THE GEODETIC NON-SPLIT DOMINATION NUMBER OF A GRAPH

P. Anto Paulin Brinto
Department of Mathematics,
antopaulin@gmail.com
Scott Christian College (Autonomous), Nagercoil - 629 003, India.

Abstract

Let G be a connected graph. A set $S \subseteq V(G)$ is called a geodetic non-split dominating set of G if S is both geodetic set and a non split dominating set of G. The geodetic non-split domination number of G is the minimum order of its geodetic non-split dominating set of G and denoted by $\gamma_{gns}(G)$. In this paper, we determined the geodetic non-split domination number of some standard graphs.

Keywords : distance , geodetic number, domination number, non-split domination number, geodetic non-split domination number.
AMS Subject Classification : 05C12, 05C69

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to [5]. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in G \,/\, uv \in E(G)\}$. The *degree* of a vertex $v \in V$ is $deg(v) = |N(v)|$. If $e = \{u, v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then we call e a *pendant edge* or *end edge*, u a leaf or end vertex and v a support. A vertex of degree $n - 1$ is called a *universal vertex*. The minimum and maximum degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *subgraph induced* by a set S of vertices of a graph G is denoted by $G[S]$ with

$V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) : u, v \in S\}$. A vertex v in a graph G is called a *extreme vertex* if the subgraph induced by its neighborhood is *complete*.

For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. For $u, v \in V$, The closed interval $I[u, v]$ consists of all vertices lying on some $u - v$ geodesic of G including the vertices u and v. For $S \subseteq V$, $I[S] = \bigcup_{u,v \in S} I[u, v]$. A set S of vertices is called a geodetic set if $I[S] = V$. The geodetic number of G is the minimum order of its geodetic set of G and denoted by $g(G)$. Any geodetic set of order $g(G)$ is g-set of G. The geodetic number of a graph is studied in [1-6, 11-15, 20-28].

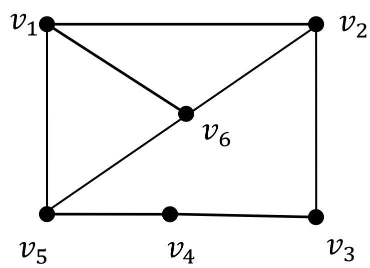
A set of vertices D in a graph G is a dominating set if each vertex of G is dominated by some vertex of D. The domination number of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$. A dominating set of size $\gamma(G)$ is said to be a γ-set. The domination number of a graph is studied in [7-10,16-19,29-31]. A dominating set D in a graph G is a non-split dominating set if $G[V - D]$ is connected. The non-split domination number of G is the minimum order of its non-split dominating set of G and denoted by $\gamma_{ns}(G)$. Any geodetic non-split dominating set of order $\gamma_{ns}(G)$ is γ_{ns}-set of G. The non-split domination number of a graph is studied in [32]. Geodetic non-split domination concepts have interesting applications in channel assignment problems in radio technologies. Also, there are useful applications of these concepts to security based communication network design. In this article we studied the concept of the geodetic non-split domination number of a graph. The following theorem is used in the sequel.

Theorem 1.1. [6] Each extreme vertex of a connected graph G belongs to every geodetic set of G.

2. The geodetic non-split domination number of a graph.

Definition 2.1. Let G be a connected graph . A set $S \subseteq V(G)$ is called a geodetic non-split dominating set of G if S is both geodetic set and a non split dominating set of G. The geodetic non-split domination number of G is the minimum order of its geodetic non-split dominating set of G and denoted by $\gamma_{gns}(G)$. Any geodetic non-split dominating set of order $\gamma_{gns}(G)$ is γ_{gns}-set of G.

Example 2.2. For the graph G of Figure 2.1, $S = \{v_2, v_3, v_5\}$ is a g-set of G so that $g(G) = 3$. Since $< S >$ is not connected, S is not a geodetic non-split dominating set of G and so $\gamma_{gns}(G) \geq 4$. Let $S_1 = \{v_2, v_3, v_4, v_5\}$. Then S_1 is γ_{ns}-set of G so that $\gamma_{gns}(G) = 4$.



