ON SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH TOUCHARD POLYNOMIALS

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Abstract:
This paper focuses on the establishment of a new subfamily of analytic functions including Touchard polynomials. Afterwards, we attempt to obtain geometric properties such as coefficient inequalities, distortion properties, extreme points, radii of starlikeness and convexity, Hadmard product and convolution and integral operators for the class.

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1.Introduction

The application of special function on Geometric function Theory is a current and interesting topic of research. It is often used in areas such as physics, engineering, and mathematics.As a result of De Branges', the classic Bieberbach problem is successfully solved by applying a generalized hypergeometric function. Several types of special functions, including generalized hypergeometric Gaussian functions (see [1,2,3]), and Gegenbauer polynomials (see[30]), have been studied extensively.

In combinatorics, the Bell numbers, usually denoted by B_k for $k \in \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, count the number of ways a set with k elements can be partitioned into disjoint and nonempty subsets. These numbers have been studied by mathematicians since the 19th century, and their roots go back to medieval Japan, but they are named after Eric Temple Bell, who wrote about them in the 1930. The Bell numbers B_k for $k \geq 0$ can be generated by

$$e^{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = 1 + t + t^2 + \frac{5}{6}t^3 + \frac{5}{8}t^4 + \frac{13}{30}t^5 + \frac{203}{720}t^6 + \cdots$$

and the first ten Bell numbers B_k for $0 \leq k \leq 9$ are

$$B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52,$$
$$B_6 = 203, B_7 = 877, B_8 = 4140, B_9 = 21147.$$

In [15], it was pointed out that there have been studies on interesting applications of the Bell polynomials $B_k(x)$ in soliton theory, including links with bilinear and trilinear forms of nonlinear differential equations which possess soliton solutions. See, for example,[9,10,11] . Therefore, applications of the Bell polynomials $B_k(x)$ to integrable nonlinear equations are greatly expected and any amendment on multilinear forms of soliton equations, even on exact solutions, would be beneficial to interested audiences in the research community. For more information about the Bell polynomials $B_k(x)$, please refer to and closely related references therein.

The Touchard polynomials, studied by Jacques Touchard (1939), also called the exponential polynomials comprise a polynomial sequence of binomial type. It is a new algorithm for solving linear and nonlinear integral equations. Touehard in his work on the cycles of permutations generalized the Bell polynomials in order to study some problems of enumeration of the permutations when the cycles possess certain properties. And he introduced and study a class of related polynomials. An exponential generating function, recurrence relations and connections with other well-known polynomials are obtained. In special cases, relations with Stifling number of the first and second kind, as well as with other numbers recently studied are derived. Finally, a combinatorial interpretation is discussed.

In general, the integral equations are difficult to be solved analytically, therefor in many equations we need to get the approximate solutions, and for this case the "Touchard Polynomials method“ for the solution linear "Volterra integro- differential equatio“ is implemented. TheTouchard polynomials method has been applied in for solving linear and nonlinear Volterra (Fredholm) integral equations.

There has been research on interesting applications of the Touchard polynomials Tn(x) in nonlinear Fredholm-Volterra integral equations[12] and soliton theory in [9,10,11], including connections with bilinear and trilinear forms of nonlinear differential equations which possess soliton solutions. Therefore, applications of the Touchard polynomials $T_n(x)$ to integrable nonlinear equations are greatly expected and any amendment on multi-linear forms of soliton equations, even on exact solutions, would be beneficial to interested audiences in the community. For more information about the Touchard polynomials $T_n(x)$, see [15] . A Touchard polynomial is also known as an exponential generating polynomial created by Jacques Touchard [26]. (see[15]) or Polynomial sequences of Bell type (see [3]) are polynomial sequences of binomial type that represent a random variable X with a Poisson distribution with an expected value $\hbar$ then its n^{th} moment is $E(X_\varrho) = \mathfrak{I}(\varrho, \hbar)$, resulting in the type:

$$\mathfrak{I}(\varrho, \hbar) = e^{\varrho} \sum_{\ell=0}^{\infty} \frac{\varrho^{\ell} \ell^{\hbar}}{\ell!} w^{\ell}, \quad w \in U. \qquad (1.1)$$

The result of the second force is presented using the coefficients of Touchard polynomials as follows:

$$\phi_{\varrho}^{\hbar}(w) = w + \sum_{\ell=2}^{\infty} \frac{(\ell-1)^{\hbar} \varrho^{\ell-1}}{(\ell-1)!} e^{-\varrho} w^{\ell}, \quad w \in U, \qquad (1.2)$$

where $\hbar \geq 0, \varrho > 0$ and by analyzing ratio tests, we find that the radius of convergence of the above series is infinity.

