of few families of trees are found. We introduce a new concept pell graceful edge number and the same is found for few standard graphs, also bounds are found in some cases.

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Keywords: Pell graceful edge number, Pell graceful vertex nuber.

[1*]Assistant Professor, Department of Mathematics R.V. Government Arts College, Chengalpattu-1
[2]Assistant Professor, Department of Mathematics Pachaiyappa's College, Chennai-30

***Corresponding Author:** S. Mangaleswaran
Assistant Professor, Department of Mathematics R.V. Government Arts College, Chengalpattu-1, mangaleswaran75@gmail.com

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1 Introduction

Graph theory is now a key branch of applied mathematics as well as research that involves multiple disciplines such as computer science, operations research and other sciences. Because of its broad spectrum of applications in all areas of life science, a graph is a very useful discrete structure. Euler attempted to solve the Konigsberg seven bridges problem in the 18^{th} century and established graph theory. Over 200 variants of graph labeling have been introduced and studied over the last 60 years, with nearly 2500 research articles published [2]. Under specific conditions, graph labeling is the assignment of integers to vertices, edges or both. In this paper pell graceful labeling are exhibited for some tree graphs and a new graph parameter has been introduced and found for few graphs. As labeled graphs are widely used mathematical models for a variety of applications, we made this study. This paper exhibits few new results on graph labeling. Throughout this paper we follow the notations as in [4].

2 Pell Graceful Labeling

Definition 2.1 (Pell sequence of numbers). *Pell numbers are numbers that look like Fibonacci numbers and are generated by the formula*
$P_{n+1} = 2P_n + P_{n-1}$ $(n \geq 1)$ *with* $P_0 = 0,\ P_1 = 1$. *The first few Pell numbers are*

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots.$$

Also, P_n *can be given as* $P_n = \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$

Definition 2.2 (Pell graceful graph - PGG). $G(V,E)$ *be a graph of order m and size n* (≥ 1). *An injection* $\sigma : V(G) \to \{0,1,2,\ldots,P_n\}$ *where* P_n *is the* n^{th} *pell number in the pell sequence is said to be pell graceful, if the induced edge labeling* $\hat{\sigma} : E \to \{P_1,P_2,\ldots,P_n\}$ *given by* $\hat{\sigma}(v_1v_2) = |\sigma(v_1) - \sigma(v_2)|$ *is a bijective function from E onto the set* $\{P_1,P_2,\ldots,P_n\}$.

A graph is a Pell graceful graph if it admits pell graceful labeling.

Definition 2.3 (m-star $St(\alpha_1,\alpha_2,\ldots,\alpha_m)$). *[3] The m-star* $St(\alpha_1,\alpha_2,\ldots,\alpha_m)$ *is a disconnected graph with m-components* $K_{1,\alpha_1},K_{1,\alpha_2},\ldots,K_{1,\alpha_m}$ *where* $K_{1,n}$ *denotes a star on* $(n+1)$ *vertices.*
Example: $St(2,3,4)$

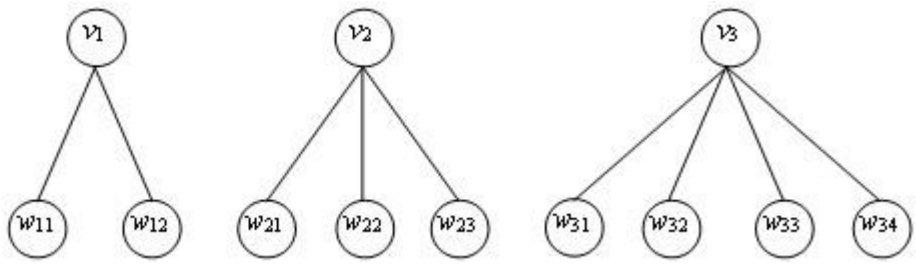

Figure 1: *It is a 3-star.*

Theorem 2.4. *The m-star* $St(\alpha_1,\alpha_2,\ldots,\alpha_m)$ *is a Pell graceful graph.*
Proof. Let v_i denote the central vertex and w_{ij}, $j = 1,2,\ldots,\alpha_i$ denote end vertices of the star K_{1,α_i}, $i = 1,2,\ldots,m$. Clearly

$$|V(St(\alpha_1,\alpha_2,\ldots,\alpha_m))| = \alpha_1 + \alpha_2 + \cdots + \alpha_m + m$$
$$|E(St(\alpha_1,\alpha_2,\ldots,\alpha_m))| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$$