G
Figure 2.1

Theorem 2.3 Each extreme vertex of G belongs to every geodetic non - split dominating set of G.

Proof. This follows from Theorem 1.1. ■

Theorem 2.4. For the complete graph $G = K_n (n \geq 2)$, $\gamma_{gns}(G) = n$.

Proof. This follows from Theorem 2.2. ■

Theorem 2.5. For the star $G = K_{1,n-1}$ $(n \geq 3)$, Then $\gamma_{gns}(G) = n - 1$.

Proof. Since the set of end vertices of G is a geodetic dominating set of G, the result follow from Theorem 2.2. ■

Theorem 2.6. For any double star of order $n \geq 4$, or a double star. Then $\gamma_{gns}(G) = n - 2$.

Proof. Since the set of end vertices of G is a geodetic dominating set of G, the result follow from Theorem 2.1.

■

Theorem 2.7. For the path $G = P_n$ $(n \geq 4)$, $\gamma_{gns}(G) = n - 2$.

Proof. Let $G = P_n: v_1, v_2, \ldots, v_n$ and $S = V - \{v_2, v_3\}$. Then S is a geodetic non-split dominating set of G and so $\gamma_{gns}(G) \leq n - 2$. We prove that $\gamma_{gns}(G) = n - 2$. On the contrary suppose that $\gamma_{gns}(G) \leq n - 3$. Then there exists a γ_{gns}-set S' such that $|S'| \leq n - 3$. Then $G[V - S']$ is not connected and so S' is not a geodetic non-split dominating set of G. Which is a contradiction. Therefore $\gamma_{gns}(G) = n - 2$. .

■

Theorem 2.8. Let G be tree with k end vertices such that every vertex of G is either an end vertex or a support vertex of G. Then $\gamma_{gns}(G) = k$.

Proof. Let S be the set of all end vertices of G. Then by Theorem 1.1, $\gamma_{gns}(G) \geq k$. Since S is a geodetic non-split dominating set of G, $\gamma_{gns}(G) = k$.

■

Theorem 2.8. For the graph $G = K_1 + (m_1K_1 \cup m_2K_2 \cup \ldots \cup m_rK_r)$ with $m_1 + m_2 + \cdots + m_r \geq 2$, $\gamma_{gns}(G) = n - 1$.

Proof. Let x be the cut vertex of G. Then $S = V - \{x\}$ is the set of all extreme vertices of G. By Theorem 2.1, $\gamma_{gns}(G) \geq n - 1$. Since S is a geodetic non-split dominating set of G, $\gamma_{gns}(G) = n - 1$. ■

Theorem 2.9. For the cycle $G = C_n$ $(n \geq 4)$,

$$\gamma_{gns}(G) = \begin{cases} n-1 & if\ \ n = 4\ or\ 5 \\ n-2 & if\ \ n \geq 6 \end{cases}$$

Proof. Let $G = C_n = v_1, v_2, \ldots, v_n, v_{1.}$ If $n = 4$ or 5, then it can be easily verified $\gamma_{gns}(G) = n - 1$. So let that $n \geq 6$. Let $S = V - \{v_2, v_3\}$. Then S is a geodetic non-split dominating set of G and so $\gamma_{gns}(G) \leq n - 2$. We prove that $\gamma_{gns}(G) = n - 2$. On the contraty suppose that $\gamma_{gns}(G) \leq n-3$. Then there exists a γ_{gns}-set S' such that $\gamma_{gns}(G) \leq n-3$. Then $G[V-S']$ is not connected and so S' is not a geodetic non-split dominating set of G. Which is a contradiction. Therefore $\gamma_{gns}(G) = n-2$.