Consider H the class of analytic functions in the unit disk $U = \{w: |w| < 1\}$. We will define A as a class of functions $\vartheta \in H$ of the type

$$\vartheta(w) = w + \sum_{\ell=2}^{\infty} a_\ell w^\ell, \quad w \in U. \qquad (1.3)$$

Let S be the subclass of A that consists of functions that are normalised, $\vartheta(0) = 0 = \vartheta'(0) = 1$, and also univalent in U. A's subclass, which consists of functions of the type

$$\vartheta(w) = w + \sum_{\ell=2}^{\infty} a_\ell w^\ell, \ \ a_\ell \geq 0. \qquad (1.4)$$

T signifies the subclass of S that consisting of mapping of the type

$$\vartheta(w) = w - \sum_{\ell=2}^{\infty} a_\ell w^\ell, \ \ a_\ell \geq 0 \text{ and } \ w \in U$$

studied extensively by Silverman [20].
For $\vartheta \in A$ given by (1.4) and $g(w)$ given by

$$g(w) = w + \sum_{\ell=2}^{\infty} b_\ell w^\ell$$

their convolution indicated by $(\vartheta * g)$, is defined as

$$(\vartheta * g)(w) = w + \sum_{\ell=2}^{\infty} a_\ell b_\ell w^\ell = (g * \vartheta)(w), \ \ w \in U.$$

The linear operator is now understood

$$\mathcal{K}_c^{\hbar}: A \to A$$

and as a consequence of convolution

$$\mathcal{K}_c^{\hbar}\vartheta(w) = \phi_\varrho^{\hbar}(w) * \vartheta(w) = w + \sum_{\ell=2}^{\infty} \Lambda_\ell^{\hbar} a_\ell w^\ell, \qquad (1.6)$$

where $\phi_\varrho^{\hbar}(w)$ is the series given by (1.2) and

$$\Lambda_\ell^{\hbar} = \frac{(\ell-1)^{\hbar} c^{\ell-1}}{(\ell-1)!} e^{-c}.$$

Now, we establish the class $TS(\upsilon, \varrho, \ell, \hbar)$ of analytic functions by using the operator $\mathcal{K}_c^{\hbar}$.
Definition 1.1. For $-1 \leq \upsilon < 1$ and $\varrho \geq 0$, we let $TS(\upsilon, \varrho, \ell, \hbar)$ be the subclass of A consisting of functions of the form (1.5) and satisfying the analytic criterion

$$\Re\left\{\frac{z\left(\mathcal{K}_c^{\hbar}\vartheta(w)\right)'}{\mathcal{K}_c^{\hbar}\vartheta(w)} - \upsilon\right\} \geq \varrho\left|\frac{z\left(\mathcal{K}_c^{\hbar}\vartheta(w)\right)'}{\mathcal{K}_c^{\hbar}\vartheta(w)} - 1\right|,$$

for $z \in E$.

The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, distortion properties, extreme points, radii of starlikness and convexity, Hadmard product and convolution and integral operators for the class.

2. Coefficient bounds

In this section we obtain a necessary and sufficient condition for function $w(z)$ is in the class $TS(\upsilon, \varrho, \ell, \hbar)$.