Define σ :

$$V(St(\alpha_1,\alpha_2,\ldots,\alpha_m)) \to \{0,1,2,\ldots,P_{\alpha_1+\alpha_2+\cdots+\alpha_m}\}$$

where

$$Pr+1 = 2Pr + Pr-1 \ (r \geq 1)$$

As

$$\sigma(v_i) = i - 1, \quad i = 1, 2, \ldots, m$$
$$\sigma(w_{11}) = P_1$$
$$\sigma(w_1\alpha_1) = P_{\alpha_1+\alpha_2+\cdots+\alpha_m}$$

$$\sigma(w_{1j}) = P_j, \quad 2 \leq j \leq \alpha_1 - 1$$
$$\sigma(w_{2j}) = P_{\alpha_1+j-1} + 1, \quad 1 \leq j \leq \alpha_2$$
$$\sigma(w_{3j}) = P_{\alpha_1+\alpha_2+j-1} + 2, \quad 1 \leq j \leq \alpha_3$$
$$\vdots \quad \vdots \quad \vdots$$
$$\sigma(w_{rj}) = P_{\alpha_1+\alpha_2+\cdots+\alpha_{r-1}+j-1} + (r-1), \quad 1 \leq j \leq \alpha_r$$
$$\vdots \quad \vdots \quad \vdots$$
$$\sigma(w_{mj}) = P_{\alpha_1+\alpha_2+\cdots+\alpha_{m-1}+j-1} + (m-1), \quad 1 \leq j \leq \alpha_m$$

By definition of σ, it is an injection and the induced function on the edge set $\hat{\sigma} : E(St(\alpha1,\alpha2,...,\alpha m)) \rightarrow \{P1,P2,...,P\alpha1+\alpha2+\cdots+\alpha m\}$ is a bijection defined by

$\hat{\sigma}(v_iw_{ij}) = |\sigma(v_i) - \sigma(w_{ij})|, \ \forall\ i,j$

$\hat{\sigma}$ gives a pell graceful labeling for $St(\alpha_1,\alpha_2,...,\alpha_m)$. Therefore $St(\alpha_1,\alpha_2,...,\alpha_m)$ is a pell graceful graph.

□

Example 2.5. *Pell graceful labeling of 3-star St*(3,3,4)

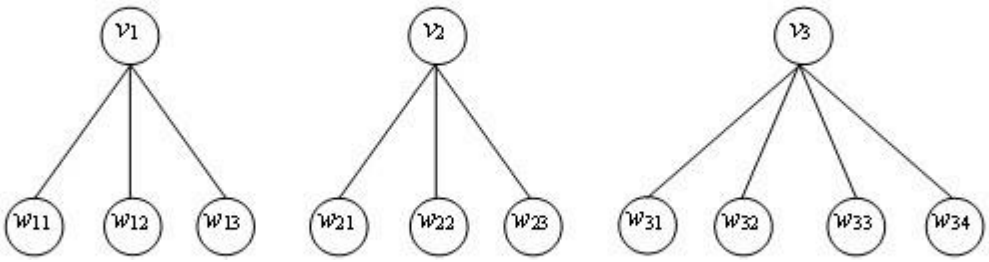

Figure 2:

Here, $m = 3;\ \alpha_1 = 3;\ \alpha_2 = 3;\ \alpha_3 = 4$

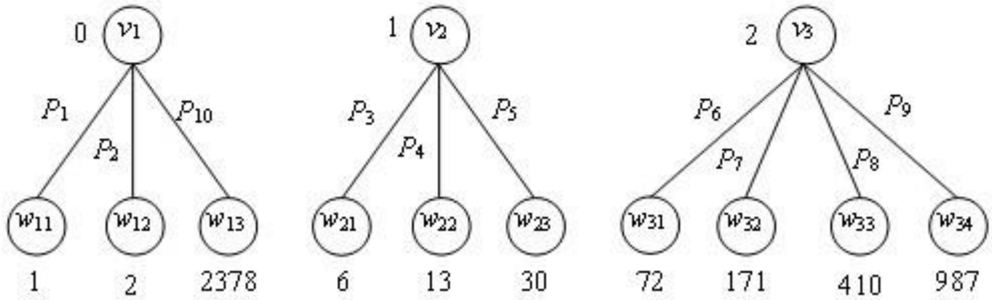

Figure 3:

Definition 2.6 (Tree t_n)**.** *The tree* t_n*,* $n \geq 1$ *is tree obtained by adding n pendent edges to each pendent vertices of star graph* $K_{1,2}$ *and* $|V(t_n)| = 2n+3$, $|E(t_n)| = 2n+2$.