Theorem 2.8. If *G* is the complete *r*-partite graph $K_{n_1,n_2,\dots,n_r}$ of order *n* with $r \geq 2$ and $1 \leq n_1 \leq n_2 \leq \dots \leq n_r$, then

(i) $\gamma_{gns}(G) = p_r$ when $n_r = 1$ and $p_r \geq 2$

(ii) $\gamma_{gns}(G) = \min\{n_t, 4\}$; when $n_{r-1} \geq 2$ and $t = \min\{i/n_i \geq 2\}$

Proof. (i) Let $X_i = \{u_i\}, 1 \leq i \leq n-1$ and $X_r = \{u_1', u_2', \dots, u_r'\}$ be the r-partite sets of *G*. Let *X* be a set of vertices of *G*. If $|X| < n_r$, then there exists at least one vertex say x such that $x \notin X_r$. Let y be a vertex of $V(G) - X_r$. Then y does not lie on a geodesic joining two vertices of *X* so that $\gamma_{gns}(G) \geq n_r$. Now, it is clear that X_r is a monophonic dominating set of *G* so that $\gamma_{gns}(G) = n_r$.

(iii) Let $X_1 = \{u_{11}, u_{12}, \dots, u_{1p_1}\}$, $X_2 = \{u_{21}, u_{22}, \dots, u_{2p_2}\}, \dots$, $X_r = \{u_{r1}, u_{r2}, \dots, u_{rn_r}\}$ be the *r*-partite sets of *G*. It is easily seen that no two element subset of *G* is a geodetic non split dominating set of *G* so that $\gamma_{gns}(G) \geq 3$. Let *X* be a set of vertices with three elements. If $|X| = 3$, then X is not a geodetic non split dominating set of *G* and so $\gamma_m(G) \geq 4$. Now, let $S = \{x, y, x', y'\}$, where $x, y \in X; x', y' \in X_2$. It is clear that S is a geodetic non split dominating set of *G* so that $\gamma_{gns}(G) = 4$. ∎

Corollary 2.9. For the complete bipartite graph $G = K_{r,s}$,

(i) $\gamma_{gns}(G) = 2$ if $r = s = 1$

(ii) $\gamma_{gns}(G) = s$ if $s \geq 2, r = 1$

(iii) $\gamma_{gns}(G) = \min\{r, s, 4\}$ if $r, s \geq 2$

Theorem 2.10. For the wheel $G = K_1 + C_{n-1}$, $(n \geq 4)$,

$$\gamma_{gns}(G) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases} .$$

Proof. Let $V(K_1) = x$ and C_{n-1} be $v_1, v_2, v_3, \dots, v_{n-1}, v_1$. Let n is even. Then $S = S = \{v_1, v_3, \dots, v_{n-1}\}$ is a geodetic non-split dominating set of G and so $\gamma_{gns}(G) \leq \frac{n}{2}$. We

prove that $\gamma_{gns}(G) = \frac{n}{2}$. On the contrary suppose that $\gamma_{gns}(G) \leq \frac{n}{2} - 1 = \frac{n-2}{2}$. Then there exists a γ_{gns}-set S' such that $|S'| \leq \frac{n-2}{2}$. Let $y \in S'$, Then $y \neq x$. Therefore $y = v_i$ for some $i (1 \leq i \leq n-1)$. Since $d(z, w) = 2$ for every $z, w \in S'$, $y \notin I[S']$. Hence it follows that S' is not a geodetic non-split dominating set of G so that $\gamma_{gns}(G) = \frac{n}{2}$.

Next assume that $n-1$ is odd. Let $W = \{v_1, v_3, \dots, v_{n-1}, v_{n-2}\}$. Then W is a minimum geodetic non-split dominating set of G so that $\gamma_{gns}(G) = \frac{n-1}{2}$.

■

Theorem 2.11. For the graph Helm graph $G = H_k$, $\gamma_{gns}(G) = k + 1$.

Proof. Let u be the central vertex of G and z be the set of r end vertices of G. By Theorem Z is a subset of G. Since u is not dominated by any vertex of Z, Z is a non-split dominating set of G and so $\gamma_{gns}(G) \geq k + 1$. Let $Z' = Z \cup \{u\}$. Then $I[Z] = V$ and $G[V - Z]$ is connected. Therefore Z is a geodetic non-split dominating set of G and so $\gamma_{gns}(G) = k + 1$.

■

Theorem 2.12. For the banana graph $G = B_{r,s}$, $\gamma_{gns}(G) = r + 2$.