Theorem 2. *The function w defined by* (1.5) *is in the class* $TS(\upsilon, \varrho, \ell, \hbar)$ *if*

$$\sum_{\ell=2}^{\infty}[\ell(1+\varrho) - (\upsilon+\varrho)]\, \Lambda_\ell^{\hbar}|a_\ell| \leq 1 - \upsilon, \qquad (2.1)$$

where $-1 \leq \upsilon < 1, \varrho \geq 0$. *The result is sharp.*

Proof. We have $w \in TS(\upsilon, \varrho, \ell, \hbar)$ if and only if the condition (1.7) satisfied. Upon the fact that

$$\Re(w) > \varrho|w-1| + \upsilon \Leftrightarrow \Re\{w\left(1+\varrho e^{i\theta}\right) - \varrho e^{i\theta}\} > \upsilon, \ -\pi \leq \theta \leq \pi.$$

Equation (1.7) may be written as

$$\Re\left\{\frac{z\left(\mathcal{K}_c^{\hbar}\vartheta(w)\right)'}{\mathcal{K}_c^{\hbar}\vartheta(w)}\left(1+\varrho e^{i\theta}\right)-\varrho e^{i\theta}\right\}$$

$$=\quad \Re\left\{\frac{z\left(\mathcal{K}_c^{\hbar}\vartheta(w)\right)'1+\varrho e^{i\theta})-\varrho e^{i\theta}\mathcal{K}_c^{\hbar}\vartheta(w)}{\mathcal{K}_c^{\hbar}\vartheta(w)}\right\}>\ \upsilon. \qquad (2.2)$$

Now, we let

$$A(z)\ =z\left(\mathcal{K}_c^{\hbar}\vartheta(w)\right)'1+\varrho e^{i\theta})-\varrho e^{i\theta}\mathcal{K}_c^{\hbar}\vartheta(w)$$
$$B(z)\ =\mathcal{K}_c^{\hbar}\vartheta(w).$$

Then (2.2) is equivalent to

$$|A(z)+(1-\upsilon)B(z)|>|A(z)-(1+\upsilon)B(z)|,\ \text{for } 0\le\upsilon<1.$$

For $A(z)$ and $B(z)$ as above, we have

$$|A(z)+(1-\upsilon)B(z)|\ge(2-\upsilon)|z|-\sum_{\ell=2}^{\infty}[\ell+1-\upsilon+\varrho(\ell-1)]\,\Lambda_\ell^{\hbar}|a_\ell|\left|z^\ell\right|$$

and similarly

$$|A(z)-(1+\upsilon)B(z)|\le\upsilon|z|-\sum_{\ell=2}^{\infty}[\ell-1-\upsilon+\varrho(\ell-1)]\,\Lambda_\ell^{\hbar}|a_\ell|\left|z^\ell\right|.$$

Therefore

$$|A(z)+(1-\upsilon)B(z)|-|A(z)-(1+\upsilon)B(z)|$$
$$\ge\ 2(1-\upsilon)-2\sum_{\ell=2}^{\infty}[\ell-\upsilon+\varrho(\ell-1)]\,\Lambda_\ell^{\hbar}|a_\ell|$$

or $$\sum_{\ell=2}^{\infty}[\ell-\upsilon+\varrho(\ell-1)]\,\Lambda_\ell^{\hbar}|a_\ell|\le(1-\upsilon),$$

which yields(2.1).

On the other hand, we must have

$$\Re\left\{\frac{z\left(\mathcal{K}_c^{\hbar}\vartheta(w)\right)'}{\mathcal{K}_c^{\hbar}\vartheta(w)}\left(1+\varrho e^{i\theta}\right)-\varrho e^{i\theta}\right\}\ge\upsilon.$$

Upon choosing the values of z on the positive real axis where $0\le|z|=r<1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r-\sum_{\ell=2}^{\infty}\left[\ell-\upsilon+\varrho e^{i\theta}(\ell-1)\right]\Lambda_\ell^{\hbar}|a_\ell|\,r^\ell}{z-\sum_{\ell=2}^{\infty}\Lambda_\ell^{\hbar}\,|a_\ell|\,r^\ell}\right\}\ge 0.$$

Since $\Re\left(-e^{i\theta}\right)\ge-\left|e^{i\theta}\right|=-1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r-\sum_{\ell=2}^{\infty}[\ell-\upsilon+\varrho(\ell-1)]\,\Lambda_\ell^{\hbar}|a_\ell|\,r^\ell}{z-\sum_{\ell=2}^{\infty}\Lambda_\ell^{\hbar}\,|a_\ell|\,r^\ell}\right\}\ge 0.$$