Theorem 2.7. *Trees* t_n ($n \geq 3$) *are Pell graceful graphs.*

Proof. t_n be the tree as defined in 2.6.

Then $|V(t_n)| = 2n + 3$, $|E(t_n)| = 2n + 2$.

$$V(t_n) = \left\{ \begin{array}{ll} v_i' & 0 \leq i \leq 2 \\ w_j' & 1 \leq j \leq n;\ u_k',\ 1 \leq k \leq n \end{array} \right\}$$

$$E(t_n) = \left\{ \begin{array}{ll} v_0'v_i' & 1 \leq i \leq 2 \\ v_1'w_j' & 1 \leq j \leq n;\ v_2'u_k',\ 1 \leq k \leq n \end{array} \right\}$$

Define

$\sigma : V(t_n) \rightarrow \{0,1,2,...,P_{2n+2}\}$ where $P_{2n+2} = 2P_{2n+1} + P_{2n}$ and $P_0 = 0$; $P_1 = 1$

as $\sigma(v_0') = 0$ $\sigma(v_1') = P_1$

$\sigma(v2') = P2n+2$

$\sigma(w_j') = P_{j+1} + 1,\ 1 \leq j \leq n$ $\sigma(u'k) = P2n+2-k + P2n+2,\ 1 \leq k \leq n$

This injection σ will induce a bijection from $E(G)$ to $\{P_1,P_2,...,P_{2n+2}\}$ defined by $\hat{\sigma}(uv) = |\sigma(u) - \sigma(v)|$, $\forall\ uv \in E(t_n)$. This $\hat{\sigma}$ gives a pell graceful labeling of t_n.

Therefore trees t_n are Pell graceful graphs.

Example 2.8. *Pell graceful labeling of tree* t_4

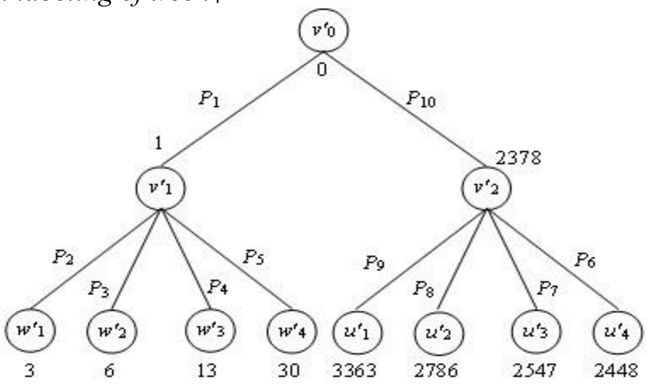

Figure 4:

Definition 2.9 (Tree $K_{1,n,n}$ $(n \geq 1)$)**.** *The graph* $K_{1,n,n}$ $(n \geq 1)$ *having* $(2n+1)$ *vertices and* $2n$ *edges as shown in below figure is called a multi star graph.*

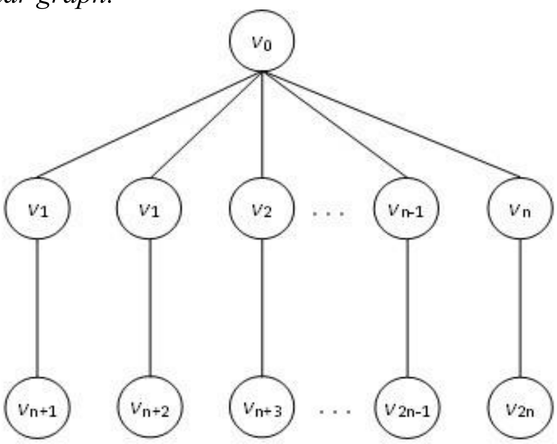

Figure 5:

Theorem 2.10. *The tree* $K_{1,n,n}$ *is a Pell graceful graph.*

Proof. This tree $K_{1,n,n}$ is of order $(2n + 1)$ and size $2n$.

$$V(K_{1,n,n}) = \{v_i'/0 \leq i \leq n, w_k'/n+1 \leq k \leq 2n\}$$

$$E(K_{1,n,n}) = \{v_0'v_i'/1 \leq i \leq n\} \cup \{v_i'w_{n+i}'/1 \leq i \leq n\}$$

Define

$\sigma : V(K_{1,n,n}) \rightarrow \{0,1,2,...,P_{2n}\}$ where $P_{2n} = 2P_{2n-1} + P_{2n-2}$ and $P_0 = 0$; $P_1 = 1$

as $\sigma(v_0') = 0$ $\sigma(v_i') = P_i$, $1 \leq i \leq n$ $\sigma(wn' + i) = Pi + Pi+n$, $1 \leq i \leq n$

This injection σ induces a pell graceful labeling $\hat{\sigma}$ from $E(K_{1,n,n})$ to $\{P1, P2, ..., P2n\}$ as $\hat{\sigma}(vi'vj') = |\sigma(vi') - \sigma(vj')|$, $\forall$ $vi'vj' \in E(K1,n,n)$. Thus $K_{1,n,n}$ admits a pell graceful labeling.