Proof. Let x be the central vertex of G and Z be the set of all end vertices of G. Then by Theorem Z is the subset of every geodetic non- split dominating set of G and so $\gamma_{gns}(G) \geq r + 1$.

Since the vertex x is not dominated by any element of x, Z is not a geodetic non-split dominating set of G and so $\gamma_{gns}(G) \geq r+1$. Let $Z' = Z \cup \{x\}$. Then Z' is a geodetic set of G and $G[V-Z']$ is connected. Therefore Z' is a geodetic non-split dominating set of G so that $\gamma_{gns}(G) = r+1$. ■

Theorem 2.13. For the Lotus inside cycle $LC_n (n \geq 3)$,

$\gamma_{gns}(LC_n) = \left\lceil \frac{n}{2} \right\rceil + 1$.

Proof. Let C_n be $v_1, v_2, v_3, \ldots, v_n$ be the inside in the graph with vertices $u_0, u_1, \ldots, u_n$ such that v_0 is the central vertex of S_n and v_i be adjacent to u_i and u_{i+1} taken modulo n, $i = 1,2,3, \ldots, n-1$. Let $n = 3$. Then $S = \{u_0, u_1, u_3\}$ is a minimum geodetic non- split dominating set of G so that $\gamma_{gns}(LC_3) = 3 = \left\lceil \frac{3}{2} \right\rceil + 1$. Next assume that $n \geq 4$. We consider the following two cases.

Case(i) n is even.

Let $n = 2k$ $(k \geq 2)$. Let $S = \{u_0, v_1, v_3, \ldots, v_{2k-1}\}$. Then S is a geodetic non-split dominating set of G and so $\gamma_{gns}(G) \geq \frac{n}{2} + 1 = \left\lceil \frac{n}{2} \right\rceil + 1$. We prove that $\gamma_{gns}(G) = \left\lceil \frac{n}{2} \right\rceil + 1$. On the contrary, suppose that $\gamma_{gns} \leq \left\lceil \frac{n}{2} \right\rceil$. Then there exists a geodetic non-split dominating set S' such that $|S'| \leq \left\lceil \frac{n}{2} \right\rceil$.

Case(ii) n is odd.

Let $n = 2k+1$ $(k \geq 2)$. Let $S = \{u_0, v_1, v_3, \ldots, v_{2k+1}\}$. Then the argument similar in case (i), we can prove that $\gamma_{gns}(G) = \left\lceil \frac{n}{2} \right\rceil + 1$. ■

References

[1] H. Abdollahzadeh Ahangar,V. Samodivkin, S.M. Sheikholeslami, A.Khodkar, The restrained number of a graph, *Bulletin of the Malaysian Mathematical Sciences Society*, 38(3), (2015), 1143-1155.

[2] H. Abdollahzadeh Ahangar, F. Fujie-Okamoto, V. Samodivkin, On the forcing

connected geodetic number and the connected geodetic number of a graph, *Ars Combinatoria* 126, (2016), 323-335.

[3] D. Anusha, J. John and S. Joseph Robin, The geodetic hop domination number of complementary prisms, *Discrete Mathematics, Algorithms and Applications* , 13(6), (2021),2150077

[4] S. Beulah Samli, J. John and S. Robinson Chellathurai, The double geo chromatic number of a graph, *Bulletin of the International Mathematical virtual Institute*, 11(1), 2021, 25 - 38.

[5] F.Buckley and F.Harary, Distance in Graphs, Addition- Wesley, Redwood City, CA, 1990.

[6] G. Chartrand, F. Harary and P. Zhang, On the geodetic number of a graph, *Networks*, 39, (2002), 1-6.

[7] J. John and P. Arul Paul Sudhahar, The monophonic domination number of a graph, *Proceeding of the International Conference on Mathematics and Business management*, 1, (2012), 142-145.

[8] J. John, G. Edwin and P. Arul Paul Sudhahar, The Steiner domination number of a graph, *International Journal of Mathematics and Computer Applications Research*, 3(3), (2013), 37 - 42.