Letting $r\to1^-$, we get the desired result. Finally the result is sharp with the extremal function w given by

$$w(z)=z-\frac{1-\upsilon}{[\ell(1+\varrho)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}}z^\ell \qquad (2.3)$$

3. Growth and Distortion Theorems

Theorem 3.1. *Let the function w defined by* (1.5) *be in the class $TS(\upsilon,\varrho,\ell,\hbar)$. Then for $|z|=r$*

$$r-\frac{1-\upsilon}{(\ell+1)(2-\upsilon+\varrho)\Lambda_2^{\hbar}}r^2\le|w(z)|\le r+\frac{1-\upsilon}{(\ell+1)(2-\upsilon+\varrho)\Lambda_2^{\hbar}}r^2. \qquad (3.1)$$

Equality holds for the function

$$w(z)=z-\frac{1-\upsilon}{(\ell+1)(2-\upsilon+\varrho)\Lambda_2^{\hbar}}z^2. \qquad (3.2)$$

Proof. We only prove the right hand side inequality in(3.1), since the other inequality can be justified using similar arguments. In view of Theorem 2.1. we have

$$\sum_{\ell=2}^{\infty}|a_\ell| \leq \frac{1-\upsilon}{(\ell+1)(2-\upsilon+\varrho)\Lambda_2^{\hbar}}.$$

Since ,

$$\begin{aligned} w(z) &= z-\sum_{\ell=2}^{\infty} a_\ell z^\ell \\ |w(z)| &= \left|z-\sum_{\ell=2}^{\infty} a_\ell z^\ell\right| \\ &\leq r+\sum_{\ell=2}^{\infty}|a_\ell| r^\ell \\ &\leq r+r^2\sum_{\ell=2}^{\infty}|a_\ell| \\ &\leq r+\sum_{\ell=2}^{\infty}\frac{1-\upsilon}{(\ell+1)(2-\upsilon+\varrho)\Lambda_2^{\hbar}} r^2 \end{aligned}$$

which yields the right hand side inequality of (3.1).
Next, by using the same technique as in proof of Theorem 3.1, we give the distortion result.

Theorem 3.2. *Let the function* w *defined by* (1.5) *be in the class* $TS(\upsilon,\varrho,\ell,\hbar)$. *Then for* $|z|=r$
$1-\frac{2(1-\upsilon)}{(\ell+1)(2-\upsilon+\varrho)\Lambda_2^{\hbar}} r \leq |w'(z)| \leq 1+\frac{2(1-\upsilon)}{(\ell+1)(2-\upsilon+\varrho)\Lambda_2^{\hbar}} r.$
Equality holds for the function given by (3.2).

Theorem 3.3. *If* $w \in TS(\upsilon,\varrho,\ell,\hbar)$ *then* $w \in TS(\gamma)$,
where

$$\gamma = 1-\frac{(\ell-1)(1-\upsilon)}{[\ell-\upsilon+\varrho(\ell-1)]\Lambda_\ell^{\hbar}-(1-\upsilon)}$$

Equality holds for the function given by (3.2)
Proof. It is sufficient to show that (2.1) implies

$$\sum_{\ell=2}^{\infty}(\ell-\gamma)\,|a_\ell| \leq 1-\gamma,$$

that is

$$\frac{\ell-\gamma}{1-\gamma} \leq \frac{[\ell-\upsilon+\varrho(\ell-1)]\Lambda_\ell^{\hbar}}{(1-\upsilon)},$$

then

$$\gamma \leq 1-\frac{(\ell-1)(1-\upsilon)}{[\ell-\upsilon+\varrho(\ell-1)]\Lambda_\ell^{\hbar}-(1-\upsilon)}.$$

The above inequality holds true for $\ell \in \mathbb{N}_0, \ell \geq 2, \varrho \geq 0$ and $0 \leq \upsilon < 1$.