$K_{1,n,n}$ is a Pell graceful graph.

□

Example 2.11. *Pell graceful labeling of* $K_{1,3,3}$

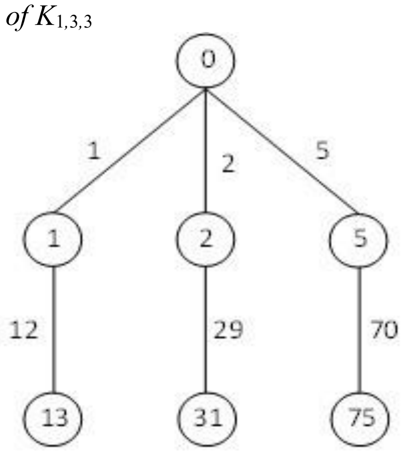

Figure 6:

3 Pell Graceful Edge Number

Definition 3.1 (Pell graceful edge number (PGEN))**.** *Pell graceful edge number of a graph G is defined as the least number of edges whose removal from the graph G makes the resulting graph pell graceful and it is denoted by PGEN(G).*

Observation 3.2. *It is proved C_3 is not a Pell graceful graph in [1]. Also it is proved in [1] that all paths P_n are Pell graceful for $n \geq 3$. We observe that $(C_3 - e)$ is a Pell graceful graph.*

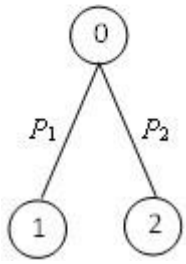

Figure 7:

Thus, $PGEN(C_3) = 1$ and in general Pell graceful number of any cycle C_n ($n \geq 3$) is given by $PGEN(C_n) \leq 1$.

Also, it is proved in [1] that complete graph K_n is not a Pell graceful graph for all $n \geq 3$ which motivates us to find $PGKN(K_n)$.

$K_3 = C_3$ is not a Pell graceful graph. Hence $PGEN(K_3) = 1$ we give a bound by the following theorem.

Theorem 3.3. *The Pell graceful edge number $PGEN(K_n)$ of a complete graph is atmost $\frac{n^2-3n+2}{2}$ for $n \geq 3$.*

Proof. It is clear that complete graph K_n is not a Pell graph for all $n \geq 3$.

Thus, $PGEN(K_n) \geq 1$. Also, $|E(K_n)| = nC_2$.
The deletion of $\frac{n^2-3n+2}{2}$ edges from K_n gives a path of length $(n - 1)$ which is a pell graceful graph.
Thus, $PGEN(K_n) \leq \frac{n^2-3n+2}{2}$. □

Theorem 3.4. *The Pell graceful edge number $PGEN(W_n)$ of a wheel graph is atmost n for $n \geq 3$.*
Proof. It is clear that wheel graph W_n is not a pell graph for all $n \geq 3$.

Thus, $PGEN(W_n) \geq 1$. Also, $|E(W_n)| = 2n$.
The deletion of all the n edges on the outer cycle of W_n results in a star graph $K_{1,n}$ which is a Pell graceful graph. Thus $PGEN(W_n) \leq n$. □

Illustration 3.5.

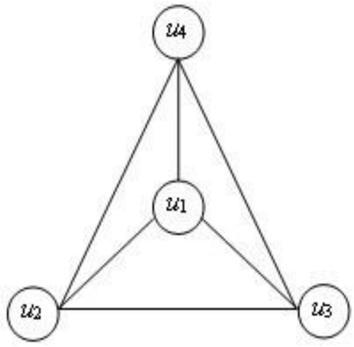

Figure 8:

Wheel W_3 is not a pell graceful graph proved in [1]. When the three edges on the outer cycle u_2u_3, u_3u_4, u_2u_4 are removed, it becomes $W_3 - 3e$.