[9] J. John and N. Arianayagam, The detour domination number of a graph, *Discrete Mathematics Algorithms and Applications*, 09, 01,(2017), 1750006.

[10] J. John, P. Arul Paul Sudhahar and D. Stalin, On the (M,D) number of a graph, *Proyecciones Journal of Mathematics*, 38(2),(2019), 255-266.

[11] J. John and D.Stalin, Edge geodetic self decomposition in Graphs, *Discrete*

Mathematics, Algorithms and Applications, 12(5), (2020), 2050064, 7 pages.

[12] J.John, The forcing monophonic and the forcing geodetic numbers of a graph, Indonesian Journal of Combinatorics .4(2), (2020) 114-125.

[13] J. John and D.Stalin,The edge geodetic self decomposition number of a graph, *RAIRO*

Operations Research, RAIRO- 55,(2021), S1935-S1947

[14] J. John and D.Stalin, Distinct edge geodetic decomposition in Graphs,

Communication in Combinatorics and Optimization, 6 (2),(2021), 185-196

[15] J. John, On the vertex monophonic, vertex geodetic and vertex Steiner numbers of graphs, Asian-European Journal of Mathematics 14 (10), (2021), 2150171

[16] J.John, and V. Sujin Flower, The edge-to-edge geodetic domination number of a graph, *Proyecciones journal of Mathematics,* 40(3), (2021), 635-658.

[17] J. John and M.S. Malchijah, The forcing non-split domination number of a graph, Korean journal of mathematics, 29(1), (2021), 1-12.

[18] J.John, and V. Sujin Flower, On the forcing domination and the forcing total domination numbers of a graph, *Graphs and Combinatorics*,38, (2022)142.

[19] S.Kavitha, S.Chellathurai and J.John, On the forcing connected domination number of a graph, Journal of Discrete Mathematical Sciences and Cryptography, 3(20) (2017) 611-624.

[20] Omar A. Abu Ghneim, Basem Al-Khamaiseh and Hasan Al-Ezeh, The geodetic, hull, and Steiner numbers of powers of paths, (2014), *Utilitas Mathematica*, 95:289-294.

[21] A. P. Santhakumaran and J. John, Edge Geodetic Number of a Graph, *Journal of Discrete Mathematical Sciences and Cryptography* 10(3), (2007) ,415-432.

[22] A. P. Santhakumaran and J. John, The edge Steiner number of a graph, *Journal*

of Discrete Mathematical Sciences and Cryptography, 10 (2007), 677 - 696.

[23] A. P. Santhakumaran, and J. John, The upper edge geodetic number and the forcing edge geodetic number of a graph, *Opuscula Mathematica* 29, 4, (2009), 427 - 441.

[24] A. P. Santhakumaran, P. Titus and J. John, The upper connected geodetic number and the forcing connected geodetic number of a graph, *Discrete Applied Mathematics*, 157 (7), (2009), 1571 - 1580.

[25] A. P. Santhakumaran, P. Titus and J. John, On the connected geodetic number of a graph, *Journal of Comb. Math. and Comb. Comp*., 69, (2009), 219 - 229.

[26] A.P. Santhakumaran and J. John, The connected edge geodetic number of a graph, *SCIENTIA Series A: Mathematical Sciences*, 17, (2009), 67 - 82.

[27] A.P. Santhakumaran and J. John, On the forcing geodetic and forcing Steiner numbers of a graph, *Discussiones Mathematicae Graph Theory*,(2011),31, 611-624

[28] A. P. Santhakumaran and J. John, The upper connected edge geodetic number of a graph, *Filomat*, 26(1), (2012), 131 - 141.

[29] D. Stalin and J. John, The forcing edge geodetic domination number of a graph, *Journal of Advanced Research in Dynamical and Control Systems*, 10(4),(2018), 172-177.

[30] D. Stalin and J. John, Edge geodetic dominations in graphs, International Journal of Pure and Applied Mathematics, 116,(22),(2017), 31-40.

[31] V. R. Kulli and B. Janakiram, The non-split domination number of a graph, Indian Journal of Pure and Applied Mathematics, 31(5), (2000), 545 - 550.