4. Extreme points

Theorem 4.1. *Let* $w_1(z) = z$ *and*

$$w_\ell(z) = z-\frac{1-\upsilon}{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}} z^\ell, \qquad (4.1)$$

for $\ell = 2,3,\cdots$. *Then* $w(z) \in TS(\upsilon,\varrho,\ell,\hbar)$ *if and only if* $w(z)$ *can be expressed in the form* $w(z) = \sum_{\ell=1}^{\infty}\zeta_\ell\, w_\ell(z)$, *where* $\zeta_\ell \geq 0$ *and* $\sum_{\ell=1}^{\infty}\zeta_\ell = 1$.
Proof. Suppose $w(z)$ can be expressed as in (4.1). Then

$$\begin{aligned} w(z) &= \sum_{\ell=1}^{\infty} \zeta_\ell\, w_\ell(z) = \zeta_1 w_1(z) + \sum_{\ell=2}^{\infty} \zeta_\ell\, w_\ell(z) \\ &= \zeta_1 w_1(z) + \sum_{\ell=2}^{\infty} \zeta_\ell \left\{ z - \frac{1-\upsilon}{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}} z^\ell \right\} \\ &= \zeta_1 z + \sum_{\ell=2}^{\infty} \zeta_\ell\, z - \sum_{\ell=2}^{\infty} \zeta_\ell \left\{ \frac{1-\upsilon}{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}} z^\ell \right\} \\ &= z - \sum_{\ell=2}^{\infty} \zeta_\ell \left\{ \frac{1-\upsilon}{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}} z^\ell \right\}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{\ell=2}^{\infty} \zeta_\ell \left(\frac{1-\upsilon}{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}} \right) \left(\frac{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}}{1-\upsilon} \right) \\ &= \sum_{\ell=2}^{\infty} \zeta_\ell = \sum_{\ell=1}^{\infty} \zeta_\ell - \zeta_1 = 1 - \zeta_1 \le 1. \end{aligned}$$

So, by Theorem 2.1, $w \in TS(\upsilon, \varrho, \ell, \hbar)$.

Conversely, we suppose $w \in TS(\upsilon, \varrho, \ell, \hbar)$. Since

$$|a_\ell| \le \frac{1-\upsilon}{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}}, \quad \ell \ge 2.$$

We may set

$$\zeta_\ell = \frac{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}}{1-\upsilon} |a_\ell|, \quad \ell \ge 2$$

and $\zeta_1 = 1 - \sum_{\ell=2}^{\infty} \zeta_\ell$. Then

$$\begin{aligned} w(z) &= z - \sum_{\ell=2}^{\infty} a_\ell\, z^\ell = z - \sum_{\ell=2}^{\infty} \zeta_\ell \frac{1-\upsilon}{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}} z^\ell \\ &= z - \sum_{\ell=2}^{\infty} \zeta_\ell\, [z - w_\ell(z)] = z - \sum_{\ell=2}^{\infty} \zeta_\ell\, z + \sum_{\ell=2}^{\infty} \zeta_\ell\, w_\ell(z) \\ &= \zeta_1 w_1(z) + \sum_{\ell=2}^{\infty} \zeta_\ell\, w_\ell(z) = \sum_{\ell=1}^{\infty} \zeta_\ell\, w_\ell(z). \end{aligned}$$

Corollary 4.2. *The extreme points of* $TS(\upsilon, \varrho, \ell, \hbar)$ *are the functions* $w_1(z) = z$ *and* $w_\ell(z) = z - \frac{1-\upsilon}{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}} z^\ell$, $\ell \ge 2$.