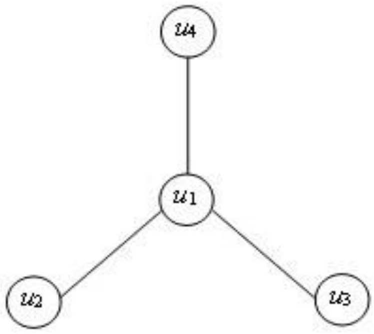

Figure 9:

Which has pell graceful labeling as given below. Define $\sigma : V(G) \rightarrow \{0,1,...,5(P_3)\}$ as $f(u_1) = 0$ $f(u_i) = P_{i-1}$, $2 \leq i \leq 4$

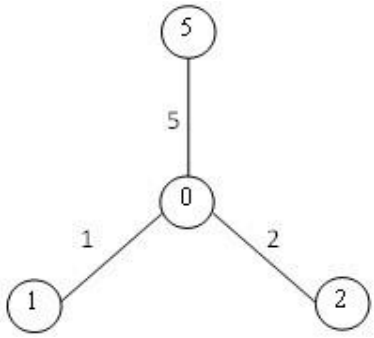

Figure 10:

In general,

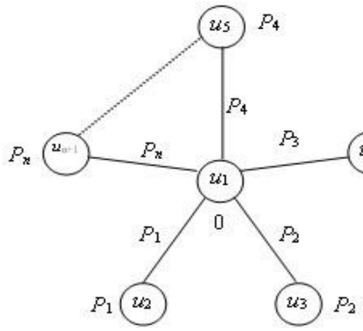

Figure 11:

Define
$\sigma(u_1) = 0$
$\sigma : V(W_n) \rightarrow \{0,1,2,...,P_n\}$
$\sigma(ui) = Pi-1,$
$2 \leq i \leq n+1$ *Thus the induced function*
$\hat{\sigma} : E(G) \rightarrow \{P_1,P_2,...,P_n\}$ $\hat{\sigma}(u_1u_i) = P_{i-1},$ 2
$\leq i \leq n+1$
is a bijection.
Hence $W_n - ne$ *is a pell graceful graph.*

4 Pell Graceful Vertex Number

Definition 4.1 (Pell graceful vertex number (PGVN))**.** *Pell graceful vertex number of a graph G is defined as the minimum number of vertices whose removal from the graph G makes it pell graceful and it is denoted by PGVN.*

Theorem 4.2. *Pell graceful vertex number of a wheel graph* W_n $(n \geq 3)$ *is* $PVGN(W_n) = 2$.
Proof. Let W_n be a wheel with n spokes. Then
$|V(W_n)| = n+1$
$|E(W_n)| = 2n$
Let
$V(W_n) = \{u_1,u_2,...,u_{n+1}\}$
Let u_1 be the centre of W_n and
$E(W_n) = \{u_1u_j; 2 \leq i \leq n+1\} \cup \{u_iu_{i+1}; 2 \leq i \leq n+1\}$

Now, a new graph from W_n by omitting u_1 and any other u_i, $2 \leq I \leq n+1$. The resulting graph G is a path which is a pell graceful graph as all paths are pell graceful. Since, removal of minimum of two vertices from W_n makes the resulting graph as pell graceful. Hence $PVGN(W_n) = 2$. □

Illustration 4.3. W_3 *is not pell graceful*

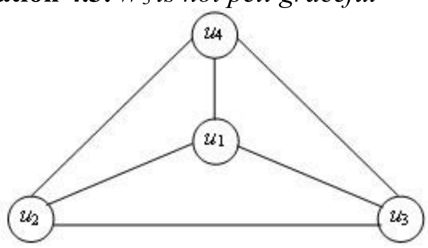

Figure 12:

Graph G is obtained from W_3 *by omitting* u_1 *and* u_2 *or* u_3 *or* u_4 *with loss of generality let it be* u_3. *Then G is*

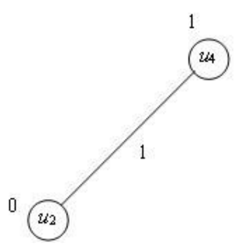

Figure 13:

Define $\sigma : \{u_2,u_4\} \rightarrow \{0,1,2\}$ *by* $\sigma(u_2) = 0$ $\sigma(u_4) = 1$
Then clearly G is pell graceful. Hence, $PGVN(W_3) = 2$.

5 Conclusion

We defined new parameter which determines when a non-pell graph becomes a pell graceful graphs. Also we have given pell graceful labeling for two different types of trees and a disconnected graph having each component as a star graph. We are interested further to introduce a new graceful labeling namely k-pell graceful labeling as a continued research work of this paper.

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