5. Radii of Close-to-convexity, Starlikeness and Convexity

A function $w \in TS(\upsilon, \varrho, \ell, \hbar)$ is said to be close-to-convex of order δ if it satisfies

$$\Re\{w'(z)\} > \delta, \quad (0 \le \delta < 1;\ z \in E).$$

Also A function $w \in TS(\upsilon, \varrho, \ell, \hbar)$ is said to be starlike of order δ if it satisfies

$$\Re\left\{\frac{zw'(z)}{w(z)}\right\} > \delta, \quad (0 \le \delta < 1;\ z \in E).$$

Further a function $w \in TS(\upsilon, \varrho, \ell, \hbar)$ is said to be convex of order δ if and only if $zu'(z)$ is starlike of order δ that is if

$$\Re\left\{1 + \frac{zw'(z)}{w(z)}\right\} > \delta, \quad (0 \le \delta < 1;\ z \in E).$$

Theorem 5.1. *Let* $w \in TS(\upsilon, \varrho, \ell, \hbar)$. *Then* w *is close-to-convex of order* δ *in* $|z| < R_1$, *where* $R_1 = \inf_{k \ge 2} \left[\frac{(1-\delta)[\ell - \upsilon + \varrho(\ell-1)]\Lambda_\ell^{\hbar}}{\ell(1-\upsilon)} \right]^{\frac{1}{\ell-1}}$.

The result is sharp with the extremal function w *is given by* (2.3).

Proof. It is sufficient to show that $|w'(z) - 1| \le 1 - \delta$, for $|z| < R_1$. We have

$$|w'(z)-1|=\left|-\sum_{\ell=2}^{\infty}\ell\, a_\ell z^{\ell-1}\right|\le\sum_{\ell=2}^{\infty}\ell\, a_\ell|z|^{\ell-1}. \qquad (5.1)$$

Thus $|w'(z)-1|\le 1-\delta$ if

$$\sum_{\ell=2}^{\infty}\frac{\ell}{1-\delta}|a_\ell||z|^{\ell-1}\le 1. \qquad (5.2)$$

But Theorem (5.1) confirms that

$$\sum_{\ell=2}^{\infty}\frac{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}}{1-\upsilon}|a_\ell|\le 1.$$

Hence (5.1) will be true if

$$\frac{\ell|z|^{\ell-1}}{1-\delta}\le\frac{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}}{1-\upsilon}.$$

We obtain

$$|z|\le\left[\frac{(1-\delta)[\ell-\upsilon+\varrho(\ell-1)]\Lambda_\ell^{\hbar}}{\ell(1-\upsilon)}\right]^{\frac{1}{\ell-1}},\ell\ge 2$$

as required.

Theorem 5.2. *Let $w\in TS(\upsilon,\varrho,\ell,\hbar)$. Then w is starlike of order δ in $|z|<R_2$, where $R_2=\inf_{k\ge 2}\left[\frac{(1-\delta)[\ell-\upsilon+\varrho(\ell-1)]\Lambda_\ell^{\hbar}}{(\ell-\delta)(1-\upsilon)}\right]^{\frac{1}{\ell-1}}$.*

*Th*e result is sharp with the extremal function w is given by (2.3)

Proof. We must show that $\left|\frac{zw\prime(z)}{w(z)}-1\right|\le 1-\delta$, for $|z|<R_2$. We have

$$\begin{aligned}\left|\frac{zw'(z)}{w(z)}-1\right| &=\left|\frac{-\sum_{\ell=2}^{\infty}(\ell-1)\,a_\ell z^{\ell-1}}{1-\sum_{\ell=2}^{\infty}a_\ell\,z^{\ell-1}}\right|\\ &\le\frac{\sum_{\ell=2}^{\infty}(\ell-1)\,|a_\ell||z|^{\ell-1}}{1-\sum_{\ell=2}^{\infty}|a_\ell|\,|z|^{\ell-1}}\\ &\le 1-\delta. \qquad (5.3)\end{aligned}$$

Hence (5.3) holds true if

$$\sum_{\ell=2}^{\infty}(\ell-1)\,|a_\ell||z|^{\ell-1}\le(1-\delta)\left(1-\sum_{\ell=2}^{\infty}|a_\ell|\,|z|^{\ell-1}\right)$$

or equivalently,

$$\sum_{\ell=2}^{\infty}\frac{\ell-\delta}{1-\delta}|a_\ell||z|^{\ell-1}\le 1. \qquad (5.4)$$

Hence, by using (5.2) and (5.4) will be true if

$$\frac{\ell-\delta}{1-\delta}|z|^{\ell-1}\le\frac{[\ell(\varrho+1)-(\upsilon+\varrho)]\Lambda_\ell^{\hbar}}{1-\upsilon}$$

$$\Rightarrow|z|\le\left[\frac{(1-\delta)[\ell-\upsilon+\varrho(\ell-1)]\Lambda_\ell^{\hbar}}{(\ell-\delta)(1-\upsilon)}\right]^{\frac{1}{\ell-1}},\ell\ge 2$$

which completes the proof.

By using the same technique in the proof of Theorem 5.2, we can show that

$$\left|\frac{zw\prime\prime(z)}{w\prime(z)}-1\right|\le 1-\delta,$$

for $|z|<R_3$, with the aid of Theorem 2.1. Thus we have the assertion of the following Theorem 5.3.

Theorem 5.3. *Let $w\in TS(\upsilon,\varrho,\ell,\hbar)$. Then w is convex of order δ in $|z|<R_3$, where $R_3=\inf_{k\ge 2}\left[\frac{(1-\delta)[\ell-\upsilon+\varrho(\ell-1)]\Lambda_\ell^{\hbar}}{\ell(\ell-\delta)(1-\upsilon)}\right]^{\frac{1}{\ell-1}}$.*

The result is sharp with the extremal function w is given by (2.3).

6. Inclusion theorem involving modified Hadamard products

For functions

$$w_j(z) = z - \sum_{\ell=2}^{\infty} |a_{\ell,j}| z^{\ell},\ j = 1,2 \qquad (6.1)$$

in the class A, we define the modified Hadamard product $w_1 * w_2(z)$ of $w_1(z)$ and $w_2(z)$ given by

$$w_1 * w_2(z) = z - \sum_{\ell=2}^{\infty} |a_{\ell,1}|\,|a_{\ell,2}| z^{\ell}.$$

We can prove the following.

Theorem 6.1. *Let the function* $w_j,\ j = 1,2,$ *given by* (6.1) *be in the class* $TS(\upsilon, \varrho, \ell, \hbar)$ *respectively. Then* $w_1 * w_2(z) \in TS(\upsilon, \varrho, \ell, \hbar, \xi)$, *where*
$\xi = 1 - \frac{(1-\upsilon)^2}{(\ell+1)(2-\upsilon)(2-\upsilon+\varrho)(1+\Lambda)-(1-\upsilon)^2}$.

Proof. Employing the technique used earlier by Schild and Silverman [20] , we need to find the largest ξ such that

$$\sum_{\ell=2}^{\infty} \frac{[\ell - \xi + \varrho(\ell - 1)]\Lambda_{\ell}^{\hbar}}{1 - \xi} |a_{\ell,1}||a_{\ell,2}| \leq 1.$$

Since $w_j \in TS(\upsilon, \varrho, \ell, \hbar),\ j = 1,2$ then we have

$$\sum_{\ell=2}^{\infty} \frac{[\ell - \upsilon + \varrho(\ell - 1)]\Lambda_{\ell}^{\hbar}}{1 - \upsilon} |a_{\ell,1}| \leq 1$$

and
$$\sum_{\ell=2}^{\infty} \frac{[\ell - \upsilon + \varrho(\ell - 1)]\Lambda_{\ell}^{\hbar}}{1 - \upsilon} |a_{\ell,2}| \leq 1,$$

by the Cauchy-Schwartz inequality, we have

$$\sum_{\ell=2}^{\infty} \frac{[\ell - \upsilon + \varrho(\ell - 1)]\Lambda_{\ell}^{\hbar}}{1 - \upsilon} \sqrt{|a_{\ell,1}||a_{\ell,2}|} \leq 1.$$

Thus it is sufficient to show that

$$\frac{[\ell - \xi + \varrho(\ell - 1)]\Lambda_{\ell}^{\hbar}}{1 - \xi} |a_{\ell,1}||a_{\ell,2}| \leq \frac{[\ell - \upsilon + \varrho(\ell - 1)]\Lambda_{\ell}^{\hbar}}{1 - \upsilon} \sqrt{|a_{\ell,1}||a_{\ell,2}|},\ \ \ell \geq 2,$$

that is

$$\sqrt{|a_{\ell,1}||a_{\ell,2}|} \leq \frac{(1 - \xi)[\ell - \upsilon + \varrho(\ell - 1)]}{1 - \upsilon)[\ell - \xi + \varrho(\ell - 1)]}.$$

Note that

$$\sqrt{|a_{\ell,1}||a_{\ell,2}|} \leq \frac{(1 - \upsilon)}{[\ell - \upsilon + \varrho(\ell - 1)]\Lambda_{\ell}^{\hbar}}.$$

Consequently, we need only to prove that

$$\frac{(1 - \upsilon)}{[\ell - \upsilon + \varrho(\ell - 1)]\Lambda_{\ell}^{\hbar}} \leq \frac{(1 - \xi)[\ell - \upsilon + \varrho(\ell - 1)]}{1 - \upsilon)[\ell - \xi + \varrho(\ell - 1)]},\ \ell \geq 2,$$

or, equivalently, that

$$\xi \leq 1 - \frac{(\ell - 1)(1 + \varrho)(1 - \upsilon)^2}{[\ell - \upsilon + \varrho(\ell - 1)]^2\Lambda_{\ell}^{\hbar} - (1 - \upsilon)^2},\ \ell \geq 2.$$

Since

$$A(k) = 1 - \frac{(\ell - 1)(1 + \varrho)(1 - \upsilon)^2}{[\ell - \upsilon + \varrho(\ell - 1)]^2\Lambda_{\ell}^{\hbar} - (1 - \upsilon)^2},\ \ell \geq 2$$

is an increasing function of $\ell, \ell \geq 2$, letting $\ell = 2$ in last equation, we obtain

$$\xi \leq A(2) = 1 - \frac{(1 + \varrho)(1 - \upsilon)^2}{[2 - \upsilon + \varrho]^2\Lambda_{\ell}^{\hbar} - (1 - \upsilon)^2}.$$

Finally, by taking the function given by (3.2), we can see that the result is sharp.

7. Convolution and Integral Operators

Let $w(z)$ be defined by (1.5) and suppose that $g(z) = z - \sum_{\ell=2}^{\infty}|b_\ell|\, z^\ell$. Then,the Hadamard product (or convolution) of $w(z)$ and $g(z)$ defined here by

$$w(z) * g(z) = w * g(z) = z - \sum_{\ell=2}^{\infty}|a_\ell|\,|b_\ell| z^\ell.$$

Theorem 12. *Let* $w \in TS(\upsilon, \varrho, \ell, \hbar)$ *and* $g(z) = z - \sum_{\ell=2}^{\infty}|b_\ell|\, z^\ell, 0 \le |b_\ell| \le 1$. *Then* $w * g \in TS(\upsilon, \varrho, \ell, \hbar)$.

Proof. In view of Theorem 2.1, we have

$$\begin{aligned} &\sum_{\ell=2}^{\infty}[\ell - \upsilon + \varrho(\ell - 1)]\, \Lambda_\ell^\hbar |a_\ell||b_\ell| \\ \le\ & \sum_{\ell=2}^{\infty}[\ell - \upsilon + \varrho(\ell - 1)]\, \Lambda_\ell^\hbar |a_\ell| \\ \le\ & (1 - \upsilon). \end{aligned}$$

Theorem 13. *Let* $w \in TS(\upsilon, \varrho, \ell, \hbar)$ *and* α *be real number such that* $\alpha > -1$. *Then the function* $F(z) = \frac{\alpha+1}{z^\alpha}\int_0^z t^{\alpha-1}\, w(t)dt$ *also belongs to the class* $TS(\upsilon, \varrho, \ell, \hbar)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{\ell=2}^{\infty}|A_\ell|\, z^\ell, \text{ where } A_\ell = \left(\frac{\alpha + 1}{\alpha + \ell}\right)|a_\ell|.$$

Since $\alpha > -1$, than $0 \le A_\ell \le |a_\ell|$. Which in view of Theorem 2.1, $F \in TS(\upsilon, \varrho, \ell, \hbar)$.

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